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WORD EQUATIONS IN SOME GEOMETRIC SEMIGROUPS

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Let S be a semigroup and let $w_1 = w_1(x_1, \cdots, x_t)$, $w_2 = w_2(x_1, \cdots, x_t)$ be two words in the variables $x_1, \cdots, x_t$. By a solution of the word equation $\{w_1, w_2\}$ in S, we mean $a_1, \cdots, a_t \in S$ such that $w_1(a_1, \cdots, a_t) = w_2(a_1, \cdots, a_t)$. Let $\mathcal{F}_{\mathbf{R}}$ denote the free product of t copies of positive reals under addition. In §3 and §5 we show that if Y is either the semigroup of certain paths in $\mathbf{R}^n$ or the semigroup of designs around the unit disc, then any solution of $\{w_1, w_2\}$ in Y can be derived from a solution of $\{w_1, w_2\}$ in $\mathcal{F}_{\mathbf{R}}$. This answers affirmatively a problem posed in Word equations of paths by Putcha. Word equations in $\mathcal{F}_{\mathbf{R}}$ are studied in §1. Using these results, it is shown that any solution in Y of $\{w_1, w_2\}$ can be approximated by a solution which is derived from a solution in a free semigroup. There are two books by Hmelevskii and Lentin on word equations in free semigroups. We also show that if $\{w_1, w_2\}$ has only trivial solutions in any free semigroup, then it has only trivial solutions in Y.

1. Preliminaries. Throughout this paper, $\mathbf{N}$, Z, Z^+, $\mathcal{Q}$, $\mathcal{Q}^+$, $\mathbf{R}$, $\mathbf{R}^+$ will denote the sets of nonnegative integers, integers, positive integers, rationals, positive rationals, reals and positive reals, respectively. For $m, n \in Z^+$, let $\mathbf{R}^{m\times n}$, $\mathcal{Q}^{m\times n}$ denote the sets of all $m \times n$ matrices over the reals and rationals, respectively. If S is a semigroup, then $S^1 = S \cup \{1\}$ with obvious multiplication if S does not have an identity element; $S^1 = S$ otherwise. If $T \subseteq S^1$, then $T^1 = T \cup \{1\}$.

DEFINITION. Let S be a semigroup and $a, b \in S$.

(1) $a \mid b$ if $b = xay$ for some $x, y \in S^1$.

(2) $a \mid_i b$ if $b = ax$ for some $x \in S^1$.

(3) $a \mid_f b$ if $b = ya$ for some $y \in S^1$.

If Γ is a nonempty set, then let $\mathcal{F} = \mathcal{F}(\Gamma)$ denote the free semigroup on Γ. If $w \in \mathcal{F}$, then let $l(w) =$ length of w. If S is a semigroup and $a_1, \cdots, a_n \in S$, then we say that $a \in S$ is a word in $a_1, \cdots, a_n$ if $a = w(a_1, \cdots, a_n)$ for some $w(x_1, \cdots, x_n) \in \mathcal{F}(x_1, \cdots, x_n)$. This is the same as saying that a is an element of the semigroup generated by $a_1, \cdots, a_n$.

Let Γ be a nonempty set. Let $\mathcal{F}_{\mathbf{R}} = \mathcal{F}_{\mathbf{R}}(\Gamma)$ denote the set of all nonempty finite sequences (also called words) of the type $w = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$

where $n \in Z^+$, $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$, $A_1, \cdots, A_n \in \Gamma$ and $A_i \neq A_{i+1}$ for $i, i+1 \in \{1, \cdots, n\}$. We define $e(w) = n$ and $l(w) = \alpha_1 + \cdots + \alpha_n$. Let $w_1, w_2 \in \mathscr{F}_{\mathbf{R}}$. Suppose $w_1 = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$, $w_2 = B_1^{\beta_1} \cdots B_m^{\beta_m}$. Then we define

$$w_1 w_2 = \begin{cases} A_1^{\alpha_1} \cdots A_n^{\alpha_n + \beta_1} B_2^{\beta_2} \cdots B_m^{\beta_m} & \text{if} \quad A_n = B_1. \\ A_1^{\alpha_1} \cdots A_n^{\alpha_n} B_1^{\beta_1} \cdots B_m^{\beta_m} & \text{if} \quad A_n \neq B_1. \end{cases}$$

Now, of course, expressions of the type $w = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$ ($\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$; $A_1, \cdots, A_n \in \Gamma$) make sense even when $A_i = A_{i+1}$ for some i, $i + 1 \in \{1, \cdots, n\}$. But note that if $n = e(w)$, then $A_i \neq A_{i+1}$ for any i, $i + 1 \in \{1, \cdots, n\}$. In such a case we call $A_1^{\alpha_1} \cdots A_n^{\alpha_n}$, the *standard form* of w. $\mathscr{F}_{\mathbf{R}}(\Gamma)$ is a semigroup and is just the free product of $|\Gamma|$ copies of $\mathbf{R}^+$ under addition (see for example [**3**; p. 411]). Let $\mathscr{N} = \mathscr{N}(\Gamma) = \{A^\alpha \mid A \in \Gamma, \alpha \in \mathbf{R}^+\}$. If $u, v \in \mathscr{F}_{\mathbf{R}}(\Gamma)$, then define $u \sim v$ if either $u = w^i$, $v = w^j$ for some $w \in \mathscr{F}_{\mathbf{R}}$, $i, j \in Z^+$ or if $u = A^\alpha$, $v = A^\beta$ for some $\alpha, \beta \in \mathbf{R}^+$, $A \in \Gamma$. Clearly, $\sim$ is an equivalence relation on $\mathscr{N}(\Gamma)$. It will follow from Theorem 1.9 that $\sim$ is in fact an equivalence relation on $\mathscr{F}_{\mathbf{R}}(\Gamma)$. Let $w \in \mathscr{F}_{\mathbf{R}}$, $w = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$ in standard form. Let $A \in \Gamma$. Then A appears *integrally* in w if for each $i \in \{1, \cdots, n\}$, $A_i = A$ implies $\alpha_i \in Z^+$. Otherwise A appears *nonintegrally* in w. A appears *rationally* in w if for each $i \in \{1, \cdots, n\}$, $A_i = A$ implies $\alpha_i \in \mathscr{Q}^+$. Let $\mathscr{F}_{\mathscr{Q}}(\Gamma) = \{w \mid w \in \mathscr{F}_{\mathbf{R}}(\Gamma)$, A appears rationally in w for each $A \in \Gamma\}$. $\mathscr{F}_{\mathscr{Q}}(\Gamma)$ is a subsemigroup of $\mathscr{F}_{\mathbf{R}}(\Gamma)$.

DEFINITION. By a word equation in variables $x_1, \cdots, x_n$ we mean $\{w_1, w_2\}$ where $w_1 = w_1(x_1, \cdots, x_n)$, $w_2 = w_2(x_1, \cdots, x_n) \in \mathscr{F}(x_1, \cdots, x_n)$. It is not necessary that each x_i appears in $w_1 w_2$. Let S be a semigroup and $a_1, \cdots, a_n \in S$. Then $(a_1, \cdots, a_n)$ is a solution of $\{w_1, w_2\}$ if $w_1(a_1, \cdots, a_n) = w_2(a_1, \cdots, a_n)$.

Let $(b_1, \cdots, b_n)$ be a solution in $\mathscr{F}(\Gamma)$ of a word equation $\{w_1, w_2\}$ in variables $x_1, \cdots, x_n$. Let S be a semigroup and $\varphi: \mathscr{F}(\Gamma) \to S$, a homomorphism. Let $a_i = \varphi(b_i)$, $i = 1, \cdots, n$. Then $(a_1, \cdots, a_n)$ is a solution of $\{w_1, w_2\}$. We say that $(a_1, \cdots, a_n)$ *follows from* $(b_1, \cdots, b_n)$.

DEFINITION. Let $\{w_1, w_2\}$ be a word equation in variables $x_1, \cdots, x_n$ and S a semigroup.

(1) Let $(a_1, \cdots, a_n)$ be a solution of $\{w_1, w_2\}$ in S. Then $(a_1, \cdots, a_n)$ is strongly resolvable if it follows from some solution of $\{w_1, w_2\}$ in $\mathscr{F}(\Gamma)$ for some nonempty set Γ. By Lentin [**2**] we can then choose $|\Gamma| \leqq n$.

(2) $\{w_1, w_2\}$ is strongly resolvable in S if every solution of $\{w_1, w_2\}$ is strongly resolvable.

Let Γ be a nonempty set and let $\xi: \Gamma \to \mathcal{Q}^+$. Then clearly there exists a unique automorphism φ of $\mathcal{F}_{\mathcal{Q}}(\Gamma)$ such that $\varphi(A) = A^{\xi(A)}$ for all $A \in \Gamma$. Now let $a_1, \cdots, a_n \in \mathcal{F}_{\mathcal{Q}}(\Gamma)$. Then there exists an automorphism φ of $\mathcal{F}_{\mathcal{Q}}(\Gamma)$ of the above type such that $b_i = \varphi(a_i) \in \mathcal{F}(\Gamma)$, $i = 1, \cdots, n$. Suppose $(a_1, \cdots, a_n)$ is a solution of a word equation. Then $(b_1, \cdots, b_n)$ is also a solution of the same equation and $a_i = \varphi^{-1}(b_i)$, $i = 1, \cdots, n$. So we have the following.

THEOREM 1.1. *Every word equation is strongly resolvable in $\mathcal{F}_{\mathcal{Q}}(\Gamma)$ for any nonempty set Γ.*

DEFINITION. Let $w_1, w_2 \in \mathcal{F}_{\mathbf{R}}(\Gamma)$. Suppose $w_1 = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$, $w_2 = B_1^{\beta_1} \cdots B_m^{\beta_m}$ in standard form. If $m = n$ and $A_i = B_i$ $(i = 1, \cdots, n)$, then let $d(w_1, w_2) = \sum_{i=1}^n |\alpha_i - \beta_i|$. Otherwise let $d(w_1, w_2) = \infty$.

LEMMA 1.2. *Let $u_1, u_2, u_3, u_4 \in \mathcal{F}_{\mathbf{R}}(\Gamma)$. Then the following are true in the extended real line.*

(i) $e(u_1u_2) = e(u_1) + e(u_2)$ *or* $e(u_1) + e(u_2) - 1$.

(ii) $d(u_1, u_2) = 0$ *if and only if* $u_1 = u_2$.

(iii) $d(u_1, u_3) \leqq d(u_1, u_2) + d(u_2, u_3)$.

(iv) $d(u_1, u_2) = d(u_2, u_1)$.

(v) $d(u_1u_2, u_3u_4) \leqq d(u_1, u_3) + d(u_2, u_4)$.

Proof. (i), (ii), (iii) and (iv) are clear. So we prove (v). Let $w_1, w_2 \in \mathcal{F}_{\mathbf{R}}(\Gamma)$, $d(w_1, w_2) < \infty$. Let $w_1 = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$, $w_2 = A_1^{\beta_1} \cdots A_n^{\beta_n}$ in standard form. Let $A \in \Gamma$. If $A \neq A_n$, then for any $\alpha \in \mathbf{R}^+$, $w_1A^\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n} A^\alpha$, $w_2 A^\alpha = A_1^{\beta_1} \cdots A_n^{\beta_n} A^\alpha$ in standard form. So $d(w_1A^\alpha, w_2A^\alpha) = d(w_1, w_2)$. If $A = A_n$, then $w_1A^\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n + \alpha}$, $w_2A^\alpha = A_1^{\beta_1} \cdots A_n^{\beta_n + \alpha}$. So again $d(w_1A^\alpha, w_2A^\alpha) = d(w_1, w_2)$. So by induction $d(w_1u, w_2u) = d(w_1, w_2)$ for all $u \in \mathcal{F}_{\mathbf{R}}(\Gamma)$. Similarly $d(uw_1, uw_2) = d(w_1, w_2)$ for all $u \in \mathcal{F}_{\mathbf{R}}(\Gamma)$. Let $u_1, u_2, u_3, u_4 \in \mathcal{F}_{\mathbf{R}}(\Gamma)$ such that $d(u_1, u_3) < \infty$ and $d(u_2, u_4) < \infty$. So $d(u_1u_2, u_3u_4) \leqq d(u_1u_2, u_3u_2) + d(u_3u_2, u_3u_4) = d(u_1, u_3) + d(u_2, u_4)$. The same holds trivially if $d(u_1, u_3) = \infty$ or $d(u_2, u_4) = \infty$.

LEMMA 1.3. (i) *Let $u \in \mathcal{F}_{\mathbf{R}}(\Gamma)$, $n \in Z^+$ such that $e(u) > 1$. Let $u = A_1^{\alpha_1} \cdots A_r^{\alpha_r}$, $u^n = B_1^{\beta_1} \cdots B_s^{\beta_s}$ in standard form. Then $\{\alpha_1, \cdots, \alpha_r\} \subseteq \{\beta_1, \cdots, \beta_s\}$.*

(ii) *Let $u, v \in \mathcal{F}_{\mathbf{R}}(\Gamma)$, $n \in Z^+$. Then $d(u, v) \leqq d(u^n, v^n) \leqq nd(u, v)$.*

Proof. (i) $1 < r \leqq s$. Since $u \,|_i u^n$, $u \,|_f u^n$ we obtain $\alpha_i = \beta_i$ $(1 \leqq i < r)$ and $\alpha_r = \beta_s$.

(ii) That $d(u^n, v^n) \leqq nd(u, v)$ follows from Lemma 1.2 (v). So we show that $d(u,v) \leqq d(u^n, v^n)$. If $d(u^n, v^n) = \infty$, this is trivial. So let $d(u^n, v^n) < \infty$. If u^n or $v^n \in \mathcal{N}(\Gamma)$, then $u, v \in \mathcal{N}(\Gamma)$ and $u \sim v$. So for some $A \in \Gamma$, $\epsilon, \delta \in \mathbf{R}^+$, $u = A^\epsilon$, $v = A^\delta$. So $d(u,v) = |\epsilon - \delta| \leqq |n\epsilon - n\delta| = d(u^n, v^n)$. Next assume $e(u^n), e(v^n) > 1$. Let $u^n = A_1^{\alpha_1} \cdots A_m^{\alpha_m}$, $v^n = A_1^{\beta_1} \cdots A_m^{\beta_m}$ in standard form with $m > 1$. Let $u = B_1^{\gamma_1} \cdots B_r^{\gamma_r}$, $v = C_1^{\delta_1} \cdots C_s^{\delta_s}$ in standard form. Then $r, s > 1$, $B_1 = A_1 = C_1$, $B_r = A_m = C_s$. If $A_1 \neq A_m$, then $rn = m = sn$. So $r = s$. If $A_1 = A_m$, then $r - n - 1 = m = ns - n - 1$. Thus in any case $r = s$. Also $B_i = A_i = C_i$, $1 \leqq i \leqq r$. For $1 \leqq i \leqq r-1$, $\gamma_i = \alpha_i$ and $\delta_i = \beta_i$. Also $\gamma_r = \alpha_m$ and $\delta_s = \beta_m$. Thus $\sum_{i=1}^r |\gamma_i - \delta_i| \leqq \sum_{i=1}^m |\alpha_i - \beta_i|$. This proves the lemma.

If $P \in \mathbf{R}^{m \times n}$, then let P^T denote the transpose of P.

LEMMA 1.4. *Let Γ be a nonempty set and let $A_1, \cdots, A_n \in \Gamma$, $\epsilon_1, \cdots, \epsilon_n \in \mathbf{R}^+$, $i_1, \cdots, i_r$, $j_1, \cdots, j_s \in \{1, \cdots, n\}$. Suppose that in $\mathcal{F}_{\mathbf{R}}(\Gamma)$,*

$$A_{i_1}^{\epsilon_{i_1}} \cdots A_{i_r}^{\epsilon_{i_r}} = A_{j_1}^{\epsilon_{j_1}} \cdots A_{j_s}^{\epsilon_{j_s}}.$$

Then there exists $P \in \mathcal{Q}^{m \times n}$ for some $m \in Z^+$ such that for any $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$, $P(\alpha_1, \cdots, \alpha_n)^T = 0$ if and only if

$$(1) \qquad A_{i_1}^{\alpha_{i_1}} \cdots A_{i_r}^{\alpha_{i_r}} = A_{j_1}^{\alpha_{j_1}} \cdots A_{j_s}^{\alpha_{j_s}}.$$

Proof. We prove by induction on $r + s$. Choose p, q maximal so that $1 \leqq p \leqq r$, $1 \leqq q \leqq s$ and for any α, β with $1 \leqq \alpha \leqq p$, $1 \leqq \beta \leqq q$, we have $A_{i_1} = A_{i_\alpha}$ and $A_{j_1} = A_{j_\beta}$. Clearly $A_{i_1} = A_{j_1}$ and $\sum_{k=1}^p \epsilon_{i_k} = \sum_{k=1}^q \epsilon_{j_k}$. Now clearly $p = r$ if and only if $q = s$. Also in this case, for any $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$, (1) holds if and only if $\sum_{k=1}^r \alpha_{i_k} = \sum_{k=1}^s \alpha_{j_k}$. We can then trivially choose a $1 \times n$ integer matrix P such that for any $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$, $P(\alpha_1, \cdots, \alpha_n)^T = 0$ if and only if $\sum_{k=1}^r \alpha_{i_k} = \sum_{k=1}^s \alpha_{j_k}$.

Thus we may assume $p < r$ adn $q < s$. Then we have

$$A_{i_{p+1}}^{\epsilon_{i_{p+1}}} \cdots A_{i_r}^{\epsilon_{i_r}} = A_{j_{q+1}}^{\epsilon_{j_{q+1}}} \cdots A_{j_s}^{\epsilon_{j_s}}.$$

If $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$, then (1) holds if and only if

$$(2) \qquad \sum_{k=1}^p \alpha_{i_k} = \sum_{k=1}^q \alpha_{j_k}$$

and

$$(3) \qquad A_{i_{p+1}}^{\alpha_{i_{p+1}}} \cdots A_{i_r}^{\alpha_{i_r}} = A_{j_{q+1}}^{\alpha_{j_{q+1}}} \cdots A_{j_s}^{\alpha_{j_s}}.$$

We can trivially choose a $1 \times n$ integer matrix P_1 such that (2) holds if and only if $P_1(\alpha_1, \cdots, \alpha_n)^T = 0$. By our induction hypothesis, we can choose $P_2 \in \mathcal{Q}^{m \times n}$ for some m such that (3) holds if and only if $P_2(\alpha_1, \cdots, \alpha_n)^T = 0$. Let $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$. Then for any $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$, $P(\alpha_1, \cdots, \alpha_n)^T = 0$ if and only if both (2) and (3) hold. This proves the lemma.

LEMMA 1.5. *Let Γ be a nonempty set and let $A_1, \cdots, A_n \in \Gamma$, $\epsilon_1, \cdots, \epsilon_n \in \mathbf{R}^+$, $i_1, \cdots, i_r, j_1, \cdots, j_s \in \{1, \cdots, n\}$. Suppose that in $\mathcal{F}_{\mathbf{R}}(\Gamma)$,*

$$A_{i_1}^{\epsilon_{i_1}} \cdots A_{i_r}^{\epsilon_{i_r}} = A_{j_1}^{\epsilon_{j_1}} \cdots A_{j_s}^{\epsilon_{j_s}}.$$

Let $\delta \in \mathbf{R}^+$. Then there exist $\alpha_1, \cdots, \alpha_n \in \mathcal{Q}^+$ such that $\Sigma_{k=1}^n |\alpha_k - \epsilon_k| < \delta$ and

$$A_{i_1}^{\alpha_{i_1}} \cdots A_{i_r}^{\alpha_{i_r}} = A_{j_1}^{\alpha_{j_1}} \cdots A_{j_s}^{\alpha_{j_s}}.$$

Proof. Choose $P \in \mathcal{Q}^{m \times n}$ as in Lemma 1.4. Let $V = \{(\beta_1, \cdots, \beta_n)^T \mid (\beta_1, \cdots, \beta_n)^T \in \mathbf{R}^{n+1}, P(\beta_1, \cdots, \beta_n)^T = 0\}$. $(\epsilon_1, \cdots, \epsilon_n)^T \in V$ and so $V \neq \{0\}$. Let

$$W = \{(\beta_1, \cdots, \beta_n)^T \mid (\beta_1, \cdots, \beta_n)^T \in \mathcal{Q}^{n \times 1}, P(\beta_1, \cdots, \beta_n)^T = 0\}.$$

Let $\mu = n -$ rank of P. Then $\dim V$ over $\mathbf{R} = \mu = \dim W$ over $\mathcal{Q}$. Since $V \neq \{0\}$, we have $\mu > 0$. W has a basis $H_1, \cdots, H_\mu$ over $\mathcal{Q}$. Let $H =$ the $n \times \mu$ matrix $[H_1, \cdots, H_\mu]$. Then rank of $H = \mu$. So $H_1, \cdots, H_\mu$ are also linearly independent over $\mathbf{R}$. Hence $H_1, \cdots, H_\mu$ form a basis of V and of course $H_1, \cdots, H_\mu \in \mathcal{Q}^{n \times 1}$. So there exist $\delta_1, \cdots, \delta_\mu \in \mathbf{R}$ such that $(\epsilon_1, \cdots, \epsilon_n)^T = \delta_1 H_1 + \cdots + \delta_\mu H_\mu$. Let $\gamma_1, \cdots, \gamma_\mu \in \mathcal{Q}$ and set $(\alpha_1, \cdots, \alpha_n)^T = \gamma_1 H_1 + \cdots + \gamma_\mu H_\mu$. Then clearly $(\alpha_1, \cdots, \alpha_n)^T \in W$. Also

$$\sqrt{\sum_{k=1}^{n} |\alpha_k - \epsilon_k|^2} \leqq \sum_{p=1}^{\mu} |\delta_p - \gamma_p| \, \|H_p\|.$$

Thus for any $\delta \in \mathbf{R}^+$ we can choose $|\delta_p - \gamma_p|$, $p = 1, \cdots, \mu$, small enough so that $|\alpha_k - \epsilon_k| < \delta/n$, $k = 1, \cdots, n$. For δ small enough we then also have $\alpha_k \in \mathcal{Q}^+$, $k = 1, \cdots, n$. This proves the lemma.

THEOREM 1.6. *Let $\{w_1, w_2\}$ be a word equation in variables $x_1, \cdots, x_n$. Let $(a_1, \cdots, a_n)$ be a solution of $\{w_1, w_2\}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma)$. Then for each $\epsilon \in \mathbf{R}^+$, there exists a solution $(b_1, \cdots, b_n)$ of $\{w_1, w_2\}$ in $\mathcal{F}_{\mathcal{Q}}(\Gamma)$ such that $\Sigma_{i=1}^n d(a_i, b_i) < \epsilon$.*

Proof. Let $a_i = A_{i1}^{\beta_{i1}} \cdots A_{im_i}^{\beta_{im_i}}$ in standard form, $i = 1, \cdots, n$. Let w_1 start with x_t and let w_2 start with x_j. Then correspondingly we have

$$A_{t1}^{\beta_{t1}} \cdots = A_{j1}^{\beta_{j1}} \cdots .$$

Choose $\alpha_{ik} \in \mathcal{Q}^+$, $i = 1, \cdots, n$, $1 \leqq k \leqq m_i$. Let $b_i = A_{i1}^{\alpha_{i1}} \cdots A_{im_i}^{\alpha_{im_i}}$, $i = 1, \cdots, n$. Then $b_1, \cdots, b_n \in \mathcal{F}_{\mathcal{Q}}(\Gamma)$. Also, $w_1(b_1, \cdots, b_n) = w_2(b_1, \cdots, b_n)$ if and only if

$$A_{t1}^{\alpha_{t1}} \cdots = A_{j1}^{\alpha_{j1}} \cdots . \tag{4}$$

But by Lemma 1.5 we can choose α_{ik}'s so that (4) holds and $|\alpha_{ik} - \beta_{ik}| < \epsilon$ for all relevant i and k. So clearly $\Sigma_{i=1}^n d(a_i, b_i) = \Sigma_{i,k} |\alpha_{ik} - \beta_{ik}| \leqq M\epsilon$ where $M = \Sigma_{i=1}^n e(a_i)$. This proves the theorem.

LEMMA 1.7. *Let* $A_1, \cdots, A_n \in \Gamma$, $\Lambda \subseteq \Gamma$. *Suppose* $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n \in \mathbf{R}^+$, $i_1, \cdots, i_r$, $j_1, \cdots, j_s \in \{1, \cdots, n\}$ *such that* $A_{i_1}^{\alpha_{i_1}} \cdots A_{i_r}^{\alpha_{i_r}} = A_{j_1}^{\alpha_{j_1}} \cdots A_{j_s}^{\alpha_{j_s}}$ *and* $A_{i_1}^{\beta_{i_1}} \cdots A_{i_r}^{\beta_{i_r}} = A_{j_1}^{\beta_{j_1}} \cdots A_{j_s}^{\beta_{j_s}}$. *Let* $\gamma_i = \alpha_i$ *if* $A_i \in \Lambda$, $\gamma_i = \beta_i$ *if* $A_i \notin \Lambda$, $i = 1, \cdots, n$. *Then* $A_{i_1}^{\gamma_{i_1}} \cdots A_{i_r}^{\gamma_{i_r}} = A_{j_1}^{\gamma_{j_1}} \cdots A_{j_s}^{\gamma_{j_s}}$.

Proof. We prove by induction on $r + s$. Choose p, q maximal such that for $1 \leqq \mu \leqq p$, $1 \leqq \nu \leqq q$, $A_{i_1} = A_{i_\mu}$ and $A_{j_1} = A_{j_\nu}$. Then

$$A_{i_1}^{\alpha_{i_1}} \cdots A_{i_p}^{\alpha_{i_p}} = A_{j_1}^{\alpha_{j_1}} \cdots A_{j_q}^{\alpha_{j_q}};$$
$$A_{i_1}^{\beta_{i_1}} \cdots A_{i_p}^{\beta_{i_p}} = A_{j_1}^{\beta_{j_1}} \cdots A_{j_q}^{\beta_{j_q}}.$$

Since $A_{i_\mu} = A_{j_\nu}$ for $1 \leqq \mu \leqq p$, $1 \leqq \nu \leqq q$, we obtain

$$A_{i_1}^{\gamma_{i_1}} \cdots A_{i_p}^{\gamma_{i_p}} = A_{j_1}^{\gamma_{j_1}} \cdots A_{j_q}^{\gamma_{j_q}}.$$

Also, if $p + q < r + s$, then $p < r$, $q < s$ and

$$A_{i_{p+1}}^{\alpha_{i_{p+1}}} \cdots A_{i_r}^{\alpha_{i_r}} = A_{j_{q+1}}^{\alpha_{j_{q+1}}} \cdots A_{j_s}^{\alpha_{j_s}};$$
$$A_{i_{p+1}}^{\beta_{i_{p+1}}} \cdots A_{i_r}^{\beta_{i_r}} = A_{j_{q+1}}^{\beta_{j_{q+1}}} \cdots A_{j_s}^{\beta_{j_s}}.$$

By our induction hypothesis we then also have,

$$A_{i_{p+1}}^{\gamma_{i_{p+1}}} \cdots A_{i_r}^{\gamma_{i_r}} = A_{j_{q+1}}^{\gamma_{j_{q+1}}} \cdots A_{j_s}^{\gamma_{j_s}}.$$

Hence $A_{i_1}^{\gamma_{i_1}} \cdots A_{i_r}^{\gamma_{i_r}} = A_{j_1}^{\gamma_{j_1}} \cdots A_{j_s}^{\gamma_{j_s}}$, proving the lemma.

We will need the following refinement of Theorem 1.6.

THEOREM 1.8. *Let* $\{w_1, w_2\}$ *be a word equation in variables*

$x_1, \cdots, x_n$. *Let* $(a_1, \cdots, a_n)$ *be a solution of* $\{w_1, w_2\}$ *in* $\mathcal{F}_{\mathbf{R}}(\Gamma)$. *Then for each* $\epsilon \in \mathbf{R}^+$, *there exists a solution* $(c_1, \cdots, c_n)$ *of* $\{w_1, w_2\}$ *in* $\mathcal{F}_{\mathcal{Q}}(\Gamma)$ *such that* $\sum_{i=1}^n d(a_i, c_i) < \epsilon$ *and so that for any* $A \in \Gamma$, A *appears integrally in each* a_i *implies* A *appears integrally in each* c_i.

Proof. Let $\Lambda = \{A \mid A \in \Gamma,\ A$ appears integrally in each $a_i\}$. Choose $(b_1, \cdots, b_n)$ as in Theorem 1.6. Let $a_i = A_{i1}^{\alpha_{i1}} \cdots A_{im_i}^{\alpha_{im_i}}$, $b_i = A_{i1}^{\beta_{i1}} \cdots A_{im_i}^{\beta_{im_i}}$, $i = 1, \cdots, n$ in standard form. Let $\gamma_{ik} = \alpha_{ik}$ if $A_{ik} \in \Lambda$, $\gamma_{ik} = \beta_{ik}$ if $A_{ik} \notin \Lambda$. Set $c_i = A_{i1}^{\gamma_{i1}} \cdots A_{im_i}^{\gamma_{im_i}}$, $i = 1, \cdots, n$. Then $c_i \in \mathcal{F}_{\mathcal{Q}}(\Gamma)$, $d(a_i, c_i) \leqq d(a_i, b_i)$. Let w_1 start with x_i, w_2 start with x_j. Then correspondingly we have,

$$A_{i1}^{\alpha_{i1}} \cdots = A_{j1}^{\alpha_{j1}} \cdots$$
$$A_{i1}^{\beta_{i1}} \cdots = A_{j1}^{\beta_{j1}} \cdots .$$

Then by Lemma 1.7 we also have

$$A_{i1}^{\gamma_{i1}} \cdots = A_{j1}^{\gamma_{j1}} \cdots .$$

So $w_1(c_1, \cdots, c_n) = w_2(c_1, \cdots, c_n)$. This proves the theorem.

Let $\{w_1, w_2\}$ be a word equation in variables $x_1, \cdots, x_n$. A solution $(a_1, \cdots, a_n)$ of $\{w_1, w_2\}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma)$ is *trivial* if either there exist $u \in \mathcal{F}_{\mathbf{R}}(\Gamma)$, $k_1, \cdots, k_n \in Z^+$ such that $u^{k_i} = a_i$, $i = 1, \cdots, n$, or if there exist $A \in \Gamma$, $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$ such that $a_i = A^{\alpha_i}$, $i = 1, \cdots, n$.

THEOREM 1.9. *Let* $\{w_1, w_2\}$ *be a word equation in variables* $x_1, \cdots, x_n$. *Suppose* $\{w_1, w_2\}$ *has only trivial solutions in any free semigroup. Then* $\{w_1, w_2\}$ *has only trivial solutions in any* $\mathcal{F}_{\mathbf{R}}(\Gamma)$.

Proof. Let $(a_1, \cdots, a_n)$ be a solution of $\{w_1, w_2\}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma)$. By Theorem 1.6, there exist solutions $(b_1^{(m)}, \cdots, b_n^{(m)})$, $m \in Z^+$ of $\{w_1, w_2\}$ in $\mathcal{F}_{\mathcal{Q}}(\Gamma)$ such that $d(a_i, b_i^{(m)}) \to 0$ as $m \to \infty$, $i = 1, \cdots, n$. By Theorem 1.1 and our hypothesis, there exist, for each $m \in Z^+$, $u_m \in \mathcal{F}_{\mathcal{Q}}(\Gamma)$, $k(m, i) \in Z^+$, $i = 1, \cdots, n$ such that $b_i^{(m)} = u_m^{k(m,i)}$, $i = 1, \cdots, n$. Now $e(b_i^{(m)}) = e(a_i)$ for all $m \in Z^+$, $i = 1, \cdots, n$. If for any $i \in \{1, \cdots, n\}$, $k(m, i) \to \infty$, then by Lemma 1.2 (i), $e(u_m) = 1$ for some $m \in Z^+$. It then follows easily (since $d(a_j, b_j^{(m)}) < \infty$, $j = 1, \cdots, n$) that $e(a_j) = 1$, $j = 1, \cdots, n$, and $a_j \sim a_r$ for all $j, r \in \{1, \cdots, n\}$. So we may assume that the $k(m, i)$'s are bounded for each $i = 1, \cdots, n$. So $\{(k(m, 1), \cdots, k(m, n)) \mid m \in Z^+\}$ is finite. Hence we can assume without loss of generality (going to a subsequence if necessary) that $k(m, i) = k(t, i)$ for all $m, t \in Z^+$, $i = 1, \cdots, n$. Thus there exist $k_1, \cdots, k_n \in Z^+$ such that for all $m \in Z^+$, $b_i^{(m)} = u_m^{k_i}$, $i = 1, \cdots, n$. If $e(u_m) = 1$ for any m, then we are done as above. So assume $e(u_m) > 1$ for all $m \in Z^+$. Now for all $m, t \in Z^+$, $d(b_1^{(m)}, b_1^{(t)}) < \infty$. So $d(u_n^{k_1}, u_t^{k_1}) < \infty$. By Lemma 1.3 (ii), $d(u_m, u_t) < \infty$. For $m \in Z^+$, let $u_m = A_1^{\alpha(m,1)} \cdots A_r^{\alpha(m,r)}$ in standard form. For any $\epsilon > 0$, $N \in Z^+$, there exist $m, t \in Z^+$, m, $t \geqq N$ such that $d(b_1^{(m)}, b_1^{(t)}) < \epsilon$. So by Lemma 1.3 (ii), $d(u_m, u_t) < \epsilon$. So for $i = 1, \cdots, r$, $\langle \alpha(m, i) \rangle$ is a Cauchy sequence in $\mathbf{R}^+$. Let $\langle \alpha(m, i) \rangle \to \alpha_i$. So $\alpha_i \in \mathbf{R}$ $(i = 1, \cdots, r)$. Let $a_1 = B_1^{\delta_1} \cdots B_t^{\delta_t}$ in standard form. Then by Lemma 1.3 (i) and the fact that $d(a_1, u_m^{k_1}) \to 0$ as $m \to \infty$, we obtain that $\{\alpha_1, \cdots, \alpha_r\} \subseteq \{\delta_1, \cdots, \delta_t\}$. Hence $\alpha_1, \cdots, \alpha_r \in \mathbf{R}^+$. Let $u = A_1^{\alpha_1} \cdots A_r^{\alpha_r}$. So $u \in \mathscr{F}_{\mathbf{R}}(\Gamma)$ and clearly $d(u_m, u) \to 0$ as $m \to \infty$. Let $i \in \{1, \cdots, n\}$. Then by Lemma 1.3(ii), $d(u_m^{k_i}, u^{k_i}) \leqq k_i d(u_m, u)$. So $d(u_m^{k_i}, u^{k_i}) \to 0$. Now $d(a_i, u_m^{k_i}) \to 0$. Also by Lemma 1.2, $d(a_i, u^{k_i}) \leqq d(a_i, u_m^{k_i}) + d(u_m^{k_i}, u^{k_i})$ for all $m \in Z^+$. So $d(a_i, u^{k_i}) = 0$ and thus by Lemma 1.2, $a_i = u^{k_i}$, $i = 1, \cdots, n$. This proves the theorem.

PROBLEM 1.10. Generalize Lentin's theory of principal solutions in the free semigroup **[2]** to $\mathscr{F}_{\mathbf{R}}$.

2. The semigroup of designs around the unit disc.

For $\alpha, \beta \in \mathbf{R}^+$, $\alpha < \beta$, let $I_{\alpha,\beta} = \{x \mid x \in \mathbf{R}^2, \alpha < \|x\| < \beta\}$. Let $\mathfrak{D} = \{(A, \alpha) \mid \alpha \in \mathbf{R}^+, \alpha > 1, A$ is a closed subset of $\bar{I}_{1,\alpha}$; for all $x \in A$ there exists a sequence $\langle x_n \rangle$ in A such that $x_n \to x$ and $\|x_n\| \neq \|x\|$ for all $n\}$. For $(A, \alpha) \in \mathfrak{D}$, let $\Phi(A, \alpha) = A$. $\mathfrak{D}$ becomes a semigroup under the following multiplication

$$(A, \alpha)(B, \beta) = (A \cup \alpha B, \alpha\beta).$$

We call $\mathfrak{D}$ the semigroup of designs around the unit disc. The multiplication above is illustrated in Figure 1. If $(A, \alpha) \in \mathfrak{D}$, then let $l(A, \alpha) = \log \alpha$. So for all $u, v \in \mathfrak{D}$, $l(uv) = l(u) + l(v)$ and $l(u) > 0$. In $\mathfrak{D}^1$, set $l(1) = 0$.

REMARK 2.1. Let $(A, \alpha) \in \mathfrak{D}$. Then $A = \overline{A \cap I_{1,\alpha}}$.

DEFINITION. Let $1 \leqq \beta < \gamma \leqq \alpha$. Then for $(A, \alpha) \in \mathfrak{D}$, $(A, \alpha)_{[\beta,\gamma]} = (\bar{B}, \gamma/\beta)$ where $B = (1/\beta)(A \cap I_{\beta,\gamma})$. Note that $(A, \alpha)_{[\beta,\gamma]} \in \mathfrak{D}$ and since $A = \bar{A}$, $\Phi((A, \alpha)_{[\beta,\gamma]}) \subseteq (1/\beta)A$. Also we define $(A, \alpha)_{[\beta,\beta]} = 1$.

Note that $l((A, \alpha)_{[\beta,\gamma]}) = \log \gamma - \log \beta$. Also by Remark 2.1, $(A, \alpha)_{[1,\alpha]} = (A, \alpha)$.

LEMMA 2.2. (i) *Let* $1 \leqq \beta < \gamma < \delta \leqq \alpha$, $(A, \alpha) \in \mathfrak{D}$. *Then* $(A, \alpha)_{[\beta,\delta]} = (A, \alpha)_{[\beta,\gamma]}(A, \alpha)_{[\gamma,\delta]}$.

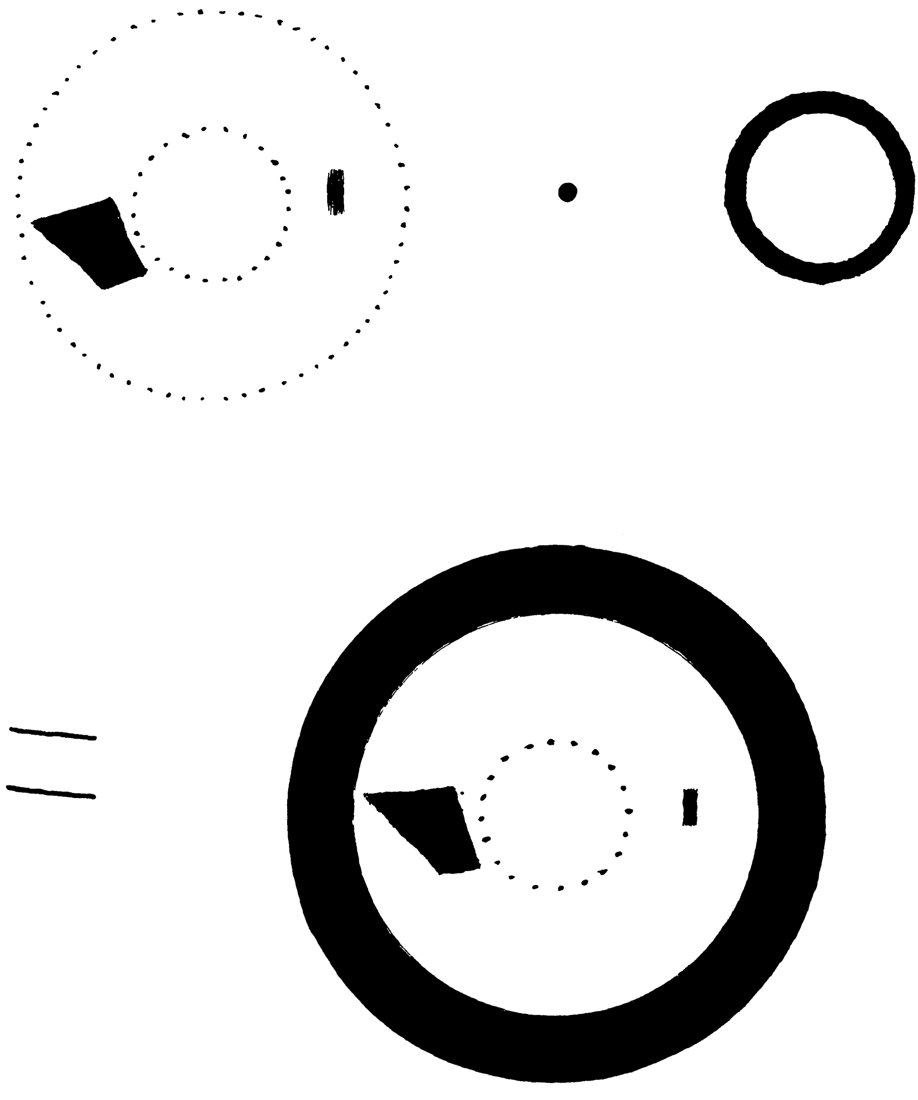

FIGURE 1. Multiplication in $\mathfrak{D}$.

(ii) *Let* $1 \leqq \beta \leqq \gamma < \delta \leqq \mu \leqq \alpha$, $(A, \alpha) \in \mathfrak{D}$. *Then* $l((A, \alpha)_{[\gamma,\delta]}) \leqq l((A, \alpha)_{[\beta,\mu]})$. *Also* $l((A, \alpha)_{[\gamma,\delta]}) = l((A, \alpha)_{[\beta,\mu]})$ *if and only if* $\beta = \gamma$ *and* $\delta = \mu$.

Proof. (i) Let $x \in A$, $\|x\| = \gamma$. Then there exists a sequence $\langle x_n \rangle$ of A such that $\|x_n\| \neq \gamma$ for all n and $x_n \to x$. So $A \cap I_{\beta,\delta} \subseteq \overline{(A \cap I_{\beta,\gamma})} \cup \overline{(A \cap I_{\gamma,\delta})}$. So if $A_1 = A \cap I_{\beta,\delta}$, $A_2 = A \cap I_{\beta,\gamma}$, $A_3 = A \cap I_{\gamma,\delta}$, then $\bar{A}_1 = \bar{A}_2 \cup \bar{A}_3$. Also $(A, \alpha)_{[\beta,\delta]} = ((1/\beta)\bar{A}_1, \delta/\beta)$, $(A, \alpha)_{[\beta,\gamma]} = ((1/\beta)\bar{A}_2, \gamma/\beta)$ and $(A, \alpha)_{[\gamma,\delta]} = ((1/\gamma)\bar{A}_3, \delta/\gamma)$. This yields the result.

(ii) This follows by noting that by (i), $(A,\alpha)_{[\beta,\mu]} = (A,\alpha)_{[\beta,\gamma]}(A,\alpha)_{[\gamma,\delta]}(A,\alpha)_{[\delta,\mu]}$.

LEMMA 2.3. *Let* (A,α), $(B,\beta) \in \mathfrak{D}$. *Set* $(C,\gamma) = (A,\alpha)(B,\beta)$. *Then* $(C,\gamma)_{[1,\alpha]} = (A,\alpha)$ *and* $(C,\gamma)_{[\alpha,\gamma]} = (B,\beta)$.

Proof. $C = A \cup \alpha B$. So $C \cap I_{1,\alpha} \subseteq A$. It follows that $\overline{C \cap I_{1,\alpha}} = \overline{A \cap I_{1,\alpha}}$. By Remark 2.1, $\Phi((C,\gamma)_{[1,\alpha]}) = \overline{C \cap I_{1,\alpha}} = \overline{A \cap I_{1,\alpha}} = A$. Thus $(C,\gamma)_{[1,\alpha]} = (A,\alpha)$. Now $C \cap I_{\alpha,\gamma} \subseteq \alpha B$. So $\overline{C \cap I_{\alpha,\gamma}} = \overline{\alpha B \cap I_{\alpha,\gamma}}$. Thus $\Phi((C,\gamma)_{[\alpha,\gamma]}) = (1/\alpha)(\overline{C \cap I_{\alpha,\gamma}}) = (1/\alpha)(\overline{\alpha B \cap I_{\alpha,\gamma}}) = (\overline{B \cap I_{1,\beta}}) = B$. It follows that $(C,\gamma)_{[\alpha,\gamma]} = (B,\beta)$.

LEMMA 2.4. *Let* $(A,\alpha) \in \mathfrak{D}$, $1 \leqq \beta < \gamma \leqq \alpha$ *and set* $(B,\gamma/\beta) = (A,\alpha)_{[\beta,\gamma]}$. *Let* $\chi : [1,\gamma/\beta] \to [\beta,\gamma]$ *be the order preserving homeomorphism* $\chi(x) = \beta x$. *Then for* $1 \leqq \delta < \mu \leqq \gamma/\beta$, $(B,\gamma/\beta)_{[\delta,\mu]} = (A,\alpha)_{[\chi(\delta),\chi(\mu)]}$.

Proof. $B = (1/\beta)(\overline{A \cap I_{\beta,\gamma}}) \subseteq (1/\beta)A$. So $B \cap I_{\delta,\mu} = I_{\delta,\mu} \cap (1/\beta)A = (1/\beta)(I_{\chi(\delta),\chi(\mu)} \cap A)$. It follows that $\Phi((B,\gamma/\beta)_{[\delta,\mu]}) = \Phi((A,\alpha)_{[\chi(\delta),\chi(\mu)]})$. Also, $\chi(\mu)/\chi(\delta) = \mu/\delta$ and the result follows.

LEMMA 2.5. *Let* $u_1, \cdots, u_n$, $(A,\alpha) \in \mathfrak{D}$ *such that* $(A,\alpha) = u_1 \cdots u_n$. *Then there exist* $\alpha_0, \cdots, \alpha_n \in \mathbf{R}^+$ *such that* $1 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \alpha$ *and* $(A,\alpha)_{[\alpha_{i-1},\alpha_i]} = u_i$, $i = 1, \cdots, n$.

Proof. Clearly we can assume $n > 1$. By Lemma 2.3, there exists $\beta \in (1,\alpha)$ such that $(A,\alpha)_{[1,\beta]} = u_1$, $(A,\alpha)_{[\beta,\alpha]} = u_2 \cdots u_n$. We are now done by induction and Lemma 2.4.

LEMMA 2.6. $\mathfrak{D}$ *is a cancellative semigroup. Let* $u_1, u_2, v_1, v_2 \in \mathfrak{D}$ *such that* $u_1u_2 = v_1v_2$. *Then exactly one of the following occurs.*

(i) $l(u_1) < l(v_1)$, $l(v_2) < l(u_2)$, $u_1 |_i v_1$ *and* $v_2 |_f u_2$.

(ii) $l(v_1) < l(u_1)$, $l(u_2) < l(v_2)$, $v_1 |_i u_1$ *and* $u_2 |_f v_2$.

(iii) $u_1 = v_1$ *and* $u_2 = v_2$.

Proof. Let $u_1, u_2, v_1, v_2 \in \mathfrak{D}$ such that $u_1u_2 = v_1v_2 = (A,\alpha)$. By Lemma 2.3, there exist $\beta, \gamma \in (1,\alpha)$ such that $(A,\alpha)_{[1,\beta]} = u_1$, $(A,\alpha)_{[1,\gamma]} = v_1$, $(A,\alpha)_{[\beta,\alpha]} = u_2$ and $(A,\alpha)_{[\gamma,\alpha]} = v_2$. Suppose $l(u_1) \leqq l(v_1)$. Then by Lemma 2.2(ii), $\beta \leqq \gamma$. So by Lemma 2.2(i), $u_1 |_i v_1$, $v_2 |_f u_2$. If $l(u_1) = l(v_1)$, then $\beta = \gamma$ and so $u_1 = v_1$, $u_2 = v_2$. We are now done by symmetry.

LEMMA 2.7. *Let* $(A,\alpha)\in\mathfrak{D}$, $x\in A$, $\|x\|=\beta$. *Then,*
(i) *If* $\beta\in(1,\alpha)$, *then for* $1\leqq\gamma<\beta<\delta\leqq\alpha$, $x\in\gamma\Phi((A,\alpha)_{[\gamma,\delta]})$.
(ii) *If* $\beta=1$, *then* $x\in\Phi((A,\alpha)_{[1,\delta]})$ *for all* $\delta\in(1,\alpha]$.
(iii) *If* $\beta=\alpha$, *then* $x\in\gamma\Phi((A,\alpha)_{[\gamma,\alpha]})$ *for all* $\gamma\in[1,\alpha)$.

Proof. (i) $x\in A\cap I_{\gamma,\delta}\subseteq\gamma\Phi((A,\alpha)_{[\gamma,\delta]})$.
(ii) There exists a sequence $\langle x_n\rangle$ in A, $\|x_n\|\neq 1$ for all n such that $x_n\to x$. So $x\in\overline{A\cap I_{1,\delta}}=\Phi((A,\alpha)_{[1,\delta]})$.
(iii) There exists a sequence $\langle x_n\rangle$ in A, $\|x_n\|\neq\alpha$ for all n such that $x_n\to x$. So $x\in\overline{A\cap I_{\gamma,\alpha}}=\gamma\Phi((A,\alpha)_{[\gamma,\alpha]})$.

DEFINITION. Let $U=\{x\mid x\in\mathbf{R}^2,\ \|x\|=1\}$.
(1) Let $K=\bar{K}\subseteq U$. Then for $\alpha\in\mathbf{R}^+$, $\alpha>1$, let $K^{(\alpha)}=(A,\alpha)$ where $A=\{\gamma x\mid x\in K,\ \gamma\in[1,\alpha]\}$. Let $\mathcal{L}=\{K^{(\alpha)}\mid K=\bar{K}\subseteq U,\alpha\in\mathbf{R}^+,\alpha>1\}$. Then $\mathcal{L}\subseteq\mathfrak{D}$. Note that $K=U\cap\Phi(K^{(\alpha)})$. So if $K^{(\alpha)}$, $L^{(\beta)}\in\mathcal{L}$ and $K^{(\alpha)}=L^{(\beta)}$, then $K=L$ and $\alpha=\beta$. Examples of elements of $\mathcal{L}$ are given in Figure 2.
(2) Let $K^{(\alpha)}\in\mathcal{L}$. Then for $\beta\in\mathbf{R}^+$, $(K^{(\alpha)})^\beta=K^{(\alpha^\beta)}$. This is well defined and agrees with the semigroup definition of power if $\beta\in Z^+$.
(3) Let $u,v\in\mathfrak{D}$. Define $u\sim v$ if either there exist $a\in\mathfrak{D}$, $i,j\in Z^+$ such that $u=a^i$, $v=a^j$, or if $u,v\in\mathcal{L}$ and $v=u^\alpha$ for some $\alpha\in\mathbf{R}^+$.

REMARK 2.8. (i) $K^{(\alpha)},K^{(\beta)}\in\mathcal{L}$. Then $K^{(\alpha)}K^{(\beta)}=K^{(\alpha\beta)}$.
(ii) Let $u\in\mathcal{L}$, $\beta,\gamma\in\mathbf{R}^+$. Then $(u^\beta)^\gamma=u^{\beta\gamma}$, $u^{\beta+\gamma}=u^\beta u^\gamma$ and $l(u^\beta)=\beta l(u)$.
(iii) Let $u\in\mathcal{L}$. Then there exists unique $v\in\mathcal{L}$ such that $u\sim v$ and $l(v)=1$. If $l(u)=\gamma$, then $v^\gamma=u$.
(iv) Let $u\in\mathfrak{D}$, $v\in\mathcal{L}$. If $u\mid v$, then $u\in\mathcal{L}$ and $u\sim v$.
(v) $\sim$ is clearly an equivalence relation on $\mathcal{L}$. If $u\in\mathfrak{D}$, $v\in\mathcal{L}$, $u\sim v$, then $u\in\mathcal{L}$. It will follow from Theorem 3.16 that $\sim$ is in fact an equivalence relation on $\mathfrak{D}$.

THEOREM 2.9. *Let T be a nonempty finite set. For* $i\in T$, $j\in Z^+$, *choose* $u_{i,j}\in\mathfrak{D}$ *such that* $u_{i,j+1}\mid u_{i,j}$ *for all* $i\in T$, $j\in Z^+$; *and* $l(u_{i,j})\to 0$ *as* $j\to\infty$ *for any fixed* $i\in T$. *Let* $(A,\alpha)\in\mathfrak{D}$. *Assume that for each* $\beta\in(1,\alpha)$, $j\in Z^+$, *there exist* $k\in Z^+$, $\gamma,\delta\in[1,\alpha]$, $i,p,q\in T$ *such that* $\gamma<\beta<\delta$, $k>j$ *and so that either* $(A,\alpha)_{[\gamma,\delta]}=u_{i,k}$ *or else* $(A,\alpha)_{[\gamma,\beta]}=u_{p,k}$ *and* $(A,\alpha)_{[\beta,\delta]}=u_{q,k}$. *Then some* $u_{i,j}\in\mathcal{L}$.

Proof. Let $U=\{x\mid x\in\mathbf{R}^2,\ \|x\|=1\}$. Let $|T|=n$. We prove by induction on n. So assume that the theorem is true for nonempty sets of order less than n (possibly none). We assume that the conclusion of the

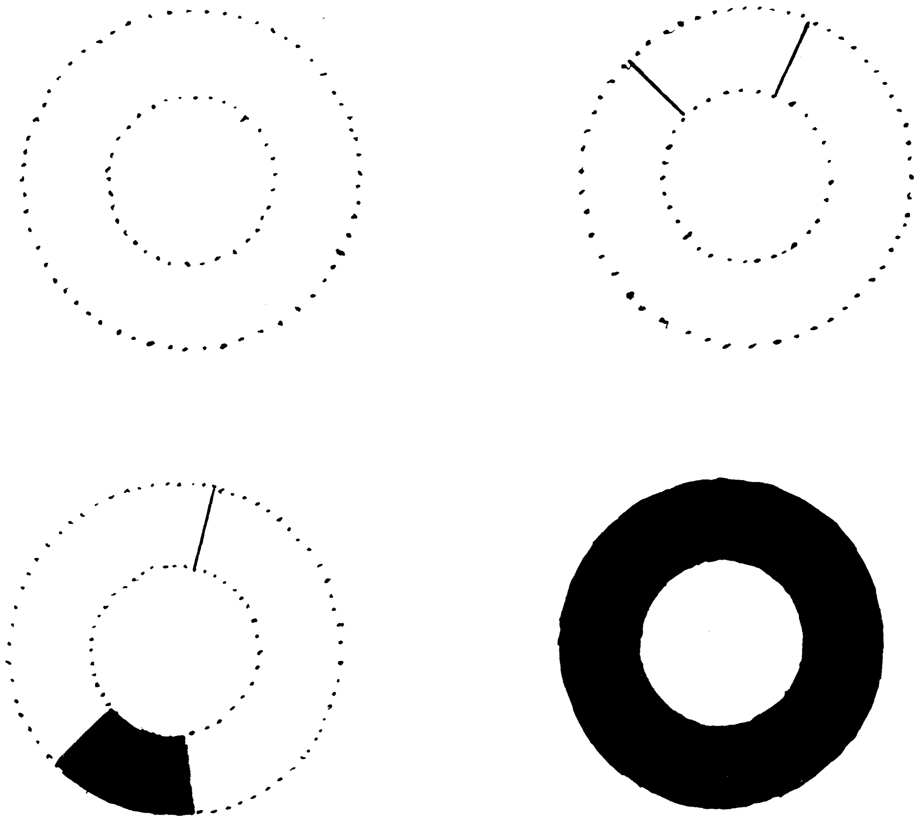

FIGURE 2. Examples of elements of $\mathscr{L}$.

theorem is false and obtain a contradiction. For $x \in U$, let $P_x = \{\gamma x \mid \gamma \in \mathbf{R}^+\}$ and $J_x = P_x \cap I_{1,\alpha}$. Then $\bar{J}_x = P_x \cap \bar{I}_{1,\alpha}$. First we claim that it suffices to show that for each $x \in U$, $J_x \subseteq A$ or $J_x \cap A = \varnothing$. In such a case, first let $J_x \subseteq A$. Then since A is closed, $\bar{J}_x \subseteq A$. Next let $J_x \cap A = \varnothing$. We claim that $\bar{J}_x \cap A = \varnothing$. For, let $y \in \bar{J}_x \cap A$. Then $\|y\| = 1$ or α. So there exists a sequence $\langle y_n \rangle$ in $A \cap I_{1,\alpha}$ such that $y_n \to y$. Let $y_n = r_n\ x_n$, $r_n \in (1, \alpha)$, $x_n \in U$. Then $x_n \to x$. Since $y_n \in J_{x_n} \cap A$, we obtain $J_{x_n} \subseteq A$. So $((\alpha + 1)/2)x_n \in A$ for all n. Since A is closed and $x_n \to x$, we get $((\alpha + 1)/2)x \in A$, contradicting the fact that $J_x \cap A = \varnothing$. We have thus shown that for all $x \in U$, $\bar{J}_x \cap A = \varnothing$ or $\bar{J}_x \subseteq A$. So letting $K = A \cap U$ we see that K is closed and that $(A, \alpha) = K^{(\alpha)} \in \mathscr{L}$. Then of course some $u_{i,j} \in \mathscr{L}$, a contradiction. This establishes our claim.

So let $x \in U$ such that $J_x \not\subseteq A$. Then $J_x \backslash A$ is nonempty and open in J_x. So there exist $\beta, \gamma \in (1, \alpha)$ such that $\beta < \gamma$ and $\bar{I}_{\beta,\gamma} \cap J_x \subseteq J_x \backslash A$. Let $\delta \in (\beta, \gamma)$ and let $j \in Z^+$. Then there exist $k \in Z^+$, $\mu, \nu \in [1, \alpha]$, $i, p, q \in T$ such that $\mu < \delta < \nu$, $k > j$ and so that either $(A, \alpha)_{[\mu,\nu]} =$

$u_{i,k}$ or else $(A, \alpha)_{[\mu,\delta]} = u_{p,k}$ and $(A, \alpha)_{[\delta,\nu]} = u_{q,k}$. If j is large enough (and hence $l(u_{i,k})$, $l(u_{p,k})$, $l(u_{q,k})$ small enough), we obtain that $\mu, \nu \in (\beta, \gamma)$. Hence by Lemma 2.4, $(A, \alpha)_{[\beta,\gamma]}$ satisfies the hypothesis of the theorem for the same T. We now claim that for each $i \in T$, there exists $j \in Z^+$, such that $u_{i,j} \mid (A, \alpha)_{[\beta,\gamma]}$. Suppose not. Then for any $j \in Z^+$, $u_{i,j}$ doesn't come into consideration in the above argument. So $n > 1$ and $(A, \alpha)_{[\beta,\gamma]}$ satisfies the theorem with $T\backslash\{i\}$ in place of T. So by our induction hypothesis some $u_{p,j} \in \mathcal{L}$, a contradiction. So our claim is established. Since $u_{i,j+1} \mid u_{i,j}$ for all relevant i, j, we see that there exists $r \in Z^+$ such that for all $i \in T$, $j \in Z^+$, $j > r$, $u_{i,j} \mid (A, \alpha)_{[\beta,\gamma]}$.

We now assume $J_x \cap A \neq \emptyset$ and obtain a contradiction. So let $a \in J_x \cap A$, $\|a\| = \delta$. So $\delta \in (1, \alpha)$. There exist $k \in Z^+$, $\mu, \nu \in [1, \alpha]$, $i, p, q \in T$ such that $\mu < \delta < \nu$, $k > r$ and so that either $(A, \alpha)_{[\mu,\nu]} = u_{i,k}$ or else $(A, \alpha)_{[\mu,\delta]} = u_{p,k}$ and $(A, \alpha)_{[\delta,\nu]} = u_{q,k}$. But $u_{i,k}, u_{p,k}, u_{q,k} \mid (A, \alpha)_{[\beta,\gamma]}$. So in any case $(A, \alpha)_{[\mu,\delta]} \mid (A, \alpha)_{[\beta,\gamma]}$ and $(A, \alpha)_{[\delta,\nu]} \mid (A, \alpha)_{[\beta,\gamma]}$. By Lemma 2.5, there exist $\xi_1, \xi_2 \in \mathbf{R}^+$ such that $\xi_1\Phi((A, \alpha)_{[\mu,\delta]}) \cup \xi_2\Phi((A, \alpha)_{[\delta,\nu]}) \subseteq \Phi((A, \alpha)_{[\beta,\gamma]})$. By Lemma 2.7(i), $a \in \mu\,\Phi((A, \alpha)_{[\mu,\nu]})$. Since $(A, \alpha)_{[\mu,\nu]} = (A, \alpha)_{[\mu,\delta]} \cdot (A, \alpha)_{[\delta,\nu]}$, there exists $\xi_3 \in \mathbf{R}^+$ such that $a \in \xi_3\Phi((A, \alpha)_{[\mu,\delta]})$ or $a \in \xi_3\Phi((A, \alpha)_{[\delta,\nu]})$. So for some $\xi \in \mathbf{R}^+$, $\xi a \in \Phi((A, \alpha)_{[\beta,\gamma]}) = (1/\beta)(A \cap I_{\beta,\gamma}) \subseteq (1/\beta)(A \cap \bar{I}_{\beta,\gamma})$. So $\beta\xi a \in A \cap \bar{I}_{\beta,\gamma}$. But $a \in J_x$ and so $\beta\xi a \in P_x$. But $\|\beta\xi a\| \in [\beta, \gamma] \subseteq (1, \alpha)$. So $\beta\xi a \in A \cap J_x \cap \bar{I}_{\beta,\gamma}$, contradicting the fact that $\bar{I}_{\beta,\gamma} \cap J_x \subseteq J_x\backslash A$. This contradiction completes the proof of the theorem.

3. Word equations in $\mathfrak{D}$. Let Γ be a nonempty set. Define $\mathcal{F}_{\mathbf{R}}(\Gamma \mid \emptyset) = \mathcal{F}_{\mathbf{R}}(\Gamma)$ and $\mathcal{F}_{\mathbf{R}}(\Gamma \mid \Gamma) = \mathcal{F}(\Gamma)$. If $\Lambda \subseteq \Gamma$, $\Lambda \neq \emptyset$, $\Lambda \neq \Gamma$, then let $\mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$ denote the subsemigroup of $\mathcal{F}_{\mathbf{R}}(\Gamma)$ generated by $\mathcal{F}_{\mathbf{R}}(\Gamma\backslash\Lambda)$ and $\mathcal{F}(\Lambda)$. Let $w \in \mathcal{F}_{\mathbf{R}}(\Gamma)$. Then for any $\Lambda \subseteq \Gamma$, $w \in \mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$ if and only if each $A \in \Lambda$ appears integrally in w.

Let $\varphi : \Gamma \to \mathfrak{D}$, $\Lambda \subseteq \Gamma$, such that $\varphi(\Gamma\backslash\Lambda) \subseteq \mathcal{L}$. Then φ extends naturally to a homomorphism $\hat{\varphi} : \mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda) \to \mathfrak{D}$. In fact let $w \in \mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$, $w = A_1^{\epsilon_1} \cdots A_n^{\epsilon_n}$ in standard form. So $A_i \in \Lambda$ implies $\epsilon_i \in Z^+$. Define $\hat{\varphi}(w) = \varphi(A_1)^{\epsilon_1} \cdots \varphi(A_n)^{\epsilon_n}$. This makes sense, since for $u \in \mathcal{L}$, $\epsilon \in \mathbf{R}^+$, u^ϵ is defined. Using Remark 2.8(ii), it is easily seen that $\hat{\varphi}$ is a homomorphism. We call $\hat{\varphi}$ the natural extension of φ to $\mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$.

Let $(u_1, \cdots, u_n)$ be a solution in $\mathcal{F}_{\mathbf{R}}(\Gamma)$ of a word equation $\{w_1, w_2\}$. Let $\Lambda = \{A \mid A \in \Gamma,\ A$ appears integrally in each $u_1, \cdots, u_n\}$. Then $u_1, \cdots, u_n \in \mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$. Let $\varphi : \Gamma \to \mathfrak{D}$ such that $\varphi(\Gamma\backslash\Lambda) \subseteq \mathcal{L}$. Let $\hat{\varphi}$ be the natural extension of φ. Let $a_i = \hat{\varphi}(u_i)$, $i = 1, \cdots, n$. Then $(a_1, \cdots, a_n)$ is a solution of $\{w_1, w_2\}$ in $\mathfrak{D}$. We say that $(a_1, \cdots, a_n)$ *follows from* $(u_1, \cdots, u_n)$.

REMARK 3.1. In the above notation suppose there exists $\Lambda_1 \subseteq \Gamma$, $\psi: \Gamma \to \mathfrak{D}$ such that $\psi(\Gamma \backslash \Lambda_1) \subseteq \mathscr{L}$. Let $\hat{\psi}$ be the natural extension of ψ to $\mathscr{F}_{\mathbf{R}}(\Gamma | \Lambda_1)$. Suppose $u_1, \cdots, u_n \in \mathscr{F}_{\mathbf{R}}(\Gamma | \Lambda_1)$ and $a_i = \hat{\psi}(u_i)$, $i = 1, \cdots, n$. Then $(a_1, \cdots, a_n)$ follows from $(u_1, \cdots, u_n)$. This is because the above implies that $\Lambda_1 \subseteq \Lambda$ and so $\Gamma \backslash \Lambda \subseteq \Gamma \backslash \Lambda_1 \subseteq \mathscr{L}$. Also it is clear that the natural extension of ψ to $\mathscr{F}_{\mathbf{R}}(\Gamma | \Lambda)$ is the restriction of $\hat{\psi}$ to $\mathscr{F}_{\mathbf{R}}(\Gamma | \Lambda)$.

Even though we are only interested in word equations, it will be convenient to introduce the concept of a constrained word equation.

DEFINITION. Let $w_1 = w_1(x_1, \cdots, x_n)$, $w_2 = w_2(x_1, \cdots, x_n) \in \mathscr{F}(x_1, \cdots, x_n)$. Let $T_1, \cdots, T_s$ denote s disjoint nonempty subsets of $\{x_1, \cdots, x_n\}$. Choose $\alpha_k \in \mathbf{R}^+$ corresponding to each $k \in T_j$, $j = 1, \cdots, s$. Let $M_j = \{(x_k, \alpha_k) | k \in T_j\}$. We call $\mathscr{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ a constrained word equation in variables $x_1, \cdots, x_n$. We allow the possibility that $m = 0$, in which case $\mathscr{A}$ is the word equation $\{w_1, w_2\}$. If $1 \leqq i \leqq n$ and $i \notin T_j$ for every j, $1 \leqq j \leqq s$, then we say that x_i is a free variable of $\mathscr{A}$. Otherwise x_i is a constrained variable. If $m = 0$, then x_i is free $(1 \leqq i \leqq n)$. Let $a_1, \cdots, a_n \in \mathfrak{D}$. Then $(a_1, \cdots, a_n)$ is a solution of $\mathscr{A}$ if the following conditions are satisfied.

(1) $w_1(a_1, \cdots, a_n) = w_2(a_1, \cdots, a_n)$.

(2) $(x_k, \alpha_k) \in M_j$ implies that $a_k \in \mathscr{L}$ and $l(a_k) = \alpha_k$, $j = 1, \cdots, s$.

(3) Let $(x_i, \alpha_i) \in M_p$, $(x_j, \alpha_j) \in M_q$. Then $a_i \sim a_j$ if and only if $p = q$.

Similarly if $a_1, \cdots, a_n \in \mathscr{F}_{\mathbf{R}}(\Gamma)$, then we say that $(a_1, \cdots, a_n)$ is a solution of $\mathscr{A}$ if (1), (2) and (3) above are satisfied with $\mathscr{L}$ replaced by $\mathscr{N}(\Gamma)$.

DEFINITION. Let $\mathscr{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ be a constrained word equation in variables $x_1, \cdots, x_n$.

(1) Let $\mu = (a_1, \cdots, a_n)$, $\nu = (b_1, \cdots, b_n)$ be solutions of $\mathscr{A}$ in $\mathfrak{D}$, $\mathscr{F}_{\mathbf{R}}$ respectively. (Note that then for each constrained variable x_i, $l(a_i) = l(b_i)$). Then we say that μ follows from ν (as solutions of $\mathscr{A}$) if μ follows from ν as solutions of the word equation $\{w_1, w_2\}$.

(2) A solution μ of $\mathscr{A}$ in $\mathfrak{D}$ is resolvable if it follows from a solution of $\mathscr{A}$ in $\mathscr{F}_{\mathbf{R}}(\Gamma)$ with $|\Gamma| \leqq r + s \leqq n$ where r is the number of free variables of $\mathscr{A}$.

(3) $\mathscr{A}$ is resolvable in $\mathfrak{D}$ if every solution of $\mathscr{A}$ in $\mathfrak{D}$ is resolvable.

LEMMA 3.2. *Let* $w_1, w_2 \in \mathscr{F}(x_1, \cdots, x_n)$. *Let* $a_1, \cdots, a_n \in \mathscr{N}(\Gamma)$ *such that* $a_i \sim a_j$ *for all* i, j. *Suppose* $l(w_1(a_1, \cdots, a_n)) = l(w_2(a_1, \cdots, a_n))$. *Then* $w_1(a_1, \cdots, a_n) = w_2(a_1, \cdots, a_n)$.

Proof. For some $A \in \Gamma$, $a_i = A^{\alpha_i}$, $\alpha_i = l(a_i)$, $i = 1, \cdots, n$. Let

$l(w_1(a_1, \cdots, a_n)) = l(w_2(a_1, \cdots, a_n)) = \beta$. Then clearly $w_1(a_1, \cdots, a_n) = A^{\beta} = w_2(a_1, \cdots, a_n)$.

LEMMA 3.3. *Let $a_1, \cdots, a_n \in \mathcal{L}$, $b_1, \cdots, b_n \in \mathcal{N}(\Gamma)$. Suppose that $a_i \sim a_j$ implies $b_i \sim b_j$ for $i, j \in \{1, \cdots, n\}$. Assume further that $l(a_i) = l(b_i)$, $i = 1, \cdots, n$. Let w_1, $w_2 \in \mathcal{F}(x_1, \cdots, x_n)$ such that $w_1(a_1, \cdots, a_n) = w_2(a_1, \cdots, a_n)$. Then $w_1(b_1, \cdots, b_n) = w_2(b_1, \cdots, b_n)$.*

Proof. We prove by induction on length of $w_1 w_2$ in $\mathcal{F}(x_1, \cdots, x_n)$. We can assume without loss of generality that each x_i appears in $w_1 w_2$. Let $w_1 = x_{i_1} \cdots x_{i_s}$, $w_2 = x_{j_1} \cdots x_{j_t}$. So

$$a_{i_1} \cdots a_{i_s} = a_{j_1} \cdots a_{j_t} = a.$$

Choose p, q maximal so that $1 \leqq p \leqq s$, $1 \leqq q \leqq t$; for $1 \leqq k \leqq p$, $a_{i_1} \sim a_{i_k}$ and for $1 \leqq k \leqq q$, $a_{j_1} \sim a_{j_k}$. Now $a_{i_1} |_i a_{j_1}$ or $a_{j_1} |_i a_{i_1}$. So by Remark 2.8(iv), $a_{i_1} \sim a_{j_1}$. Let $u = a_{i_1} \cdots a_{i_p}$ and $v = a_{j_1} \cdots a_{j_q}$. Then $u, v \in \mathcal{L}$. Also $a = ub = vc$ for some $b, c \in \mathfrak{D}^1$. First assume $p = s$. Then $b = 1$. If $q \neq t$, then $a_{j_{q+1}} | u$ and so $a_{j_{q+1}} \sim u \sim a_{j_1}$, a contradiction. So $q = t$. Then $a_i \sim a_j$ for all i, j. Hence $b_i \sim b_j$ for all i, j. Since $l(b_i) = l(a_i)$ for all i, we obtain that $l(w_1(b_1, \cdots, b_n)) = l(w_1(a_1, \cdots, a_n)) = l(w_2(a_1, \cdots, a_n)) = l(w_2(b_1, \cdots, b_n))$. We are then done by Lemma 3.2. Similarly we are done if $q = t$. So assume $p < s$ and $q < t$. We claim that $u = v$. Otherwise, by symmetry, let $v = uv_1$, $v_1 \in \mathcal{L}$. Then $b = v_1 c$. Since $a_{i_{p+1}} |_i b$, we see that $a_{i_{p+1}} |_i v_1$ or $v_1 |_i a_{i_{p+1}}$. So $a_{i_{p+1}} \sim v_1 \sim a_{i_1}$, a contradiction. So $u = v$ and $b = c$. Thus

$$a_{i_1} \cdots a_{i_p} = a_{j_1} \cdots a_{j_q};\, a_{i_{p+1}} \cdots a_{i_s} = a_{j_{q+1}} \cdots a_{j_t}.$$

By our induction hypothesis,

$$b_{i_1} \cdots b_{i_p} = b_{j_1} \cdots b_{j_q} \quad \text{and} \quad b_{i_{p+1}} \cdots b_{i_s} = b_{j_{q+1}} \cdots b_{j_t}.$$

So $b_{i_1} \cdots b_{i_s} = b_{j_1} \cdots b_{j_t}$ and we are done.

LEMMA 3.4. *Let $\mathcal{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Suppose for some $w_3, w_4, w_5, w_6 \in \mathcal{F}(x_1, \cdots, x_n)$, $w_1 = w_3 w_4$, $w_2 = w_5 w_6$ such that w_3 and w_5 involve only constrained variables. Let $(a_1, \cdots, a_n)$ be a solution of $\mathcal{A}$ in $\mathfrak{D}$. Suppose $w_3(a_1, \cdots, a_n) = w_5(a_1, \cdots, a_n)$. Let $\mathcal{B} = \{w_4, w_6; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Then $(a_1, \cdots, a_n)$ is a solution of $\mathcal{B}$. If $(a_1, \cdots, a_n)$ is resolvable as a solution of $\mathcal{B}$, then it is resolvable as a solution of $\mathcal{A}$.*

Proof. Note that the free and constrained variables of $\mathcal{A}$ and $\mathcal{B}$ are the same. Clearly $w_4(a_1, \cdots, a_n) = w_6(a_1, \cdots, a_n)$ and so $(a_1, \cdots, a_n)$ is a solution of $\mathcal{B}$. Let $(b_1, \cdots, b_n)$ be a solution of $\mathcal{B}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma)$ from which $(a_1, \cdots, a_n)$ follows. It suffices to show that $w_1(b_1, \cdots, b_n) = w_2(b_1, \cdots, b_n)$. Let x_j be a variable appearing in $w_3 w_5$. Then x_j is constrained and so $a_j \in \mathcal{L}$, $b_j \in \mathcal{N}(\Gamma)$ and $l(a_j) = l(b_j)$. For the same reason if x_j, x_k appear in $w_3 w_5$, then $a_j \sim a_k$ if and only if $b_j \sim b_k$. So by Lemma 3.3, $w_3(b_1, \cdots, b_n) = w_5(b_1, \cdots, b_n)$. Since $(b_1, \cdots, b_n)$ is a solution of $\mathcal{B}$, $w_4(b_1, \cdots, b_n) = w_6(b_1, \cdots, b_n)$. So $w_1(b_1, \cdots, b_n) = w_2(b_1, \cdots, b_n)$.

LEMMA 3.5. *Let $\mathcal{A} = \{w_1, w_1; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Then $\mathcal{A}$ is resolvable in $\mathfrak{D}$.*

Proof. Let $(a_1, \cdots, a_n)$ be a solution of $\mathcal{A}$ in $\mathfrak{D}$. Let $c_i = a_i$ if x_i is a free variable, and otherwise let $c_i \in \mathcal{L}$ such that $c_i \sim a_i$, $l(c_i) = 1$. Then for constrained x_i we have $a_i = c_i^{l(a_i)}$. Let $\Gamma = \{A_1, \cdots, A_n\}$ where $A_i = A_j$ if and only if $i = j$ or x_i, x_j are constrained and $a_i \sim a_j$. Then $|\Gamma| = r + s$ where r is the number of free variables of $\mathcal{A}$. Let $b_i = A_i$ if x_i is free and otherwise let $b_i = A_i^{l(a_i)}$. Then $(b_1, \cdots, b_n)$ is a solution of $\mathcal{A}$. Let $\Lambda = \{A_i \mid x_i \text{ is free}\}$. Then $b_i \in \mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$, $i = 1, \cdots, n$. Let $\varphi : \Gamma \to \mathfrak{D}$ be given by $\varphi(A_i) = c_i$, $i = 1, \cdots, n$. Then φ is well defined and $\varphi(\Gamma \backslash \Lambda) \subseteq \mathcal{L}$. Let $\hat{\varphi}$ be the natural extension of φ to $\mathcal{F}_{\mathbf{R}}(\Gamma \mid \Lambda)$. Then $\hat{\varphi}(b_i) = a_i$, $i = 1, \cdots, n$. So $(a_1, \cdots, a_n)$ follows from $(b_1, \cdots, b_n)$.

LEMMA 3.6. *Any constrained word equation without free variables is resolvable in $\mathfrak{D}$.*

Proof. Let $\mathcal{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$ with all variables being constrained. Let $(a_1, \cdots, a_n)$ be a solution of $\mathcal{A}$ in $\mathfrak{D}$. So each $a_i \in \mathcal{L}$. Choose $c_i \in \mathcal{L}$ so that $c_i \sim a_i$, $l(c_i) = 1$. So $a_i = c_i^{l(a_i)}$. Let $\Gamma = \{A_1, \cdots, A_n\}$ with $A_i = A_j$ if and only if $a_i \sim a_j$. So $|\Gamma| = s$. Let $b_i = A_i^{l(a_i)}$, $i = 1, \cdots, n$. By Lemma 3.3, $(b_1, \cdots, b_n)$ is a solution of $\mathcal{A}$. Define $\varphi : \Gamma \to \mathfrak{D}$ by $\varphi(A_i) = c_i$, $i = 1, \cdots, n$. Then φ is well defined and $\varphi(\Gamma) \subseteq \mathcal{L}$. Let $\hat{\varphi}$ be the natural extension of φ to $\mathcal{F}_{\mathbf{R}}(\Gamma)$. Then $\hat{\varphi}(b_i) = a_i$, $i = 1, \cdots, n$. So $(a_1, \cdots, a_n)$ follows from $(b_1, \cdots, b_n)$.

LEMMA 3.7. *Let $\mathcal{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Let $w_3 \in \mathcal{F}(x_1, \cdots, x_n)$ and let $\mathcal{B} = \{w_3 w_1, w_3 w_2; M_1, \cdots, M_s\}$ in the same variables. Let $(a_1, \cdots, a_n)$ be a solution of $\mathcal{B}$. Then $(a_1, \cdots, a_n)$ is a solution of $\mathcal{A}$. If $(a_1, \cdots, a_n)$ is resolvable as a solution of $\mathcal{A}$, then it is resolvable as a solution of $\mathcal{B}$.*

Proof. This follows by noting that in $\mathfrak{D}$ as well as in any $\mathcal{F}_{\mathbf{R}}(\Gamma)$, the solutions of $\mathcal{A}$ and $\mathcal{B}$ are the same.

LEMMA 3.8. *Let $\mathcal{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Suppose x_1 is a free variable not occuring in w_1w_2. Let $\mathcal{B} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_2, \cdots, x_n$. If $\mathcal{B}$ is resolvable in $\mathfrak{D}$, then so is $\mathcal{A}$.*

Proof. Let $(a_1, \cdots, a_n)$ be a solution of $\mathcal{A}$ in $\mathfrak{D}$. Then $(a_2, \cdots, a_n)$ is a solution of $\mathcal{B}$ in $\mathfrak{D}$. So $(a_2, \cdots, a_n)$ follows from some solution $(b_2, \cdots, b_n)$ of $\mathcal{B}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma)$ with $|\Gamma| \leqq r + s$ where r is the number of free variables of $\mathcal{B}$. Correspondingly there exist $\Lambda \subseteq \Gamma$, $\varphi: \Gamma \to \mathfrak{D}$ such that $b_2, \cdots, b_n \in \mathcal{F}_{\mathbf{R}}(\Gamma | \Lambda)$, $\varphi(\Gamma \backslash \Lambda) \subseteq \mathcal{L}$ and the natural extension $\hat{\varphi}$ of φ to $\mathcal{F}_{\mathbf{R}}(\Gamma | \Lambda)$ satisfies $\hat{\varphi}(b_i) = a_i$, $i = 2, \cdots, n$. Let $b_1 \notin \mathcal{F}_{\mathbf{R}}(\Gamma)$ and set $\Gamma_1 = \Gamma \cup \{b_1\}$, $\Lambda_1 = \Lambda \cup \{b_1\}$. Then $(b_1, \cdots, b_n)$ is a solution of $\mathcal{A}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma_1)$. Extend φ to φ_1 by setting $\varphi_1(b_1) = a_1$. Then $b_1, b_2, \cdots, b_n \in \mathcal{F}_{\mathbf{R}}(\Gamma_1 | \Lambda_1)$, $\varphi_1(\Gamma_1 \backslash \Lambda_1) \subseteq \mathcal{L}$ and the natural extension $\hat{\varphi}_1$ of φ_1 to $\mathcal{F}_{\mathbf{R}}(\Gamma_1 | \Lambda_1)$ satisfies $\hat{\varphi}_1(b_i) = a_i$, $i = 1, \cdots, n$. So $(a_1, \cdots, a_n)$ follows from $(b_1, \cdots, b_n)$, $|\Gamma_1| \leqq r + 1 + s$ and the number of free variables of $\mathcal{A}$ is $r + 1$.

LEMMA 3.9. *Let $\mathcal{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Suppose $(a_1, \cdots, a_n)$ is a solution of $\mathcal{A}$ in $\mathfrak{D}$. Assume that for some $i \neq j$, x_i and x_j are free variables and $a_i = a_j$. Let $w'_t(x_1, \cdots, x_n) = w_t(x_1, \cdots, x_{j-1}, x_i, x_{j+1}, \cdots, x_n)$, $t = 1, 2$. Then x_j does not appear in $w'_1 w'_2$. Let $\mathcal{B} = \{w'_1, w'_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. If $\mathcal{B}$ is resolvable in $\mathfrak{D}$, then the solution $(a_1, \cdots, a_n)$ of $\mathcal{A}$ is resolvable in $\mathfrak{D}$.*

Proof. Clearly $(a_1, \cdots, a_n)$ is also a solution of $\mathcal{B}$. Let $(b_1, \cdots, b_n)$ be a solution of $\mathcal{B}$ in $\mathcal{F}_{\mathbf{R}}(\Gamma)$ from which $(a_1, \cdots, a_n)$ follows. Then $\mu = (b_1, \cdots, b_{j-1}, b_i, b_{j+1}, \cdots, b_n)$ is also a solution of $\mathcal{A}$ and $(a_1, \cdots, a_n)$ follows from μ.

LEMMA 3.10. *Let $\mathcal{A} = \{w_1, w_2; M_1, \cdots, M_s\}$ in variables $x_1, \cdots, x_n$. Let $(a_1, \cdots, a_n)$ be a solution of $\mathcal{A}$ in $\mathfrak{D}$. Suppose that for some i, x_i is free and $a_i \in \mathcal{L}$. If $a_i \sim a_j$ for some $(x_j, \alpha_j) \in M_p$, then let $M'_p = M_p \cup \{(x_i, l(a_i))\}$, $M'_q = M_q$ for $q \neq p$ and set $\mathcal{B} = \{w_1, w_2; M'_1, \cdots, M'_s\}$ in variables $x_1, \cdots, x_n$. If $a_i \not\sim a_j$ for any constrained variable x_j, then set $\mathcal{B} = \{w_1, w_2; M_1, \cdots, M_s, \{(x_i, l(a_i))\}\}$ in variables $x_1, \cdots, x_n$. Then $\mathcal{B}$ has lesser number of free variables than $\mathcal{A}$. If $\mathcal{B}$ is resolvable in $\mathfrak{D}$ then so is the solution $(a_1, \cdots, a_n)$ of $\mathcal{A}$.*

Proof. Let r be the number of free variables of $\mathcal{A}$. Then $\mathcal{B}$ has

$r-1$ free variables. Clearly $(a_1,\cdots,a_n)$ is also a solution of $\mathscr{B}$. Let $(a_1,\cdots,a_n)$ follow from a solution $(b_1,\cdots,b_n)$ of $\mathscr{B}$ in $\mathscr{F}_{\mathbf{R}}(\Gamma)$ with $|\Gamma|\leqq(r-1)+(s+1)=r+s$. Then clearly $(b_1,\cdots,b_n)$ is also a solution of $\mathscr{A}$ and hence the result follows.

LEMMA 3.11. *Let $\mathscr{A}=\{w_1,w_2;M_1,\cdots,M_s\}$. Let $\mu=(a_1,\cdots,a_n)$ be a solution of $\mathscr{A}$ in $\mathfrak{D}$. Suppose $(x_i,\alpha_i)\in M_k$. Assume $a_i=a_i'a_i''$ for some $a_i',a_i''\in\mathfrak{D}$. Introduce new variables x_i', x_i'' and set*

$$\begin{aligned}&w_t'(x_1,\cdots,x_{i-1},x_i',x_i'',x_{i+1},\cdots,x_n)\\&\quad=w_t(x_1,\cdots,x_{i-1},x_i'x_i'',x_{i+1},\cdots,x_n)\\&\quad\in\mathscr{F}(x_1,\cdots,x_{i-1},x_i',x_i'',x_{i+1},\cdots,x_n),\qquad t=1,2.\end{aligned}$$

Let $M_j'=M_j$ for $j\neq k$, $M_k'=\{(x_i',l(a_i')),\ (x_i'',l(a_i''))\}\cup(M_k\backslash\{(x_i,\alpha_i)\})$. Let $\mathscr{B}=\{w_1',w_2';M_1',\cdots,M_s'\}$ in variables $x_1,\cdots,x_{i-1},x_i',x_i'',x_{i+1},\cdots,x_n$. Then $\mathscr{B}$ has the same number of free variables as $\mathscr{A}$. Also $\nu=(a_1,\cdots,a_{i-1},a_i',a_i'',a_{i+1},\cdots,a_n)$ is a solution of $\mathscr{B}$. If ν is resolvable in $\mathfrak{D}$ then so is μ.

Proof. Let r be the number of free variables of $\mathscr{A}$ (and hence $\mathscr{B}$). First note that since $a_i',a_i''\,|\,a_i$, $a_i'\sim a_i''\sim a_i$. It is then obvious that ν is a solution of $\mathscr{B}$. Let ν follow from a solution $(b_1,\cdots,b_{i-1},b_i',b_i'',b_{i+1},\cdots,b_n)$ of $\mathscr{B}$ in $\mathscr{F}_{\mathbf{R}}(\Gamma)$ with $|\Gamma|\leqq r+s$. Let $b_i=b_i'b_i''$ and let $\xi=(b_1,\cdots,b_{i-1},b_i,b_{i+1},\cdots,b_n)$. It is then clear that ξ is a solution of $\mathscr{A}$ and that μ follows from ξ.

LEMMA 3.12. *Let $\mathscr{A}=\{w_1,w_2;M_1,\cdots,M_s\}$ in variables $x_1,\cdots,x_n$. Let $\mu=(a_1,\cdots,a_n)$ be a solution of $\mathscr{A}$ in $\mathfrak{D}$. Suppose $i\neq j$, x_j is a free variable and $a_j=a_ia_j'$ for some $a_j'\in\mathfrak{D}$. Introduce a new variable x_j'. Let*

$$\begin{aligned}&w_t'(x_1,\cdots,x_{j-1},x_j',x_{j+1},\cdots,x_n)\\&\quad=w_t(x_1,\cdots,x_{j-1},x_ix_j',x_{j+1},\cdots,x_n)\\&\quad\in\mathscr{F}(x_1,\cdots,x_{j-1},x_j',x_{j+1},\cdots,x_n),\qquad t=1,2.\end{aligned}$$

Let $\mathscr{B}=\{w_1',w_2';M_1,\cdots,M_s\}$ in variables $x_1,\cdots,x_{j-1},x_j',x_{j+1},\cdots,x_n$. Then $\nu=(a_1,\cdots,a_{j-1},a_j',a_{j+1},\cdots,a_n)$ is a solution of $\mathscr{B}$. If ν is resolvable then so is μ.

Proof. Let r be the number of free variables of $\mathscr{A}$ (and hence $\mathscr{B}$). It is clear that ν is a solution of $\mathscr{B}$. Let ν follow from a solution

$(b_1, \cdots, b_{j-1}, b'_j, b_{j+1}, \cdots, b_n)$ of $\mathscr{B}$ in $\mathscr{F}_{\mathbf{R}}(\Gamma)$ with $|\Gamma| \leqq r+s$. Let $b_j = b_t b'_j$. Then $\delta = (b_1, \cdots, b_{j-1}, b_j, b_{j+1}, \cdots, b_n)$ is a solution of $\mathscr{A}$ and μ follows from δ.

Let $r \in \mathbf{N}$ and consider the following:

(*) Every constrained word equation in less than r free variables (possibly none) is resolvable in $\mathfrak{D}$.

LEMMA 3.13. *Assume* (*). *Let* $\mathscr{A} = \{w_1, w_2; \cdots\}$ *in variables* $x_1, \cdots, x_n$. *Assume* $\mathscr{A}$ *has exactly r free variables and that* w_1 *and* w_2 *start with different variables, at least one of which is free. Then* $\mathscr{A}$ *is resolvable in* $\mathfrak{D}$.

Proof. Let $(a_1, \cdots, a_n)$ be a solution of $\mathscr{A}$ in $\mathfrak{D}$. Assume $(a_1, \cdots, a_n)$ is not resolvable. We will obtain a contradiction. Let $T = \{i \mid x_i \text{ is a constrained variable}\}$. So by (*) and Lemma 3.8, each free variable occurs in w_1w_2. Let x_i appear $m_i^{(1)}$ times in w_1w_2, $i = 1, \cdots, n$. Then $m_i^{(1)} \in \mathbf{N}$ for $i \in T$ and $m_i^{(1)} \in Z^+$ for $i \notin T$. Let $u = w_1w_2(a_1, \cdots, a_n)$. So u is a word in $a_1, \cdots, a_n$ with a_i appearing $m_i^{(1)}$ times, $i = 1, \cdots, n$. Now let $\mathscr{A}^{(1)} = \mathscr{A}$, $w_1^{(1)} = w_1$, $w_2^{(1)} = w_2$, $x_i^{(1)} = x_i$, $a_i^{(1)} = a_i$, $i = 1, \cdots, n$. We will construct a sequence of constrained word equations $\mathscr{A}^{(k)} = \{w_1^{(k)}, w_2^{(k)}; \cdots\}$ in variables $x_1^{(k)}, \cdots, x_n^{(k)}$ with solutions $(a_1^{(k)}, \cdots, a_n^{(k)})$ in $\mathfrak{D}$ such that the following properties are true for all $k \in Z^+$.

(I) The constrained variables of $\mathscr{A}^{(k)}$ are exactly $x_i^{(k)}$, $i \in T$. Also for $i \in T$, $a_i^{(k)} = a_i^{(1)}$.

(II) u is a word in $a_1^{(k)}, \cdots, a_n^{(k)}$ with $a_i^{(k)}$ appearing $m_i^{(k)}$ times. If $k > 1$, then $m_i^{(k)} \geqq m_i^{(k-1)}$, $i = 1, \cdots, n$ and $\Sigma_{i=1}^n m_i^{(k)} > \Sigma_{i=1}^n m_i^{(k-1)}$.

(III) If $k > 1$, then $a_i^{(k-1)}$ is a word in $a_1^{(k)}, \cdots, a_n^{(k)}$, $i = 1, \cdots, n$.

(IV) If $k > 1$, then $a_i^{(k)} |_f a_i^{(k-1)}$, $i = 1, \cdots, n$.

(V) $w_1^{(k)}$ and $w_2^{(k)}$ start with different variables, at least one of which is free.

(VI) $(a_1^{(k)}, \cdots, a_n^{(k)})$ is not resolvable.

Clearly $\mathscr{A}^{(1)}$ satisfies (I) to (VI). We proceed by induction. So having constructed $\mathscr{A}^{(j)}$, $1 \leqq j \leqq k$, satisfying (I) to (VI), we proceed to construct $\mathscr{A}^{(k+1)}$. Let $w_1^{(k)} = x_p^{(k)} \cdots$, $w_2^{(k)} = x_q^{(k)} \cdots$. So $p \neq q$ and either x_p or x_q is free. We have correspondingly

$$a_p^{(k)} \cdots = a_q^{(k)} \cdots. \tag{5}$$

First consider the case that $a_p^{(k)} = a_q^{(k)}$. If both $x_p^{(k)}$ and $x_q^{(k)}$ are free, then by applying first Lemma 3.9, and then Lemma 3.8 and (*), we see that

$(a_1^{(k)}, \cdots, a_n^{(k)})$ is resolvable, a contradiction. Next assume $x_q^{(k)}$ is constrained. Then $x_p^{(k)}$ is free and $a_p^{(k)} \in \mathcal{L}$. Then by Lemma 3.10 and (*), $(a_1^{(k)}, \cdots, a_n^{(k)})$ is resolvable, a contradiction. So $l(a_p^{(k)}) \neq l(a_q^{(k)})$. By symmetry, assume $l(a_p^{(k)}) < l(a_q^{(k)})$. Then $a_p^{(k)}|_i a_q^{(k)}$. First suppose $x_q^{(k)}$ is constrained. Then $x_p^{(k)}$ is free and $a_p^{(k)} \in \mathcal{L}$. We then get a contradiction as above. So $x_q^{(k)}$ is free. Now $a_q^{(k)} = a_p^{(k)} a_q^{(k+1)}$ for some $a_q^{(k+1)} \in \mathfrak{D}$. Set $a_i^{(k+1)} = a_i^{(k)}$ for $i \neq q$. Clearly $a_i^{(k+1)}|_f a_i^{(k)}$, $i = 1, \cdots, n$. Also since $q \notin T$, $a_i^{(k)} = a_i^{(k+1)}$ for $i \in T$. Trivially, each $a_i^{(k)}$ is a word in $a_1^{(k+1)}, \cdots, a_n^{(k+1)}$. So u is a word in $a_1^{(k+1)}, \cdots, a_n^{(k+1)}$. Let $a_i^{(k+1)}$ appear $m_i^{(k+1)}$ times in this word. Then $m_i^{(k+1)} = m_i^{(k)}$ for $i \neq p$ and $m_p^{(k+1)} = m_p^{(k)} + m_q^{(k)} \geqq m_p^{(k)} + m_q^{(1)} > m_p^{(k)}$. So $\Sigma_{i=1}^n m_i^{(k+1)} > \Sigma_{i=1}^n m_i^{(k)}$. Now the left hand side of (5) must include more than just $a_p^{(k)}$ (as $l(a_p^{(k)}) < l(a_q^{(k)})$). So let the left side of (5) be $a_p^{(k)} a_t^{(k)} \cdots$. If $t \neq q$, then (5) becomes

$$a_t^{(k+1)} \cdots = a_q^{(k+1)} \cdots, \qquad t \neq q. \tag{6}$$

If $t = q$, then (5) becomes

$$a_p^{(k+1)} a_q^{(k+1)} \cdots = a_q^{(k+1)} \cdots, \qquad p \neq q. \tag{7}$$

Now introduce a new variable $x_q^{(k+1)}$ and set $x_i^{(k+1)} = x_i^{(k)}$ for $i \neq q$. If (6) holds, then correspondingly let $w_1^{(k+1)} = x_t^{(k+1)} \cdots$, $w_2^{(k+1)} = x_q^{(k+1)} \cdots$. If (7) holds, then correspondingly let $w_1^{(k+1)} = x_p^{(k+1)} x_q^{(k+1)} \cdots$, $w_2^{(k+1)} = x_q^{(k+1)} \cdots$. Now applying Lemma 3.12 and then Lemma 3.7 we can construct a constrained word equation $\mathscr{A}^{(k+1)} = \{w_1^{(k+1)}, w_2^{(k+1)}; \cdots\}$ in variables $x_1^{(k+1)}, \cdots, x_n^{(k+1)}$ such that $(a_1^{(k+1)}, \cdots, a_n^{(k+1)})$ is an unresolvable solution of $\mathscr{A}^{(k+1)}$. Also a close examination of the construction shows that the constrained variables of $\mathscr{A}^{(k+1)}$ are exactly $x_i^{(k+1)}$, $i \in T$. This completes the induction step of our construction.

Now by (II), $\Sigma_{i=1}^n m_i^{(k)} \to \infty$ as $k \to \infty$. So at least one $m_t^{(k)} \to \infty$. So $l(a_t^{(k)}) \to 0$. Let $K = \{i \mid l(a_i^{(k)}) \to 0\}$. By (I), $T \cap K = \varnothing$. There exists $\epsilon \in \mathbf{R}^+$ such that for $i \notin K$, $l(a_i^{(k)}) > \epsilon$ for all $k \in Z^+$. Choose k large enough so that $l(a_t^{(k)}) < \epsilon$. Let $a = a_t^{(k)}$. Then by (III), for all $\alpha \in Z^+$, $\alpha > k$, a is a word in $a_i^{(\alpha)}$, $i \in K$. Let $P_\alpha = \{a_i^{(\alpha)} \mid i \in K\}$. Let $a = (A, \xi)$. Then by Lemma 2.5, for each $\alpha \in Z^+$, $\alpha > k$, there exist $\xi_0, \cdots, \xi_m$ such that $1 = \xi_0 < \xi_1 < \cdots < \xi_m = \xi$ and for $j = 1, \cdots, m$, $(A, \xi)_{[\xi_{j-1}, \xi_j]} \in P_\alpha$. So we see that the hypothesis of Theorem 2.9 is satisfied. So $a_i^{(\alpha)} \in \mathcal{L}$ for some $i \in K$, $\alpha \in Z^+$. Then since $T \cap K = \varnothing$, $x_i^{(\alpha)}$ is a free variable of $\mathscr{A}_i^{(\alpha)}$. So by Lemma 3.10 and (*), $(a_1^{(\alpha)}, \cdots, a_n^{(\alpha)})$ is resolvable, contradicting (VI). This completes the proof of Lemma 3.13.

THEOREM 3.14. *Every constrained word equation is resolvable in* $\mathfrak{D}$.

Proof. Let $r \in \mathbf{N}$ and assume (*). We must show that every constrained word equation with r free variables is resolvable. Let $\mathscr{A} = \{w_1, w_2; \cdots\}$ in variables $x_1, \cdots, x_n$ with r free variables. We prove by induction on length of $w_1 w_2$ in $\mathscr{F}(x_1, \cdots, x_n)$ that $\mathscr{A}$ is resolvable. Let $T = \{i \mid x_i \text{ is constrained}\}$. Let $(a_1, \cdots, a_n)$ be a solution of $\mathscr{A}$ in $\mathfrak{D}$. If w_1 and w_2 start with the same variable, then by our induction hypotheses, Lemma 3.7 and Lemma 3.5, we are done. So let w_1, w_2 start with different variables. If some free variable does not appear in $w_1 w_2$ then since (*) holds, we are done by Lemma 3.8. So assume that each free variable occurs in $w_1 w_2$. If either w_1 or w_2 starts with a free variable, then we are done by Lemma 3.13. So assume that both w_1 and w_2 start with constrained variables. Let $w_1 = x_{i_1} \cdots x_{i_m}$ and $w_2 = x_{j_1} \cdots x_{j_t}$. Choose p, q maximal so that $1 \leqq p \leqq m$, $1 \leqq q \leqq t$ and for $1 \leqq \alpha \leqq p$, $1 \leqq \beta \leqq q$ we have $i_\alpha, j_\beta \in T$. Clearly,

$$a_{i_1} \cdots a_{i_m} = a_{j_1} \cdots a_{j_t}. \tag{8}$$

By symmetry assume that $l(a_{i_1} \cdots a_{i_p}) \leqq l(a_{j_1} \cdots a_{j_q})$. Choose α minimal such that $1 \leqq \alpha \leqq q$ and $l(a_{i_1} \cdots a_{i_p}) \leqq l(a_{j_1} \cdots a_{j_\alpha})$. Then $a_{j_\alpha} = a'_{j_\alpha} a''_{j_\alpha}$ for some $a'_{j_\alpha} \in \mathscr{L}$, $a''_{j_\alpha} \in \mathscr{L}^1$ such that

$$a_{i_1} \cdots a_{i_p} = \begin{cases} a_{j_1} \cdots a_{j_{\alpha-1}} a'_{j_\alpha} & \text{if} \quad \alpha > 1 \\ a'_{j_1} & \text{if} \quad \alpha = 1. \end{cases} \tag{9}$$

First consider the case $a''_{j_\alpha} = 1$. Then $a'_{j_\alpha} = a_{j_\alpha}$ and $a_{i_1} \cdots a_{i_p} = a_{j_1} \cdots a_{j_\alpha}$. Now by (8), $p = m$ if and only if $\alpha = t$ and in such a case we are done by Lemma 3.6. So let $p < m$, $\alpha < t$. But now we are done by Lemma 3.4 and our induction hypothesis on $l(w_1 w_2)$ in $\mathscr{F}(x_1, \cdots, x_n)$.

So we are left with the case $a''_{j_\alpha} \neq 1$. Then $p < m$ and $x_{i_{p+1}}$ is free. Also by (8), (9) we have

$$a_{i_{p+1}} \cdots = a''_{j_\alpha} \cdots. \tag{10}$$

Now as in Lemma 3.11 introduce new variables x'_{j_α}, x''_{j_α}. Corresponding to (10), let $w'_1 = x_{i_{p+1}} \cdots$ and $w'_2 = x''_{j_\alpha} \cdots$. Now an application of Lemma 3.11 followed by Lemma 3.4 (because of (9)) yields a constrained word equation $\mathscr{B} = \{w'_1, w'_2, \cdots\}$ with same free variables as $\mathscr{A}$ (though the total number of variables is $n + 1$) such that (10) represents a solution of $\mathscr{B}$ and the resolvability of $\mathscr{B}$ implies the resolvability of $(a_1, \cdots, a_n)$. Also in this construction, $x_{i_{p+1}}$ is free and x''_{j_α} is constrained. So by Lemma 3.13, $\mathscr{B}$ is resolvable. So $(a_1, \cdots, a_n)$ is resolvable and our proof of Theorem 3.14 is complete.

COROLLARY 3.15. *Every word equation is resolvable in* $\mathfrak{D}$.

Let $\{w_1, w_2\}$ be a word equation in variables $x_1, \cdots, x_n$. A solution $(a_1, \cdots, a_n)$ in $\mathfrak{D}$ of $\{w_1, w_2\}$ is *trivial* if either there exist $u \in \mathfrak{D}$, $k_1, \cdots, k_n \in Z^+$ such that $a_i = u^{k_i}$, $i = 1, \cdots, n$ or if there exist $a \in \mathcal{L}$, $\alpha_1, \cdots, \alpha_n \in \mathbf{R}^+$ such that $a^{\alpha_i} = a_i$, $i = 1, \cdots, n$. Then Theorem 1.9 and Corollary 3.15 imply the following.

THEOREM 3.16. *Let* $\{w_1, w_2\}$ *be a word equation in variables* $x_1, \cdots, x_n$ *having only trivial solutions is any free semigroup. Then* $\{w_1, w_2\}$ *has only trivial solutions in* $\mathfrak{D}$.

4. An approximation theorem for $\mathfrak{D}$.

For the definition of a pseudo-metric, see for example [**5**; p. 129]. Consider the following properties for a function $\varphi: \mathfrak{D} \times \mathfrak{D} \to \mathbf{R}^+ \cup \{0\}$.

(a) φ is a pseudo-metric on $\mathfrak{D}$.

(b) For any $u_1, u_2 \in \mathfrak{D}$, $\epsilon \in \mathbf{R}^+$, there exists $\delta \in \mathbf{R}^+$ such that for all $v_1, v_2 \in \mathfrak{D}$, $\varphi(u_i, v_i) < \delta$, $i = 1, 2$, implies $\varphi(u_1u_2, v_1v_2) < \epsilon$.

(c) For any $u \in \mathcal{L}$, $\varphi(u, u^\delta) \to 0$ as $\delta \to 1$.

If the above hold, then it is easy to see that for all $u_1, \cdots, u_m \in \mathfrak{D}$, $\epsilon \in \mathbf{R}^+$, there exists $\delta \in \mathbf{R}^+$ such that for any $v_1, \cdots, v_n \in \mathfrak{D}$, $\varphi(u_i, v_i) < \delta$, $i = 1, \cdots, m$ implies $\varphi(u_1 \cdots u_m, v_1 \cdots v_m) < \epsilon$.

Using Corollary 3.15, Theorems 1.1 and 1.8, we obtain the following

THEOREM 4.1. *Let* φ *satisfy* (a), (b) *and* (c) *above. Let* $(a_1, \cdots, a_n)$ *be a solution in* $\mathfrak{D}$ *of a word equation* $\{w_1, w_2\}$. *Then for every* $\epsilon \in \mathbf{R}^+$, *there exists a strongly resolvable solution* $(b_1, \cdots, b_n)$ *of* $\{w_1, w_2\}$ *in* $\mathfrak{D}$ *such that* $\varphi(a_i, b_i) < \epsilon$, $i = 1, \cdots, n$.

DEFINITION. Let ρ be the pseudo-metric on compact subsets of $\mathbf{R}^2$ given by $\rho(A, B) = m(A \backslash B \cup B \backslash A)$ where m denotes the Lebesgue measure. Let λ be pseudo-metric on $\mathfrak{D}$ given by $\lambda((A, \alpha), (B, \beta)) = \rho(A, B) + |\alpha - \beta|$.

THEOREM 4.2. *Let* $(a_1, \cdots, a_n)$ *be a solution in* $\mathfrak{D}$ *of a word equation* $\{w_1, w_2\}$. *Then for every* $\epsilon \in \mathbf{R}^+$, *there exists a strongly resolvable solution* $(b_1, \cdots, b_n)$ *of* $\{w_1, w_2\}$ *in* $\mathfrak{D}$ *such that* $\lambda(a_1, b_i) < \epsilon$, $i = 1, \cdots, n$.

Proof. By Theorem 4.1 we must show that λ satisfies (a), (b) and (c). First note that ρ satisfies the following.

1. $\rho(A \cup B, C \cup D) \leqq \rho(A, C) + \rho(B, D)$.
2. $\rho(\alpha A, A) \to 0$ as $\alpha \to 1$ and A is fixed.

Now let $(A_1, \alpha_1), (A_2, \alpha_2), (B_1, \beta_1), (B_2, \beta_2) \in \mathfrak{D}$. Then $(A_1, \alpha_1)(A_2, \alpha_2) =$

$(A_1 \cup \alpha_1 A_2, \alpha_1\alpha_2)$ and $(B_1, \beta_1)(B_2, \beta_2) = (B_1 \cup \beta_1 B_2, \beta_1\beta_2)$. So

$$\rho(A_1 \cup \alpha_1 A_2, B_1 \cup \beta_1 B_2) \leqq \rho(A_1, B_1) + \rho(\alpha_1 A_2, \beta_1 A_2) + \rho(\beta_1 A_2, \beta_1 B_2).$$

Let (A_1, α_1), (A_2, α_2) be fixed and suppose $\lambda((A_1, \alpha_1), (B_1, \beta_1)) \to 0$, $\lambda((A_2, \alpha_2), (B_2, \beta_2)) \to 0$. Then $\rho(A_1, B_1) \to 0$, $\beta_1 \to \alpha_1$, $\beta_2 \to \alpha_2$, $\rho(A_2, B_2) \to 0$. So $\rho(A_1 \cup \alpha_1 A_2, B_1 \cup \beta_1 B_2) \to 0$ and $\beta_1\beta_2 \to \alpha_1\alpha_2$. Thus $\lambda((A_1, \alpha_1)(A_2, \alpha_2), (B_1, \beta_1)(B_2, \beta_2)) \to 0$. This establishes (b). Next let $K = \bar{K} \subseteq U = \{x \mid x \in \mathbf{R}^2, \|x\| = 1\}$, $\alpha, \beta \in \mathbf{R}^+$, $1 < \alpha < \beta$. Then $\Phi(K^{(\beta)}) \backslash \Phi(K^{(\alpha)}) \subseteq \bar{I}_{\alpha,\beta}$. So for α fixed, $\lambda(K^{(\alpha)}, K^{(\beta)}) \to 0$ as $\beta \to \alpha$. This establishes (c). (a) is of course trivial and the theorem is proved.

5. Word equations of paths. In this section let $n \in Z^+$ be fixed and let $\mathcal{D}_1$ denote the groupoid of paths in $\mathbf{R}^n$ mentioned in the problem at the end of **[4]**. Also let $*$, $\equiv$, $f_{[\alpha,\beta]}$ have the same meaning as in **[4]**. Let $\mathcal{L}_1$ denote the set of lines in $\mathcal{D}_1$. Let $\mathcal{L}_1^* = \{f * \mid f \in \mathcal{L}_1\}$ and let $\mathcal{D}_1^* = \{f * \mid f \in \mathcal{D}_1\}$. So $\mathcal{D}_1^*$ is a semigroup. We start off with an analogue of Theorem 2.9.

THEOREM 5.1. *Let T be a nonempty finite set. For $i \in T$, $j \in Z^+$, choose $f_{i,j} \in \mathcal{D}_1$ such that $f_{i,j+1} |_f f_{i,j}$ for all $i \in T$, $j \in Z^+$ and $l(f_{i,j}) \to 0$ as $j \to \infty$ for any fixed $i \in T$. Let $f \in \mathcal{D}_1$. Assume that for each $\beta \in [0,1]$, $j \in Z^+$, there exist $\alpha, \gamma \in [0,1]$, $i \in T$ such that $\alpha < \gamma$, $\beta \in [\alpha, \gamma]$ and $f_{[\alpha,\gamma]} \equiv f_{i,j}$. Then some $f_{p,q} \in \mathcal{L}_1$.*

Proof. The second part of the proof of [4; Theorem 2.1] shows that there exist $\mu, \nu \in [0,1]$, $\mu < \nu$ such that $f_{[\mu,\nu]} \in \mathcal{L}_1$. Choose $\beta \in (\mu, \nu)$. For any $j \in Z^+$, there exist $\alpha, \gamma \in [0,1]$, $i \in T$ such that $\alpha < \gamma$, $\beta \in [\alpha, \gamma]$ and $f_{[\alpha,\gamma]} \equiv f_{i,j}$. We can choose j big enough (and hence $l(f_{i,j})$ small enough) so that we must have $\alpha > \mu$, $\gamma < \nu$. Then $f_{i,j} \equiv f_{[\alpha,\gamma]} \in \mathcal{L}_1$.

For $a \in \mathcal{L}_1^*$, $\alpha \in \mathbf{R}^+$, let a^α denote the line in $\mathcal{L}_1^*$ in the same direction as a but with length $\alpha l(a)$. Let $u, v \in \mathcal{D}_1^*$. Then define $u \sim v$ if either there exist $a \in \mathcal{D}_1^*$, $i, j \in Z^+$ such that $u = a^i$, $v = a^j$ or if $u, v \in \mathcal{L}_1^*$ and $v = u^\alpha$ for some $\alpha \in \mathbf{R}^+$. Because of Theorem 5.1, we can repeat §3 (including all the definitions) with $\mathfrak{D}$ replaced by $\mathcal{D}_1^*$ and $\mathcal{L}$ replaced by $\mathcal{L}_1^*$. We then obtain the following theorem which answers affirmatively a problem posed at the end of **[4]**.

THEOREM 5.2. *Every word equation is resolvable in $\mathcal{D}_1^*$.*

Using Theorem 1.9, we now obtain,

THEOREM 5.3. *Let $\{w_1, w_2\}$ be a word equation which has only* *trivial solutions in any free semigroup. Then $\{w_1, w_2\}$ has only trivial solutions in $\mathcal{D}_1^*$.*

For continuous f: $[0,1] \to \mathbf{R}^n$, let $\|f\| = \sup_{t\in[0,1]} \|f(t)\|$.

DEFINITION. For $u, v \in \mathcal{D}_1^*$, let $\eta(u,v) = \inf\{\|f - g\| \mid f, g \in \mathcal{D}_1, f \equiv u, g \equiv v\}$.

Then η can be shown to have the following properties:

(a) η is a pseudo-metric on $\mathcal{D}_1^*$.

(b) For any $u_1, u_2 \in \mathcal{D}_1^*$, $\epsilon \in \mathbf{R}^+$, there exists $\delta \in \mathbf{R}^+$ such that for all $v_1, v_2 \in \mathcal{D}_1^*$, $\eta(u_i, v_i) < \delta$, $i = 1,2$ implies $\eta(u_1u_2, v_1v_2) < \epsilon$.

(c) For any $u \in \mathcal{L}_1^*$, $\eta(u, u^\delta) \to 0$ as $\delta \to 1$.

As in §4, Theorems 1.1, 1.8 and 5.2 easily imply the following.

THEOREM 5.4. *Let $(a_1, \cdots, a_m)$ be a solution in $\mathcal{D}_1^*$ of a word equation $\{w_1, w_2\}$. Then for every $\epsilon \in \mathbf{R}^+$, there exists a strongly resolvable solution $(b_1, \cdots, b_m)$ of $\{w_1, w_2\}$ in $\mathcal{D}_1^*$ such that $\eta(a_i, b_i) < \epsilon$, $i = 1, \cdots, m$.*

Note added in the proof. Problem 1.10 has recently been solved by the author.

REFERENCES

1. Ju. I. Hmelevskiĭ, *Equations in free semigroups*, Proc. Steklov Inst. Math., **107** (1971), 270.
2. A. Lentin, *Équations dans les monoides libres*, Mouton/Gauthier-Villars, 1972.
3. E. S. Ljapin, *Semigroups*, Amer. Math. Soc. Transl., (1963). Translation of Polugruppy (1960).
4. M. S. Putcha, *Word equations of paths*, J. Algebra, (to appear).
5. H. L. Royden, *Real Analysis*, Second Edition, MacMillan, New York, 1968.

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