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PEAK-INTERPOLATION SETS OF CLASS C^1

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Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$, with C^2-boundary ∂D. Let $A(D)$ be the algebra of all $f \in C(\bar{D})$ that are holomorphic in D. Let M be a C^1-submanifold of ∂D whose tangent space $T_w(M)$ lies in the maximal complex subspace of $T_w(\partial D)$, for every $w \in M$.

The principal result of the present paper is that every compact subset of M is then a peak-interpolation set for $A(D)$.

This will be stated again, in slightly different form, as Theorem 3.1. It should be stressed that the smoothness assumptions made on ∂D and on M are quite weak. Under stronger regularity assumptions, the same conclusion has been reached earlier by Henkin **[6]** and, independently, by Nagel **[9]** (if ∂D and M are of class C^3), as well as by Burns and Stout **[3]** who dealt with a real-analytic interpolation problem. These three proofs are quite different from each other. The basic idea of **[9]**—to exhibit appropriate functions in $A(D)$ by means of integrals—is used in the present paper, but in a way that is simpler and requires less differentiability. In part, this simplicity is achieved by establishing the theorem first for strictly convex domains. The general case follows then from Fornaess' embedding theorem **[4]**.

I thank Alexander Nagel for many interesting conversations on this subject. My proof was originally designed for C^1-manifolds in the boundary of the unit ball of $\mathbf{C}^n$, and it was his prodding that made me push it to its present generality.

I. Definitions and terminology.

1.1 Throughout this paper, n will be a fixed positive integer and $\mathbf{C}^n$ will be the vector space of n complex variables, with the usual inner product $\langle z, w\rangle = \Sigma_1^n z_j \bar{w}_j$ and norm $|z| = \langle z, z\rangle^{\frac{1}{2}}$. For $1 \leqq j \leqq n$, we write

$$(1) \qquad D_j = \partial/\partial z_j, \quad \bar{D}_j = \partial/\partial \bar{z}_j.$$

1.2. Throughout this paper, W will be an open set in $\mathbf{C}^n$, and $\rho: W \to \mathbf{R}$ will be a function of class C^2, i.e., a function all of whose second-order derivatives are continuous. For each ρ, and for each $w \in W$, we define

$$(2) \qquad N(w) = ((\bar{D}_1\rho)(w), \cdots, (\bar{D}_n\rho)(w)),$$

$$P_w(\zeta) = \sum_{j,k=1}^{n} (D_jD_k\rho)(w)\zeta_j\zeta_k \qquad (\zeta \in \mathbf{C}^n), \tag{3}$$

$$\langle H_w\zeta, \eta\rangle = \sum_{j,k=1}^{n} (D_j\bar{D}_k\rho)(w)\zeta_j\bar{\eta}_k \qquad (\zeta, \eta \in \mathbf{C}^n), \tag{4}$$

and

$$Q_w(\zeta) = P_w(\zeta) + \langle H_w\zeta, \zeta\rangle. \tag{5}$$

The vector $N(w)$ is perpendicular to the level surface of ρ through w (see §1.7). P_w is a homogeneous polynomial of degree 2, H_w is a hermitian operator on $\mathbf{C}^n$ (the so-called *complex Hessian* of ρ at w), and the Taylor expansion of ρ, about any $w \in W$, can be written in the form

$$\begin{aligned}\rho(z) = \rho(w) + 2\operatorname{Re}\langle z - w, N(w)\rangle + \operatorname{Re} Q_w(z - w)\\ + |z - w|^2\epsilon(z, w)\end{aligned} \tag{6}$$

where $\epsilon\colon W \times W \to \mathbf{R}$ is continuous, and $\epsilon(w, w) = 0$.

1.3. A bounded open set $D \subset \mathbf{C}^n$ is said to be *strictly pseudoconvex, with C^2-boundary ∂D*, if there is an open set $W \supset \bar{D}$ and a C^2-function $\rho\colon W \to \mathbf{R}$, as in §1.2, such that

(i) $D = \{z \in W\colon \rho(z) < 0\}$,
(ii) $N(w) \neq 0$ for all $w \in \partial D$, and
(iii) there is a constant $\beta > 0$ which makes the inequality

$$\langle H_w\zeta, \zeta\rangle \geqq \beta|\zeta|^2 \tag{7}$$

true for all $w \in W$ and for all $\zeta \in \mathbf{C}^n$.

REMARK. Strict pseudoconvexity is often defined locally. See, for instance, pp. 262–263 of **[5]**. However, if the local definition is satisfied, then there exists a global ρ with the above properties. This is proved on p. 169 of **[8]**.

1.4. An open set $D \subset \mathbf{C}^n$ is said to be *strictly convex* if there is a function $\rho\colon \mathbf{C}^n \to \mathbf{R}$, as in §1.2, such that
(i) $D = \{z \in \mathbf{C}^n\colon \rho(z) < 0\}$,
(ii) $\rho(z) \to \infty$ as $|z| \to \infty$, and
(iii) there is a constant $\alpha > 0$ which makes the inequality

$$\operatorname{Re} Q_w(\zeta) \geqq \alpha|\zeta|^2 \tag{8}$$

true for all $w \in \mathbf{C}^n$ and all $\zeta \in \mathbf{C}^n$.

1.5. Let D be a bounded open set in $\mathbf{C}^n$. As usual, $A(D)$ denotes the algebra of all continuous functions $f: \bar{D} \to \mathbf{C}$ that are holomorphic in D. A compact set $E \subset \partial D$ is said to be a *peak-interpolation set for* $A(D)$—or simply a *PI*-set—if every $g \in C(E)$ extends to an $f \in A(D)$ that satisfies

$$|f(z)| < \max\{|g(w)|: w \in E\} \tag{9}$$

for all $z \in \bar{D} \setminus E$.

(The function $g \equiv 0$ must of course be excluded in (9).)

If μ is a complex Borel measure on ∂D such that $\int f d\mu = 0$ for every $f \in A(D)$, we shall write: $\mu \perp A(D)$.

The following well-known theorem of Bishop **[1]** will be used:

A compact set $E \subset \partial D$ is a (PI)-set for $A(D)$ if and only if $\mu(E_0) = 0$ for every $\mu \perp A(D)$ and for every compact $E_0 \subset E$.

1.6. Throughout this paper, Ω will be an open set in $\mathbf{R}^m$ and $\Phi: \Omega \to \mathbf{C}^n$ will be a mapping of class C^1. This means that to every $x \in \Omega$ corresponds an $\mathbf{R}$-linear operator $\Phi'(x): \mathbf{R}^m \to \mathbf{C}^n (\equiv \mathbf{R}^{2n})$, the so-called Fréchet derivative of Φ at x, which gives the Taylor expansion

$$\Phi(y) = \Phi(x) + \Phi'(x)(y - x) + |y - x|\eta(x, y) \quad (x, y \in \Omega). \tag{10}$$

Here $\eta: \Omega \times \Omega \to \mathbf{C}^n$ is continuous, and $\eta(x, x) = 0$.

We say that Φ is *nonsingular* if the rank of $\Phi'(x)$ is m for every $x \in \Omega$. In that case, every $x \in \Omega$ has a neighborhood in which Φ is one-to-one.

1.7. Suppose now that D and ρ are related as in §1.3 (except that (iii) is not needed at present), so that ∂D is the level surface of ρ given by $\rho(z) = 0$. Let $\Omega \subset \mathbf{R}^m$ be open, and consider a C^1-map $\Phi: \Omega \to \partial D$. For $x \in \Omega$ and $v \in \mathbf{R}^m$, differentiation of

$$\rho(\Phi(x + tv)) = 0$$

with respect to the real variable t, at $t = 0$, gives

$$\operatorname{Re}\langle \Phi'(x)v, N(\Phi(x))\rangle = 0, \tag{11}$$

where N is defined by (2).

Setting $w = \Phi(x)$, it follows that the equation

$$\operatorname{Re}\langle \zeta, N(w)\rangle = 0 \tag{12}$$

describes the vectors $\zeta \in \mathbf{C}^n$ that form the *real* tangent space $T_w(\partial D)$; its $\mathbf{R}$-dimension is $2n-1$. The equation

$$\langle \zeta, N(w)\rangle = 0 \tag{13}$$

defines the maximal *complex* subspace of $T_w(\partial D)$; its $\mathbf{C}$-dimension is $n-1$.

We shall be concerned with mappings $\Phi\colon \Omega \to \partial D$ that satisfy, in place of (11), the more stringent analogue of (13), namely

$$\langle \Phi'(x)v, N(\Phi(x))\rangle = 0 \qquad (x \in \Omega, v \in \mathbf{R}^m). \tag{14}$$

This orthogonality condition (14) is an analytic reformulation of the geometric requirement (stated in the opening paragraph) that the tangent vectors $\Phi'(x)v$ should lie in the maximal complex subspace of $T_w(\partial D)$.

II. Some lemmas.

2.1. LEMMA. *If D is strictly convex and if α is the constant that occurs in* (8), *then*

$$2\operatorname{Re}\langle w-z, N(w)\rangle \geqq \alpha |w-z|^2 \tag{15}$$

for all $w \in \partial D$, $z \in \bar{D}$.

Proof. Put $h(t) = \rho((1-t)w + tz), t \in \mathbf{R}$. Then $h(0) = \rho(w) = 0, h(1) = \rho(z) \leqq 0$; by the chain rule,

$$h'(0) = 2\operatorname{Re}\langle z-w, N(w)\rangle$$

and

$$h''(t) = 2\operatorname{Re} Q_u(z-w) \geqq 2\alpha |z-w|^2 \tag{16}$$

where $u = (1-t)w + tz$. If these data are inserted into the Taylor formula

$$h(1) = h(0) + h'(0) + \tfrac{1}{2}h''(t),$$

which holds for some $t \in (0,1)$, the result is (15).

Note. By (16), h is a convex function. This shows that "strictly convex" domains are indeed geometrically convex. Also, (15) implies that $N(w) \neq 0$ if $w \in \partial D$.

2.2. LEMMA. *Suppose*
(a) $\rho: W \to \mathbf{R}$ *is of class* C^2,
(b) $\Phi: \Omega \to W$ *is of class* C^1,
(c) $\Psi(x) = N(\Phi(x))$ *for* $x \in \Omega$. *Then*

$$\langle \Phi'(x)v, \Psi'(x)v \rangle = Q_{\Phi(x)}(\Phi'(x)v) \tag{17}$$

for all $x \in \Omega, v \in \mathbf{R}^m$.

Proof. Fix x and v, put $\gamma(t) = \Phi(x + tv), \Gamma(t) = N(\gamma(t))$, for those real t for which $x + tv \in \Omega$. Then $\Gamma(t) = \Psi(x + tv)$, so that the left side of (17) is $\langle \gamma'(0), \Gamma'(0) \rangle$.

The chain rule shows that the jth component Γ_j' of Γ' is

$$\frac{d}{dt}(\bar{D}_j\rho)(\gamma(t)) = \sum_{k=1}^{n} (\bar{D}_k\bar{D}_j\rho) \cdot \bar{\gamma}_k' + (D_k\bar{D}_j) \cdot \gamma_k'.$$

Hence, referring to §1.2,

$$\begin{aligned}\langle \gamma'(0), \Gamma'(0) \rangle &= \sum_{J=1}^{n} \gamma_J'(0)\bar{\Gamma}_J'(0) \\ &= P_{\gamma(0)}(\gamma'(0)) + \langle H_{\gamma(0)}\gamma'(0), \gamma'(0) \rangle\end{aligned}$$

which, by (5), is equal to the right side of (17).

Our next lemma is crucial for the main theorem. It is here—and only here—that the orthogonality condition (14) is used.

2.3. LEMMA. *Assume, in addition to the hypotheses of Lemma* 2.2, *that* Φ *satisfies* (14). *Then the inner products*

$$\left\langle \frac{\Phi(y + \delta v) - \Phi(y + \delta u)}{\delta^2}, N(\Phi(y + \delta v)) \right\rangle \tag{18}$$

converge to

$$\tfrac{1}{2}Q_{\Phi(y)}(\Phi'(y)(v - u)) \tag{19}$$

as $\delta \to 0$, *for* $y \in \Omega, u \in \mathbf{R}^m, v \in \mathbf{R}^m$.

Note that the denominator in (18) is δ^2, not δ!

Proof. Fix y, u, v. Fix $\delta > 0$ for the moment, small enough to ensure that convex combinations of $y + \delta u$ and $y + \delta v$ are in Ω. For $0 \leqq t \leqq 1$, define

$$x(t) = y + (1-t)\delta u + t\delta v \tag{20}$$

and put $\gamma(t) = \Phi(x(t)), \Gamma(t) = N(\Phi(x(t))) = \Psi(x(t))$, where $\Psi = N \circ \Phi$, as in Lemma 2.2. Note that

$$\gamma'(t) = \Phi'(x(t))x'(t) = \delta\Phi'(x(t))(v-u). \tag{21}$$

Thus (14) implies the relation

$$\langle \gamma'(t), \Gamma(t) \rangle = 0 \qquad (0 \leqq t \leqq 1) \tag{22}$$

which is used in the following computation:

$$\begin{aligned}
\langle \Phi(y+\delta v) - \Phi(y+\delta u), N(\Phi(y+\delta v)) \rangle &= \langle \gamma(1) - \gamma(0), \Gamma(1) \rangle \\
&= \int_0^1 \langle \gamma'(t), \Gamma(1) \rangle dt \\
&= \int_0^1 \langle \gamma'(t), \Gamma(1) - \Gamma(t) \rangle dt \\
&= \int_0^1 dt \int_t^1 \langle \gamma'(t), \Gamma'(s) \rangle ds \\
&= \delta^2 \int_0^1 dt \int_t^1 \langle \Phi'(x(t))(v-u), \Psi'(x(s))(v-u) \rangle ds.
\end{aligned}$$

As $\delta \to 0$, (20) shows that $x(t) \to y$, uniformly for $t \in [0,1]$. Since $\rho \in C^2$, we have $N \in C^1$, hence $\Psi \in C^1$. The last integrand converges therefore uniformly to

$$\langle \Phi'(y)(v-u), \Psi'(y)(v-u) \rangle \tag{23}$$

as $\delta \to 0$. Since the double integral extends over one half of the unit square in the (s,t)-plane, the desired conclusion folows from (23) and Lemma 2.2.

2.4. Lemma. *If $F\colon \mathbf{R}^m \to C$ is a homogeneous polynomial of degree 2, such that* $\operatorname{Re} F(x) > 0$, *unless* $x = 0$, *then*

$$\int_{\mathbf{R}^m} \frac{dx}{[1+F(x)]^m} \neq 0. \tag{24}$$

Here, and later, dx denotes Lebesgue measure.

Proof. The hypotheses imply that $\operatorname{Re} F(x) \geqq c|x|^2$ for some $c > 0$. The integrand in (24) is thus in $L^1(\mathbf{R}^m)$. Writing $F(x)$ in the form

(25)
$$F(x)=\sum_{j,k=1}^{m} c_{jk}x_jx_k$$

with $c_{jk}\in\mathbf{C}, c_{jk}=c_{kj}$, we associate a matrix (c_{jk}) to each F. Put $a_{jk}=\operatorname{Re} c_{jk}$. Our hypothesis is then that the symmetric real matrix (a_{jk}) is strictly positive-definite, i.e., that all of its eigenvalues $\lambda_1,\cdots,\lambda_m$ are positive.

Put $J=\int_{\mathbf{R}^m}[1+|x|^2]^{-m}dx$.

We claim that

(26)
$$\det(c_{jk})\left\{\int_{\mathbf{R}^m}\left[1+\sum c_{jk}x_jx_k\right]^{-m}dx\right\}^2=J^2$$

whenever (a_{jk}) is strictly positive-definite.

Since $J>0$, (26) implies (24).

To prove (26), suppose first that $c_{jk}\in\mathbf{R}$, i.e., that $c_{jk}=a_{jk}$. An orthogonal transformation of $\mathbf{R}^m$ will diagonalize (a_{jk}) and will transform the integrand to $[1+\Sigma\lambda_jx_j^2]^{-m}$. Since the determinant is $\Pi\lambda_j$, (26) follows, for real c_{jk}, by replacing $\lambda_j^{\frac{1}{2}}x_j$ by y_j.

To prove (26) in general, regard the symmetric matrices (c_{jk}) as points in $\mathbf{C}^N$, where $N=m(m+1)/2$. Let $T\subset\mathbf{C}^N$ be the tube domain that consists of all (c_{jk}) for which (a_{jk}) is strictly positive-definite. We just proved that (26) holds if $(c_{jk})\in T\cap\mathbf{R}^N$. Since the integral in (26) is a holomorphic function of (c_{jk}) in T, (26) holds for all $(c_{jk})\in T$.

III. The main theorem.

3.1. THEOREM. *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$, with C^2-boundary.*

Let Ω be an open set in $\mathbf{R}^m$ and let $\Phi\colon\Omega\to\partial D$ be a nonsingular C^1-mapping that satisfies the orthogonality condition

(27)
$$\langle\Phi'(x)v, N(\Phi(x))\rangle=0$$

for all $x\in\Omega, v\in\mathbf{R}^m$.

Let K be a compact subset of Ω.

Then $\Phi(K)$ is a peak-interpolation set for $A(D)$.

REMARKS. (i) Theorem 3.1 implies the one stated in the Introduction, if we think of M as being parametrized by Φ. Note, however, that Φ is not assumed to be globally one-to-one in Ω. Thus $\Phi(\Omega)$ need not be a manifold.

(ii) Theorem 3.1 has a converse: *If $\Phi\colon\Omega\to\partial D$ is of class C^1, if Φ' satisfies a Lipschitz condition of some positive order, and if $\Phi(K)$ is a (PI)-set for $A(D)$, for every compact $K\subset\Omega$, then (27) holds.*

This is contained in **[10]**. Whether the Lipschitz condition can be removed from the converse is an unanswered question.

3.2. *Proof for strictly convex D.*

We shall prove that *every $p \in \Omega$ has a neighborhood Ω_p such that $\mu(\Phi(K)) = 0$ for all compact $K \subset \Omega_p$ and for all $\mu \perp A(D)$.*

By Bishop's theorem, quoted in §1.5, this gives the conclusion of Theorem 3.1.

Fix $p \in \Omega$. Since $\Phi'(x)$ has rank m for all $x \in \Omega$ and since Φ' is continuous, we can find a constant $c > 0$ and a ball B, centered at p, with $\bar{B} \subset \Omega$, so that

$$(28) \qquad |\Phi'(x)v| \geqq c|v| \qquad (x \in \bar{B}, v \in \mathbf{R}^m)$$

and

$$(29) \qquad |\Phi(x) - \Phi(y)| \geqq c|x - y| \qquad (x, y \in \bar{B}).$$

We shall prove the above statement with $\Omega_p = B$.

Choose $\alpha > 0$ so that (8) holds. Then (28) implies

$$(30) \qquad \operatorname{Re} Q_{\Phi(y)}(\Phi'(y)v) \geqq \alpha c^2 |v|^2 \qquad (y \in \bar{B}, v \in \mathbf{R}^m).$$

The absolute values of the integrands in

$$(31) \qquad g(y) = \int_{\mathbf{R}^m} \{1 + \tfrac{1}{2} Q_{\Phi(y)}(\Phi'(y)v)\}^{-m} dv \qquad (y \in \bar{B})$$

are thus dominated by $\{1 + \frac{1}{2}\alpha c^2 |v|^2\}^{-m}$, which is in $L^1(R^m)$. Moreover, $g(y) \neq 0$, by (30) and Lemma 2.4.

Now let $f: \mathbf{R}^m \to \mathbf{C}$ be continuous, with support in B. For $\delta > 0$, define

$$(32) \qquad h_\delta(z) = \int_B \frac{\delta^m (f/g)(x)\,dx}{\{\delta^2 + \langle \Phi(x) - z, N(\Phi(x))\rangle\}^m}.$$

By Lemma 2.1, the real part of the inner product in (32) is nonnegative if $z \in \bar{D}$. For each $\delta > 0$, the integrand is thus bounded, and we see that $h_\delta \in A(D)$.

We claim that $\{h_\delta\}$ has the following properties:

(I) *$\{h_\delta\}$ is uniformly bounded on $\bar{D}$.*

(II) $\lim_{\delta \to 0} h_\delta(z) = 0$ *for all $z \in \bar{D} \backslash \Phi(\bar{B})$.*

(III) $\lim_{\delta \to 0} h_\delta(\Phi(y)) = f(y)$ *for all $y \in \bar{B}$.*

Proof of (I). Fix $z \in \bar{D}$, choose $y \in \bar{B}$ so that $|\Phi(x) - z| \geqq |\Phi(y) - z|$ for all $x \in B$. Then

$$2|\Phi(x) - z| \geqq |\Phi(x) - \Phi(y)| \qquad (x \in B). \tag{33}$$

Define the integrand in (32) to be 0 when $x \notin B$. Then B can be replaced by $\mathbf{R}^m$ in (32), and we can rewrite (32) in the form

$$h_\delta(z) = \int_{\mathbf{R}^m} \frac{(f/g)(y + \delta v)dv}{\{1 + \delta^{-2}\langle\Phi(y + \delta v) - z, N(\Phi(y + \delta v))\rangle\}^m} \tag{34}$$

by the change of variable $x = y + \delta v$. If $y + \delta v \in B$, it follows from Lemma 2.1, (33), and (29) that

$$\operatorname{Re}\langle\Phi(y + \delta v) - z, N(\Phi(y + \delta v))\rangle \geqq c_1\delta^2|v|^2,$$

where $8c_1 = \alpha c^2$. The integrands in (34) are thus dominated in absolute value by

$$\|f/g\|_\infty\{1 + c_1|v|^2\}^{-m} \tag{35}$$

which is in $L^1(\mathbf{R}^m)$. Since the bound (35) is the same for all $z \in \bar{D}, \delta > 0$, we have (I).

Proof of (II). Fix $z \in \bar{D}\backslash\Phi(\bar{B})$, and choose $y \in \bar{B}$ as in the proof of (I). If $y + \delta v \in B$, it follows from Lemma 2.1 and the minimizing property of y that

$$2\operatorname{Re}\langle\Phi(y + \delta v) - z, N(\Phi(y + \delta v))\rangle \geqq \alpha|\Phi(y) - z|^2 > 0.$$

Thus, (II) follows from the dominated convergence theorem, applied to (34).

Proof of (III). Replace z by $\Phi(y)$ in (34), and use the dominated convergence theorem once more. The numerator of the integrand tends to $f(y)/g(y)$, as $\delta \to 0$. Apply Lemma 2.3 (with $u = 0$) to the denominator, and compare with (31), the definition of $g(y)$. (III) follows.

Having proved (I), (II), and (III), pick a compact $K \subset B$, pick $\mu \perp A(D)$. There are continuous functions $f_i\colon \mathbf{R}^m \to [0, 1]$, with compact supports $K_i \subset K_{i-1} \subset B$, so that $K = \cap K_i$ and $f_i(x) = 1$ for $x \in K$. Since Φ is one-to-one on B, there are continuous functions F_i on $E = \Phi(\bar{B})$ given by $F_i(\Phi(x)) = f_i(x), x \in \bar{B}$. Construct $\{h_\delta\}$ as above, with f_i in place of f. Since $\int h_\delta d\mu = 0$ for every $\delta > 0$, properties (I), (II),

(III) imply that $\int_E F_i d\mu = 0$. Since $F_i(w) = 1$ if $w \in \Phi(K)$ and $F_i(w) \to 0$ if $w \in E \backslash \Phi(K)$, as $i \to \infty$, another passage to the limit gives $\mu(\Phi(K)) = 0$.

This completes the proof for strictly convex D.

3.3 *Proof of the general case.* Let D be a bounded strictly pseudoconvex domain, with defining function ρ, as in §1.3. For sufficiently small $\epsilon > 0$, the domain

$$D_\epsilon = \{z \in W : \rho(z) < \epsilon\} \tag{36}$$

has compact closure $\bar{D}_\epsilon \subset W$, and D_ϵ is also strictly pseudoconvex. To apply Fornaess' embedding theorem **[4]** we need two facts about D and D_ϵ:

(i) There is a biholomorphic mapping of D_ϵ onto a closed submanifold of some $\mathbf{C}^k$.

(ii) $\bar{D}$ is holomorphically convex in D_ϵ.

The first of these is true because D_ϵ is a domain of holomorphy (**[7]**, Theorem 4.2.8), hence a Stein manifold (**[7]**, p. 105), and thus Bishop's embedding theorem (**[12]**, or Theorem 5.3.9 of **[7]**) gives (i) with $k = 2n + 1$.

As regards (ii), note that ρ is plurisubharmonic, by (7) and (4), so that $\bar{D}$ is equal to its $P(D_\epsilon)$-hull (see Definition 2.6.6 in **[7]**). Thus (ii) follows from Theorem 4.3.4 in **[7]**.

Since D is strictly pseudoconvex, Fornaess' Theorem 9 asserts the existence of a positive integer p, of a biholomorphic map ψ taking D_ϵ onto a closed submanifold of $\mathbf{C}^p$, and of a strictly convex domain $\tilde{D}$ in $\mathbf{C}^p$, such that $\psi(D) \subset \tilde{D}$, $\psi(\partial D) \subset \partial\tilde{D}$, and $\psi(D_\epsilon \backslash \bar{D})$ lies outside the closure of $\tilde{D}$.

Now let $\Phi: \Omega \to \partial D$ be as in the statement of Theorem 3.1. Then $\tilde{\Phi} = \psi \circ \Phi$ is a nonsingular C^1-map of Ω into $\partial\tilde{D}$. Fix $x \in \Omega$, $v \in \mathbf{R}^m$, put

$$\zeta = \Phi'(x)v, \quad w = \Phi(x), \quad \tilde{w} = \tilde{\Phi}(x), \quad \tilde{\zeta} = \tilde{\Phi}'(x)v.$$

By (27), both ζ and $i\zeta$ are in $T_w(\partial D)$. Since $\psi(\partial D) \subset \partial\tilde{D}$, $\psi'(w)$ maps $T_w(\partial D)$ into $T_{\tilde{w}}(\partial\tilde{D})$. Since ψ is holomorphic, $\psi'(w)$ is $\mathbf{C}$-linear. Thus both $\tilde{\zeta}$ and $i\tilde{\zeta} = i\psi'(w)\zeta = \psi'(w)(i\zeta)$ lie in $T_{\tilde{w}}(\partial D)$. This shows that $\tilde{\Phi}$ and $\tilde{D}$ satisfy the hypotheses of Theorem 3.1.

Let $K \subset \Omega$ be compact. By §3.2, $\tilde{\Phi}(K)$ is a (PI)-set for $A(\tilde{D})$. If $g \in C(\Phi(K))$ then there exists $G \in C(\tilde{\Phi}(K))$ given by $G(\psi(w)) = g(w)$, $w \in K$, and G has a peak-interpolation extension $F \in A(\tilde{D})$. Finally, the function $f = F \circ \psi \in A(D)$ is an extension of g with the properties required in §1.5.

This completes the proof.

IV. The dimension of (*PI*)-sets.

4.1. Suppose that the hypotheses of Theorem 3.1 hold. Associate to each $x \in \Omega$ the real vector space

$$V_x = \{\Phi'(x)u : u \in \mathbf{R}^m\}. \tag{37}$$

Since $\Phi'(x)$ has rank m, $\dim_{\mathbf{R}} V_x = m$. It will be shown, in Theorem 4.2, that $V_x \cap (iV_x) = \{0\}$, i.e., that V_x contains no complex subspace of positive dimension. (Such vector spaces are said to be totally real. A different proof, using stronger smoothness assumptions, appears in **[3]**.)

By (27), both V_x and iV_x lie in the maximal complex subspace of $T_{\Phi(x)}(\partial D)$, whose real dimension in $2n-2$. This leads to the (perhaps surprising) conclusion that *the hypotheses of Theorem* 3.1 *can only hold when* $m \leqq n-1$.

It seems thus reasonable to conjecture that the topological dimension of no (*PI*)-set in ∂D exceeds $n-1$, if D is any bounded strictly pseudoconvex domain in $\mathbf{C}^n$.

The conjecture is open even when D is a ball.

4.2. THEOREM. *If the hypotheses of Theorem* 3.1 *hold and if* V_x *is defined by* (37), *then*

$$V_x \cap (iV_x) = \{0\}. \tag{38}$$

Proof. Take $x = 0$, without loss of generality, and write V in place of V_0.

Choose $\zeta \in V, \eta \in V$. Then $\zeta = \Phi'(0)u$ and $\eta = \Phi'(0)v$, for some $u, v \in \mathbf{R}^m$. Put $\Psi = N \circ \Phi$, as in Lemma 2.2, and define $\tilde{\eta} = \Psi'(0)v$. Then

$$\begin{aligned}\langle \zeta, \tilde{\eta}\rangle &= \lim_{\delta \to 0} \left\langle \frac{\Phi(\delta u) - \Phi(0)}{\delta}, \frac{\Psi(\delta v) - \Psi(0)}{\delta}\right\rangle \\ &= \lim_{\delta \to 0} [L_\delta(u,0) + L_\delta(0,v) - L_\delta(u,v)]\end{aligned}$$

where $L_\delta(u,v) = \delta^{-2}\langle \Phi(\delta v) - \Phi(\delta u), \Psi(\delta v)\rangle$.

Hence it follows from Lemma 2.3 (with $y = 0$) that

$$\begin{aligned}2\langle \zeta, \tilde{\eta}\rangle &= Q(\zeta) + Q(\eta) - Q(\zeta - \eta) \\ &= P(\zeta) + P(\eta) - P(\zeta - \eta) + 2\mathrm{Re}\langle H\zeta, \eta\rangle\end{aligned}$$

where we have written Q, P, H in place of $Q_{\Phi(0)}, P_{\Phi(0)}, H_{\Phi(0)}$. (See §1.2.)

Suppose now that η generates a complex subspace of V. We can then replace ζ by $\lambda\eta$ in the preceding calculation, for any $\lambda \in \mathbf{C}$. Since P is homogeneous, of degree 2, and since $\lambda^2 + 1 - (\lambda - 1)^2 = 2\lambda$, it follows that

$$\lambda\langle \eta, \bar{\eta}\rangle = \lambda P(\eta) + \mathrm{Re}\{\lambda\langle H\eta, \eta\rangle\} \tag{39}$$

for every $\lambda \in \mathbf{C}$. Thus $\lambda[\langle \eta, \bar{\eta}\rangle - P(\eta)]$ is real, for all λ; this forces $\langle \eta, \bar{\eta}\rangle = P(\eta)$, and hence (39), with $\lambda = 1$, gives $\langle H\eta, \eta\rangle = 0$. But $\langle H\eta, \eta\rangle \geqq \beta|\eta|^2$ for some $\beta > 0$, since D is strictly pseudoconvex (see §1.3). Thus $\eta = 0$. This implies (38).

4.3. We conclude with examples of Theorem 3.1 for the case $D = B_n$, the unit ball of $\mathbf{C}^n$, and $\Omega = \mathbf{R}^{n-1}$.

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a nonsingular C^2-map of $\mathbf{R}^{n-1}$ onto a hypersurface in $\mathbf{R}^n$ whose normal has all components positive. This implies that there are positive functions $F_j: \mathbf{R}^{n-1} \to \mathbf{R}$ that satisfy

$$\sum_{j=1}^{n} F_j^2(x)\frac{\partial \alpha_j}{\partial x_k}(x) = 0 \quad (1 \leqq k \leqq n-1) \tag{40}$$

and are of class C^1. Moreover, one can adjust them so that

$$\sum_{j=1}^{n} F_j^2(x) = 1. \tag{41}$$

Now put $\Phi = (\varphi_1, \cdots, \varphi_n)$, where

$$\varphi_j(x) = F_j(x)\exp\{i\alpha_j(x)\} \qquad (1 \leqq j \leqq n). \tag{42}$$

Then Φ is a nonsingular C^1-map of $\mathbf{R}^{n-1}$ into ∂B_n that satisfies

$$\sum_{j=1}^{n} \frac{\partial \varphi_j}{\partial x_k}(x)\bar{\varphi}_j(x) = 0 \qquad (1 \leqq k \leqq n-1), \tag{43}$$

because of (40) and (41).

Since $D = B_n$, we can take $\rho(z) = |z|^2 - 1$, hence $N(w) = w$. Thus (43) gives the orthogonality condition (27).

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