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SEVERAL DIMENSIONAL PROPERTIES OF THE SPECTRUM OF A UNIFORM ALGEBRA

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The author has previously introduced a generalized Šilov boundary which seems useful in studying analytic structure of several dimensions in the spectrum of a uniform algebra $\mathfrak{A}$. Related generalizations of $\mathfrak{A}$-convexity, $\mathfrak{A}$-polyhedra, etc. are developed here. Several different but equivalent approaches to these various generalizations are described. The generalized boundaries discussed here are related to the "q-holomorphic functions" of the author, and to $\mathfrak{A}$-holomorphic convexity.

The generalized Šilov boundary was introduced by the author [2] to study multi-dimensional analytic structure in the spectrum of a uniform algebra. Related but more extensive applications of this boundary were developed by Sibony [13]. Kramm [10] has utilized this boundary to help obtain a characterization of Stein algebras. The definition of the Šilov boundary of order q in [2] was motivated by consideration of $\mathfrak{A}$-varieties of codimension q in the spectrum of $\mathfrak{A}$.

Here we show how extending $\mathfrak{A}$ by the conjugates of q functions from $\mathfrak{A}$, decomposing the spectrum of $\mathfrak{A}$ into $q+1$ pieces, or generalizing the idea of an $\mathfrak{A}$-polyhedron all lead to the same circle of ideas as the qth order boundary. We also relate this boundary to "q-holomorphic" functions. (In [3], [4] the author defined a function f to be q-holomorphic if $\bar{\partial} f \wedge (\partial\bar{\partial} f)^q = 0$, and developed some elementary properties of such functions.) Finally, we establish a connection between the first order boundary and the $\mathfrak{A}$-holomorphic convexity studied by Rickart [11].

We refer the reader to Stout's book, [14], for notation, terminology, and basic results concerning function algebras and uniform algebras.

1. Generalized boundaries and extension algebras.

Let A be a function algebra on the compact Hausdorff space X (although the results of this section also apply if X is locally compact). Let $\partial_0 A$ denote the usual Šilov boundary for A. For a subset S of A let $\sharp S$ denote the cardinality of S and let

$$V(S) = \{x \in X \mid \forall f \in S,\ f(x) = 0\}\ .$$

If K is a closed subset of X define the restriction algebra

$$A|K = \{f|_K: f \in A\}$$

and let A_K denote the uniform closure of $A|K$ in $C(K)$.

DEFINITION. Let q be a nonnegative integer. A subset Γ of X is a qth order boundary for A if given $S \subseteq A$ with $\# S \leqq q$, $V(S) \neq \emptyset$, we have:

$$\forall f \in A, \exists x \in \Gamma \cap V(S) \quad \text{such that} \quad |f(x)| = \max_{V(S)} |f| .$$

We then define the qth order Šilov boundary for A by

$$\partial_q A = \text{Closure}\,[\cup \{\partial_0[A|V(S)]: S \subseteq A, \# S \leqq q\}] .$$

Evidently $\partial_q A$ is the smallest closed qth order boundary for A, and the two definitions for $\partial_0 A$ are consistent.

DEFINITION. If $\mathfrak{B}$ is a commutative Banach algebra with unit, let $M = M(\mathfrak{B})$ denote its spectrum and $\hat{B}$ its algebra of Gelfand transforms. Since $\hat{B}$ is a function algebra on M we may define $\partial_q \mathfrak{B} = \partial_q \hat{B}$.

Now suppose that A is a uniform algebra on the compact Hausdorff space X. We denote the corresponding commutative Banach algebra by $\mathfrak{A}$, and we identify X with the corresponding subset of its spectrum M. Evidently $\partial_q A = \partial_q \mathfrak{A}$ if and only if $\partial_q \mathfrak{A} \subseteq X$. Of course X contains the usual Šilov boundary of A, so this always holds for $q = 0$, but it need not hold when $q > 0$. (Let $\Delta = \{z \in \boldsymbol{C}: |z| \leqq 1\}$. Take $X = \partial\Delta$, $A = P(X)$. Then $\partial_q A = X$ for all q, $\partial_0 \mathfrak{A} = X$, but $\partial_q \mathfrak{A} = \Delta$ for $q > 0$.) The generalized Šilov boundary used in [2], [10], and [13] is $\partial_q \mathfrak{A}$, but we shall sometimes find it more convenient here to use $\partial_q A$. For examples of $\partial_q \mathfrak{A}$, see [13], pp. 145-147.

Sibony apparently arrived at his definition of $\partial_q \mathfrak{A}$ by considering the behavior of plurisubharmonic functions. We include his definition here for completeness.

THEOREM 1 (*Sibony*, [13] *Theorem* 3). *If A is a uniform algebra on the compact Hausdorff space X, then $\partial_q \mathfrak{A}$ is the smallest compact subset of M which satisfies the condition: whenever $f, g_1, \cdots, g_q \in A$ and* $\text{Re}\, f \leqq \sum_{j=1}^q |g_j|$ *on K, then* $\text{Re}\, f \leqq \sum_{j=1}^q |g_j|$ *on M.*

When $\mathfrak{B}$ is a commutative Banach algebra with unit, $\partial_q \mathfrak{B}$ has an interpretation in terms of quotient algebras. To see this, recall that when I is a closed ideal in $\mathfrak{B}$, the spectrum of $\mathfrak{B}/I$ is naturally identified with $V(\hat{I}) = \{\varphi \in M(\mathfrak{B}) \,|\, \forall f \in I, \hat{f}(\varphi) = 0\}$. Thus we obtain:

THEOREM 2. *For a commutative Banach algebra $\mathfrak{B}$ with unit,*

$\partial_q\mathfrak{B} = \text{Closure}\,[\cup\{\partial_0(\mathfrak{B}/I)$: *I is an ideal of codimension at most q in* $\mathfrak{B}\}]$.

For the remainder of this section, we consider a function algebra A on a compact Hausdorff space X, and show how the qth order boundaries for A are related to extensions of A by conjugates of functions in A.

NOTATION. If $S \subseteqq C(X)$, let $A(S)$ denote the function algebra on X generated by A and S; i.e.,

$$A(S) = \left\{ \sum_{|I| \leqq N} g_I f_1^{i_1} \cdots f_r^{i_r} | f_1, \cdots, f_r \in S,\ g_I \in A,\ 0 \leqq r, N < \infty \right\}$$

where $I = (i_1, \cdots, i_r)$ and $|I| = i_1 + \cdots + i_r$; $i_1, \cdots, i_r \geqq 0$.

THEOREM 3. *Let Γ be a closed subset of X. Then Γ is a qth order boundary for A if and only if for all $S \subseteqq A$ with $\# S \leqq q$, Γ is a boundary for $A(\bar{S})$.*

Proof. First assume that Γ is a qth order boundary for A. Let $S = \{f_1, \cdots, f_q\} \subset A$, and let $F \in A(\bar{S})$, so that

$$F = \sum_I g_I \bar{f}_1^{i_1} \cdots \bar{f}_q^{i_q}$$

for some $g_I \in A$. Choose $y \in X$ with $|F(y)| = \max_X |F|$, and let

$$\begin{aligned} h_j &= f_j - f_j(y) \qquad j = 1, \cdots, q\,; \\ T &= \{h_1, \cdots, h_q\} \subseteqq A\,; \\ f &= \sum_I g_I \overline{f_1(y)}^{i_1} \cdots \overline{f_q(y)}^{i_q} \in A\,. \end{aligned}$$

Then $y \in V(T)$, so $V(T) \neq \varnothing$. Since Γ is a qth order boundary for A, $\max_{V(T)} |f| = \max_{V(T) \cap \Gamma} |f|$. But $y \in V(T)$ and $f = F$ on $V(T)$, whence $\max_X |F| = |F(y)| = \max_\Gamma |F|$ as desired.

Now suppose that for all $S \subseteqq A$ with $\# S \leqq q$, Γ is a boundary for $A(\bar{S})$. Let $S \subseteqq A$, $\# S \leqq q$, $V(S) \neq \varnothing$. Given $f \in A$ we will show that $\max_{V(S)} |f| = \max_{V(S) \cap \Gamma} |f|$.

Let $S = \{f_1, \cdots, f_q\}$ and let $M = 1 + \max_X \sum_{j=1}^q |f_j|^2$. Set

$$F = \frac{1}{M}\left(M - \sum_{j=1}^q |f_j|^2\right),$$

and observe that $F = 1$ on $V(S)$ while $0 < F < 1$ on $X \backslash V(S)$. For each $m \geqq 0$ we have $fF^m \in A(\bar{S})$, so that

$$\max_X |fF^m| = \max_\Gamma |fF^m|\,.$$

Since F peaks on $V(S)$, it follows that

$$\max_{V(S)} |f| = \max_{V(S)\cap\Gamma} |f| .$$

2. Relationship with q-holomorphic functions. In [3], [4] we defined a function f on C^n to be q-holomorphic if $\bar{\partial} f \wedge [\partial\bar{\partial} f]^q = 0$. The motivating example of such a function is one which is holomorphic in $(n - q)$ variables and arbitrary in the other q variables. (Compare Example 4 and Theorem 1 in [3].) We showed that an $(n - 1)$-holomorphic function on C^n satisfies the maximum principle, and we related "q-holomorphic convexity" to q-pseudoconvexity (Theorems 2 and 3 of [3]). Hunt and Murray [9] have since related these q-holomorphic functions to the complex Monge-Ampere equations, obtaining results which extend Bremermann's work [6] on a generalized Dirichlet problem.

In order to develop some of the connections between the generalized Šilov boundary and the q-holomorphic functions, let us define

$$A(K) = \{f \in C(K) \mid f \text{ is holomorphic on } \operatorname{int} K\}$$
$$A^q(K) = \{f \in C(K) \mid f|_{\operatorname{int} K} \in C^{(2)}(\operatorname{int} K),\ f \text{ is } q\text{-holomorphic on } \operatorname{int} K\}$$

for K an arbitrary compact subset of C^n. So, for example, $A^0(K) = A(K)$ and $A^n(K) = \{f \in C(K) \mid f|_{\operatorname{int} K} \in C^{(2)}(\operatorname{int} K)\}$. $A(K)$ is a uniform algebra but $A^q(K)$ is not even a linear space when $0 < q < n$, although it does have some algebraic closure properties; for example, if $f \in A(K)$ and $g \in A^q(K)$, then $f + g, fg, g^2 \in A^q(K)$ ([3], Proposition 4). We will still say that a subset Γ of K is a boundary for $A^q(K)$ if for all $f \in A^q(K)$, $\max_K |f|$ is achieved on Γ. The maximum principle for q-holomorphic functions mentioned above shows that ∂K is always a boundary for $A^q(K)$ when $0 \leqq q < n$, and certainly K is the only boundary for $A^q(K)$ when $q \geqq n$. Similarly, it is clear that ∂K is a qth order boundary for $A(K)$ when $0 \leqq q < n$, and that the only qth order boundary for $A(K)$ when $q \geqq n$ is K. One reason for this similarity is given by the following result.

THEOREM 4. *Let Γ be a closed subset of the compact set $K \subseteq C^n$. If Γ is a boundary for $A^q(K)$, then Γ is a* qth *order boundary for $A(K)$.*

Proof. Let $S \subseteq A$, $\# S \leqq q$. It is easy to verify that $A(K)(\bar{S}) \subseteq A^q(K)$. Since Γ is a boundary for $A^q(K)$, it is a boundary for $A(K)(\bar{S})$. By Theorem 3, Γ is a qth order boundary for $A(K)$.

Now suppose that Ω is a bounded open subset of C^n with C^2

boundary. Recall that Ω is (strictly) q-pseudoconvex at a point $x \in \partial\Omega$ if the Levi form in the complex tangent space to Ω at x of a defining function for Ω has at least $n - 1 - q$ nonnegative (positive) eigenvalues. Let

$$F_{q,\Omega} = \text{Closure}\,\{x \in \partial\Omega \,|\, \Omega \text{ is strictly } q\text{-pseudoconvex at } x\}\,.$$

THEOREM 5. *Let Ω be a bounded open subset of C^n with C^2 boundary. Then $F_{q,\Omega}$ is a boundary for $A^q(\bar{\Omega})$.*

Proof. For $q = 0$, see Epe [7] (or [5] or [8]). The same argument used in, say, [5] can be applied when $q > 0$. We outline a proof, based on this argument, for the case $0 < q < n$.

Let $f \in A^q(\bar{\Omega})$; we will show that $\max_{\bar{\Omega}} |f| = \max_{F_{q,\Omega}} |f|$. By the closure properties of $A^q(\bar{\Omega})$ mentioned above, we know that $A(\bar{\Omega})(\{f\}) \subseteqq A^q(\bar{\Omega})$. Let B denote the uniform closure of $A(\bar{\Omega})(\{f\})$, so that B is a uniform algebra on $\bar{\Omega}$. We will show that $F_{q,\Omega}$ contains $\partial_0 B$, which will complete the proof. For this it suffices to show that any peak point $x \in \partial\Omega$ for B is a limit of strictly q-pseudoconvex boundary points of Ω. Now given any small neighborhood U of such an x, there is a $g \in A(\bar{\Omega})(\{f\})$ for which $\operatorname{Re} g$ achieves its maximum value, say 1, only in U. Since $\operatorname{Re} g$ is q-plurisubharmonic on Ω (Theorem 3.3 of [9]), $\varphi(z) = -1 + \varepsilon \sum_{j=1}^{n} |z_j|^2 + \operatorname{Re} g(z)$ is strictly q-plurisubharmonic on Ω for any positive ε. If we choose ε to be a small positive number, and c to be a small negative number for which $W = \{z \in \Omega \,|\, \varphi(z) = c\}$ is smooth, and if we then translate the hypersurface W in the outward normal direction to Ω at x until W is externally tangent to Ω, any point of tangency of W provides a strictly q-pseudoconvex boundary point of Ω near x.

Note. There does not seem to be a simple way to apply the above argument directly to the original function $f \in A^q(\bar{\Omega})$, as the set $\{z \in \partial\Omega \,|\, \operatorname{Re} f(z) = \max_{\bar{\Omega}} \operatorname{Re} f\}$ may extend over a large portion of $\partial\Omega$. Then we cannot simply translate a level hypersurface to make it externally tangent.

Putting Theorems 4 and 5 together, we see that $F_{q,\Omega}$ always contains $\partial_q A(\bar{\Omega})$. In fact, Sibony has shown that $\partial_q A(\bar{\Omega}) = F_{q,\Omega}$ when Ω is a C^∞ pseudoconvex domain which is an "S_δ". ([13], Proposition 4.) In this case $\bar{\Omega}$ is the spectrum of the corresponding Banach algebra $\mathfrak{A}(\bar{\Omega})$, so we also have $\partial_q \mathfrak{A}(\bar{\Omega}) = F_{q,\Omega}$. Furthermore, it is easy to see that $F_{q,\Omega}$ is the smallest closed boundary for $A^q(\Omega)$ in this case. For an arbitrary bounded Ω with C^2 boundary it would seem to be a difficult question to determine whether a given strictly q-pseudoconvex boundary point x of Ω must be included in every closed boundary for $A^q(\bar{\Omega})$ or in $\partial_q A(\bar{\Omega})$, as these involve global existence questions; but it is not hard to see that for any such x there is a closed ball B centered at x for which $x \in \partial_q A(\bar{\Omega} \cap B)$ and for which x is any closed boundary for $A^q(\bar{\Omega} \cap B)$. (See the proof of Theorem 3 in [3] for the construction of an appropriate peaking function.)

3. Generalizations of $\mathfrak{A}$-convexity. Throughout this section let A be a uniform algebra on the compact Hausdorff space X. As in section one, $\mathfrak{A}$ denotes the corresponding Banach algebra and M denotes its spectrum; we will also regard $\mathfrak{A}$ as a uniform algebra on M. K, K_j, etc. will always denote closed subsets of M. We recall briefly some facts about $\mathfrak{A}$-convexity.

The $\mathfrak{A}$-convex hull of K is defined by

$$h(K) = \left\{x \in M \mid \forall f \in \mathfrak{A},\ |f(x)| \leqq \max_K |f|\right\},$$

and the rational $\mathfrak{A}$-convex hull of K is

$$rh(K) = \{x \in M \mid \forall f \in \mathfrak{A},\ f(x) \in f(K)\}.$$

K is a boundary for $\mathfrak{A}$ if and only if $h(K) = M$. One says that a set K is $\mathfrak{A}$-convex if and only if $h(K) = K$. The simplest $\mathfrak{A}$-convex sets are the $\mathfrak{A}$-polyhedra. If $D = \{|z| \leqq 1\}$ and if $F_1, \cdots, F_r \in \mathfrak{A}$, the corresponding $\mathfrak{A}$-polyhedron is

$$\pi(F_1, \cdots, F_r) = \{x \in M \mid F_j(x) \in D,\ j = 1, \cdots, r\}.$$

$h(K) = \bigcap \{\pi : \pi \supseteqq K,\ \pi \text{ is an } \mathfrak{A}\text{-polyhedron}\}$.

There is an obvious generalization of $h(K)$ parallel to the generalized Šilov boundary.

DEFINITION.

$$h_q(K) = \Big\{x \in M \mid \forall S \subseteqq \mathfrak{A},\ \text{if } \# S \leqq q \text{ and } x \in V(S),$$
$$\text{then } \forall f \in \mathfrak{A},\ |f(x)| \leqq \max_{V(S) \cap K} |f|\Big\}.$$

(Here $V(S) = \{x \in M \mid \forall f \in S,\ f(x) = 0\}$.) Evidently K is a qth order boundary for the algebra $\mathfrak{A}$ on M if and only if $h_q(K) = M$.

A similar generalization of $\mathfrak{A}$-polyhedron is also possible, and in fact one was made by Rothstein [12] in studying Hartogs' theorems for analytic varieties. Our definition is based on his. Let

$$D^n = \{z \in \boldsymbol{C}^n \mid z = (z_1, \cdots, z_n),\ \text{and for some } j,\ |z_j| \leqq 1\},$$

and let $\mathfrak{A}^n = \{F = (f_1, \cdots, f_n) \mid f_1, \cdots, f_n \in \mathfrak{A}\}$.

DEFINITION. If $F_1, \cdots, F_r \in \mathfrak{A}^{q+1}$, the corresponding q-polyhedron is

$$\pi(F_1, \cdots, F_r) = \{x \in M \mid F_j(x) \in D^{q+1},\ j = 1, \cdots, r\} .$$

Note for future reference that the q-polyhedra are precisely the subsets of M which are finite intersections of unions of $q + 1$ $\mathfrak{A}$-polyhedra; for example, if $F = (f_1, \cdots, f_{q+1})$, then $\pi(F) = \bigcup_{j=1}^{q+1} \pi(f_j)$.

The q-polyhedra are related to $h_q(K)$ in the same way that $\mathfrak{A}$-polyhedra are related to $h(K)$. In proving this we will make use of some alternative descriptions of $h_q(K)$, two of which are based on decomposing K into $q + 1$ pieces and examining their hulls. We need a preliminary lemma which describes this kind of decomposition in C^q.

LEMMA. *If $B^n = \{z \in C^n \mid |z| \leqq 1\}$, then there are compact polynomially convex sets $L_0, L_1, \cdots, L_n \subseteqq B^n$ such that:*

(i) *$B^n = \bigcup_{j=0}^n L_j$ and*

(ii) *0 is a peak point for $P(L_j)$, $j = 0, \cdots, n$.*

Such a decomposition is not possible with fewer than $n + 1$ subsets of B^n. (Here $|z| = (\Sigma |z_j|^2)^{1/2}$.)

Proof. Let

$$M_j = \left\{z \in C^n \mid \text{for each nonzero coordinate } z_i \text{ of } z ,\ \frac{2\pi j}{n+1} \leqq \arg z_i \leqq \frac{2\pi(n+j)}{n+1}\right\} , \quad k = 0, \cdots, n .$$

Each M_j is a product of one dimensional sectors about the origin, and $\bigcup_{j=0}^n M_j = C^n$. It follows that

$$L_j = M_j \cap B^n , \quad j = 0, \cdots, n$$

yields the desired decomposition. That $n + 1$ pieces are needed will follow from the next result applied to $P(B^n)$, since $\partial_{n-1}(P(B^n)) = \partial B^n$.

As a final preliminary, suppose $S \subseteqq \mathfrak{A}$ and define

$$h_S(K) = \left\{x \in M \mid \forall f \in \mathfrak{A}(\bar{S}),\ |f(x)| \leqq \max_K |f|\right\} .$$

Of course this is just the $\mathfrak{B}$-convex hull of K, where B is the uniform algebra generated by A and $\{\bar{f}\colon f \in S\}$.

THEOREM 6. *For any closed subset K of M, the following sets are equal:*

$H_1 = h_q(K)$;
$H_2 = \bigcap \{h_S(K) \mid S \subseteq \mathfrak{A}, \# S \leqq q\}$;
$H_3 = \bigcap \{\pi \mid \pi \text{ is a } q\text{-polyhedron containing } K\}$;
$H_4 = \left\{x \in M \mid \text{for any decomposition } K = \bigcup_{j=0}^{q} K_j,\ x \in \bigcup_{j=0}^{q} h(K_j)\right\}$;
$H_5 = \left\{x \in M \mid \text{if } K_1, \cdots, K_q \subseteq K \text{ and } x \notin \bigcup_{j=1}^{q} rh(K_j), \text{ then there is a compact set } L \subseteq K \backslash \bigcup_{j=1}^{q} K_j \text{ with } x \in h(L)\right\}$.

Proof. $H_1 = H_2$: This follows readily from the definitions by considering $A|_{H_1}$ and $A|_{H_2}$ together with Theorem 3.

$H_1 \subseteq H_5$: Let $x \in h_q(K)$, let $K_1, \cdots, K_q \subseteq K$, and assume $x \notin \bigcup_{j=1}^{q} rh(K_j)$. We will exhibit a compact set $L \subseteq K$, L disjoint from $K_1, \cdots, K_q$, with $x \in h(L)$.

For $j = 1, \cdots, q$ choose $f_j \in \mathfrak{A}$ with $0 = f_j(x) \notin f_j(K_j)$. Let $S = \{f_1, \cdots, f_q\}$. Then $x \in V(S) \cap h_q(K)$, so $\forall f \in \mathfrak{A}$, $|f(x)| \leqq \max_{V(S) \cap K} |f|$. $L = V(S) \cap K$ has the desired properties.

$H_5 \subseteq H_4$: This is obvious.

$H_4 \subseteq H_1$: Let $x \in H_4$, let $S = \{f_1, \cdots, f_q\} \subseteq \mathfrak{A}$, and assume $x \in V(S)$. We will show that $x \in h(V(S) \cap K)$. Assume $\Sigma |f_i|^2 \leqq 1$.

By the above lemma there are compact polynomially convex sets $L_0, \cdots, L_q \subseteq B^q$ with $B^q = \bigcup_{j=0}^{q} L_j$ and 0 a peak point for $P(L_j)$, $j = 0, \cdots, q$. Let

$$K_j = \{x \in K \mid (f_1(x), \cdots, f_q(x)) \in L_j\}\ , \quad j = 0, \cdots, q\ .$$

Since $x \in H_4$, there is a j such that $x \in h(K_j)$. Let ψ be a function in $P(L_j)$ which peaks at 0, and let $\Psi = \psi(f_1, \cdots, f_q)$. Then $\Psi \in \mathfrak{A}_{K_j}$, the uniform closure of the restriction algebra $\mathfrak{A}|_{K_j}$. From the facts that $x \in V(S) \cap h(K_j)$ and that Ψ peaks on $V(S) \cap h(K_j)$, it follows that any representing measure for x on K_j is supported on $V(S) \cap K_j$. Thus $x \in h(V(S) \cap K_j) \subseteq h(V(S) \cap K)$ as desired.

$H_4 \subseteq H_3$: Suppose $x \notin H_3$. Let π be a q-polyhedron for which $K \subseteq \pi$ but $x \notin \pi$. As noted above, π can be written in the form $\pi = \bigcap_i \bigcup_{j=0}^{q} \pi_{ij}$, where the π_{ij} are $\mathfrak{A}$-polyhedra. Then for some i we have $x \notin \bigcup_{j=0}^{q} \pi_{ij}$. Let $K_j = K \cap \pi_{ij}$, $j = 0, \cdots, q$. Evidently $K = \bigcup_{j=0}^{q} K_j$ and $x \notin \bigcup_{j=0}^{q} \pi_{ij} \supseteq \bigcup_{j=0}^{q} h(K_j)$, so $x \notin H_4$.

$H_3 \subseteq H_4$: Suppose $x \notin H_4$. Then there are $K_0, \cdots, K_q$ with $K =$

$\bigcup K_j$, $x \notin \bigcup h(K_j)$. Choose $f_j \in \mathfrak{A}$ with $|f_j(x)| > 1 \geqq \max_{K_j} |f_j|$, $j = 0, \cdots, q$. Let $F = (f_0, \cdots, f_q)$. Then $x \notin \pi(F) \supseteqq K$, so $x \notin H_3$.

COROLLARY. *$\partial_q \mathfrak{A}$ is the smallest compact subset K of M having the property: for every decomposition of K into $q + 1$ compact subsets, $K = \bigcup_{j=0}^{q} K_j$, one has $\bigcup_{j=0}^{q} h(K_j) = M$.*

4. **$\mathfrak{A}$-holomorphic convexity and the first order boundary.** Again let A denote a uniform algebra on X, with M, $\mathfrak{A}$ as in section three. Since the higher order boundaries reflect higher dimentional structure in M, and since holomorphic convexity first becomes interesting in C^2, it is reasonable to expect some connection between the first order boundary and uniform algebra generalizations of holomorphic convexity. An appropriate notion of $\mathfrak{A}$-holomorphic convexity was studied by Rickart [11], which we now recall.

DEFINITION. Let U be an *open* subset of M and let $\mathcal{O}(U)$ denote the locally $\mathfrak{A}$-holomorphic functions on U, i.e., $\mathcal{O}(U) = \{f \in C(U) \mid \forall x \in U \exists$ a compact neighborhood N of x such that $f|_N \in \mathfrak{A}_N\}$. For a compact set $K \subseteqq U$, set

$$\hat{K} = \left\{x \in U \mid \forall f \in \mathcal{O}(U), |f(x)| \leqq \max_K |f|\right\} .$$

Then U is called $\mathfrak{A}$-holomorphically convex if for all compact sets $K \subseteqq U$, $\hat{K}$ is compact.

THEOREM 7. *There are no proper $\mathfrak{A}$-holomorphically convex open subsets of M containing $\partial_1 \mathfrak{A}$.*

Proof. Let U be an open subset of M contaiging $\partial_1 \mathfrak{A}$. Assume $K = M \backslash U \neq \emptyset$. We will show that U is not $\mathfrak{A}$-holomorphically convex by showing that $(\partial_1 \mathfrak{A})^\wedge$ is not compact.

Let x be a peak point for $\mathfrak{A}_K$. Then $x \in K$, and the local maximum modulus principle implies that $x \in \partial[h(K)]$. Choose $x_\alpha \in M \backslash h(K)$ with $x_\alpha \to x$, and for each α choose $f_\alpha \in \mathfrak{A}$ with $f_\alpha(x_\alpha) = 1 > \max_K |f_\alpha|$. Fix α and take $S = \{f_\alpha - 1\}$. Then $x_\alpha \in V(S) \subseteqq U$, and $\partial_0[\mathfrak{A}_{V(S)}] \subseteqq \partial_1 \mathfrak{A}$, whence (using, say, Corollary 28.9 in [14]) $x_\alpha \in (\partial_1 \mathfrak{A})^\wedge$. Thus $(\partial_1 \mathfrak{A})^\wedge$ is not compact.

Let us say that a compact set $K \subseteqq M$ is "large" when the only $\mathfrak{A}$-holomorphically convex open set containing K is M, so that the content of Theorem 7 is that $\partial_1 \mathfrak{A}$ is always large. Clearly any large set must contain $\partial_0 \mathfrak{A}$, so that when $\partial_0 \mathfrak{A} = \partial_1 \mathfrak{A}$, this is the smallest large subset of M. (This happens, e.g., for $A = P(B^n)$, $n \geqq 2$.) When $\partial_0 \mathfrak{A} \neq \partial_1 \mathfrak{A}$, it may happen that there is a smallest large set K with either $K = \partial_0\mathfrak{A}$ or $K = \partial_1\mathfrak{A}$ or $\partial_0\mathfrak{A} \subsetneqq K \subsetneqq \partial_1\mathfrak{A}$; or there may be no smallest large set. For example, if $A = P(\Delta^1)$ (where $\Delta^n = \{z \in \boldsymbol{C}^n \,|\, |z_j| \leqq 1\}$), then $\partial_0\mathfrak{A} = \partial\Delta^1$, but $\partial_1\mathfrak{A} = \Delta^1$ is the smallest large set. If $A = R(X)$ where X is one of the compact subsets of $\partial\Delta^2$ in [1] or [15], $\partial_0\mathfrak{A} = X$ is the smallest large set while $\partial_1\mathfrak{A} = h_r(X) \neq X$. Finally, consider $A = P(\Delta^2)$, $K_1 = \partial_1\mathfrak{A} = \partial\Delta^2$, $K_2 = \{(z, w) \in \Delta^2 \,|\, |z| = 1 \text{ or } |z| = |w|\}$, $K_3 = \{(z, w) \in \Delta^2 \,|\, |w| = 1 \text{ or } |w| = |z|\}$. Then K_1, K_2, K_3 are all large, but $K_1 \cap K_2 \cap K_3 = \partial_0\mathfrak{A}$ is not large.

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