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# DUFFIN'S FUNCTION AND HADAMARD'S CONJECTURE

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**The purpose of the present paper is to apply our "beta densities" to Hadamard's conjecture on the constant sign of the biharmonic Green's function of a clamped plate. In particular, we will examine in detail Duffin's function $w$ from our view point of beta densities. We will show that $w$ is a potential of $\Delta^2 w \geqq 0$ with respect to the Green's kernel of a clamped plate. As a consequence, the Green's function of the clamped infinite strip is of nonconstant sign along with $w$. On the other hand, we show using beta densities that the Green's function of any clamped bounded subregion exhausting the strip tends to that of the clamped strip and, therefore, takes on both positive and negative values. Since the infinite strip can be exhausted by ellipses, we have at once, without carrying out any numerical computations, the Garabedian result: a sufficiently eccentric ellipse is a counterexample to Hadamard's conjecture. Since the strip can also be exhausted by rectangles, we can add a sufficiently long *rectangle* to counterexamples to Hadamard's conjecture. If this may be called a new example, then countless "new" examples can be produced by exhausting the strip by "new" subregions.**

Hadamard made the following conjecture in his 1908 prize memoir [3]: the deflection of a thin, flat, elastic plane plate, horizontally clamped at its boundary, is of the same sign at all points of the plate if a perpendicular force is applied at some point of the plate. The conjecture is known to be correct if the plate is a disk. In the general case, the problem remained open until Duffin [1] showed in 1949 that a solution of a biharmonic Poisson equation with a nonnegative density on an infinite strip clamped at the edges takes on both positive and negative values. This work of Duffin contains rich physical intuition and skillful though elementary calculation which produces surprisingly interesting results and suggestions for further development. Obviously motivated by this work, Loewner [5] and subsequently Szegö [9] constructed, by means of conformal mapping techniques, finite but nonconvex analytic Jordan regions as further counterexamples to Hadamard's conjecture. The simplest counterexample, a sufficiently eccentric ellipse, was given by Garabedian [2], who used an eigenfunction expansion approach.

We give here a rough description of the contents of the present paper. First we give an outline of the definition and properties of beta densities on simply connected plane regions. We then consider, in particular, the case of an infinite strip $S$ and discuss the space $H_2(S)$ of square integrable harmonic functions on it. For this space, the ideal boundary of $S$ is negligible. We show that, as a consequence, Duffin's function is a biharmonic Green's potential. Using this result we discuss in the final part of our study Hadamard's conjecture.

Last but not least, an acknowledgement is in order in this introduction. The authors consider it quite helpful for the completion of the present work that their younger colleagues, especially Professors H. Imai and S. Segawa at Daido Institute of Technology, always showed their keen interest in the authors' seminar lectures on this subject and made valued comments.

**Beta densities.**

1. Since we will make essential use of beta densities [7], we start by discussing those fundamentals of their theory that are pertinent in our present setting. We denote by $C$ the finite complex plane $|z|<\infty$, $z = x+iy$, and by $M$ a simply connected subregion, to be called a *plate*, of $C$. For convenience, we say that a plate $M$ is *smooth* (or *piecewise smooth*) if $M$ is relatively compact and the relative boundary $\partial M$ is a smooth (i.e., $C^\infty$) (or piecewise smooth) Jordan curve. Assume that $M$ is a *smooth plate* and set $\bar{M} = M \cup \partial M$. It is well known that there exists a unique function $\beta_M(z,\zeta)$ on $M \times M$ such that

$$(1)\qquad \begin{cases} \Delta_z^2\beta_M(z,\zeta) = \Delta_z(\Delta_z\beta_M(z,\zeta)) = \delta_\zeta & (z \in M) \\ \beta_M(z,\zeta) = \dfrac{\partial}{\partial n_z}\beta_M(z,\zeta) = 0 & (z \in \partial M), \end{cases}$$

where $\Delta_z = -(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the Laplace–Beltrami operator, $\delta_\zeta$ the Dirac delta at $\zeta \in M$, and $\partial/\partial n$ the *inner* normal derivative at $\partial M$ with respect to $M$. The function $\beta_M(\cdot,\zeta)$, which is of class $C^\infty$ on $\bar{M}-\zeta$ (e.g., Hörmander [4]), is referred to as the (biharmonic) *Green's function* of the clamped plate $M$ with pole $\zeta$.

2. On a smooth plate $M$, we call $H_M(\cdot,\zeta) \equiv \Delta\beta_M(\cdot,\zeta)$ the *beta density* with pole $\zeta$. Let $g_M(\cdot,\zeta)$ be the harmonic Green's function on $M$ with the singularity $-(1/2\pi)\log|z-\zeta|$ at $\zeta$. By (1), $\Delta H_M(\cdot,\zeta) = \Delta^2\beta_M(\cdot,\zeta) = \delta_\zeta$ and a fortiori $H_M(\cdot,\zeta) - g_M(\cdot,\zeta)$ belongs to the class $H(M)$ of harmonic functions on $M$. By the first boundary condition (1),

$$(2)\qquad \beta_M(z,\zeta) = \int_M g_M(s,z)H_M(s,\zeta)\,dp\,dq \quad (s = p+iq).$$

If $\beta_M(\cdot,\zeta)$ is viewed as a potential with respect to the harmonic Green's function, then $H_M(\cdot,\zeta)$ is the density of $\beta_M(\cdot,\zeta)$. Since $H_M(\cdot,\zeta)$ is of class $C^2$ on $\bar{M}-\zeta$, we have (e.g., Miranda **[6]**)

$$\frac{\partial}{\partial n_z}\beta_M(z,\zeta)=\int_M \frac{\partial}{\partial n_z} g_M(s,z)H_M(s,\zeta)dpdq.$$

Multiply both sides by an $h\in H(M)\cap C(\bar{M})$ and integrate with respect to the line element $|dz|$ on $\partial M$. By the Fubini theorem and the Poisson type representation of harmonic functions,

$$\int_{\partial M} h(z)\frac{\partial}{\partial n_z}\beta_M(z,\zeta)|dz|=\int_M h(s)H_M(s,\zeta)dpdq.$$

Therefore, the second condition (1) is equivalent to

$$\int_M h(s)H_M(s,\zeta)dpdq=0. \tag{3}$$

This relation is true for every $h\in H(M)\cap C(\bar{M})$ if and only if it is true for every $h\in H_2(M)\equiv H(M)\cap L_2(M)$, since $H(M)\cap C(\bar{M})$ is dense in $H_2(M)$ with respect to the $L_2$ norm $\|\cdot\|$ on $M$. In terms of the inner product $(\cdot,\cdot)$ on $L_2(M)$, we write (3) simply as $H_M(\cdot,\zeta)\perp H_2(M)$. Since $g_M(\cdot,\zeta)-H_M(\cdot,\zeta)$ belongs to $H_2(M)$, (2) and (3) imply that

$$\beta_M(z,\zeta)=(H_M(\cdot,z),H_M(\cdot,\zeta))=\int_M H_M(s,z)H_M(s,\zeta)dpdq. \tag{4}$$

3. We claim that the beta density $H_M(\cdot,\zeta)$ is characterized by the following properties:

$$\begin{cases} \Delta H_M(\cdot,\zeta)=\delta_\zeta \\ H_M(\cdot,\zeta)\in L_2(M) \\ H_M(\cdot,\zeta)\perp H_2(M). \end{cases} \tag{5}$$

That $H_M(\cdot,\zeta)$ satisfies the first and third of these relations was explicitly shown in No. 2. On setting $z=\zeta$ in (4) and observing that $\beta_M(\zeta,\zeta)=\lim_{z\to\zeta}\beta(z,\zeta)<\infty$, we conclude that the second relation (5) is satisfied. Conversely, suppose a function $\bar{H}$ on $M$ satisfies (5). Then, since $h=\bar{H}-H_M(\cdot,\zeta)\in H_2(M)$, we have $(h,\bar{H})=0$ and $(h,H_M(\cdot,\zeta))=0$ and a fortiori $(h,\bar{H}-H_M(\cdot,\zeta))=\|h\|^2=0$. Hence $h\equiv 0$, and $\bar{H}$ is the beta density on $M$.

4. The importance of (5) lies in the fact that it contains no reference to the boundary $\partial M$ of the plate $M$. Therefore, we can define the *beta density* $H_M(\cdot, \zeta)$, if it exists, even for a *general* plate $M$ by (5). Reversing the usual process, we subsequently define the (biharmonic) *Green's function* $\beta_M(z, \zeta)$, or the *Green's kernel*, of a general clamped plate by (4),

$$\beta_M(z,\zeta) = \int_M H_M(s, z) H_M(s, \zeta) dpdq \tag{6}$$

on $M \times M$. At this point the biharmonic classification theory must come in: We classify plates into two categories, according as the beta density does or does not exist, in analogy with Riemann's classification of plates into hyperbolic and parabolic types. It would not be difficult to carry out this classification; however, what we really need is not the mere existence but detailed information on properties of (6). To this end, we consider what we call a *fundamental kernel* $K(z, \zeta)$ on $M$ characterized by

$$\begin{cases} K(\cdot, \zeta), K(\zeta, \cdot) \in H(M - \zeta) \\ K(z, \zeta) + \dfrac{1}{2\pi} \log|z - \zeta| \in H(M) \\ K(\cdot, \zeta) \in L_2(M) \\ \lim\limits_{\zeta \to \zeta_0} \| K(\cdot, \zeta) - K(\cdot, \zeta_0) \| = 0. \end{cases} \tag{7}$$

5. Suppose there exists a fundamental kernel $K(z, \zeta)$ on $M$. We claim that there then exists a beta density $H_M(\cdot, \zeta)$ for every $\zeta \in M$ and a Green's kernel $\beta_M(z, \zeta)$ of the clamped plate $M$ with the following properties: $\Delta^2 \beta_M(\cdot, \zeta) = \delta_\zeta$; $\beta_M \in C(M \times M)$ (*joint continuity*); $\lim_\iota \sup_{F \times F} |\beta_{M_\iota} - \beta_M| = 0$, where $\{M_\iota\}$ is any directed set of plates $M_\iota \subset M$ exhausting $M$ and $F$ is any compact subset of $M$ (*consistency relation*).

For a proof we recall that $H_2(M)$ is a locally bounded Hilbert space and consider the functional $k_\zeta(u) = (u, K(\cdot, \zeta))$ on $H_2(M)$ for any fixed $\zeta \in M$. It is easily seen that $k_\zeta$ is bounded and thus $k_\zeta \in H_2(M)$. It is also readily verified that $\lim_{\zeta \to \zeta_0} \| k_\zeta - k_{\zeta_0} \| = 0$. As a consequence, $H_M(\cdot, \zeta) = K(\cdot, \zeta) - k_\zeta$ is the beta density on $M$ with pole $\zeta \in M$. By means of the properties of $K(\cdot, \zeta)$ and $k_\zeta$ it is not difficult to ascertain that $\beta_M(z, \zeta) \equiv (H_M(\cdot, z), H_M(\cdot, \zeta))$ is continuous on $M \times M$. From $(H_M(\cdot, \zeta) - H_{M_\iota}(\cdot, \zeta), H_{M_\iota}(\cdot, z)) = 0$ we obtain on setting $H_{M_\iota}(\cdot, \zeta) = 0$ on $M - M_\iota$

$$\begin{cases} \| H_M(\cdot, \zeta) - H_{M_\iota}(\cdot, \zeta) \|^2 = \| H_M(\cdot, \zeta) \|^2 - \| H_{M_\iota}(\cdot, \zeta) \|^2 \\ |\beta_M(z, \zeta) - \beta_{M_\iota}(z, \zeta)| \leqq \| H_M(\cdot, z) - H_{M_\iota}(\cdot, z) \| \cdot \| H_M(\cdot, \zeta) - H_{M_\iota}(\cdot, \zeta) \|. \end{cases}$$

Using these relations we deduce $\lim_\iota \|H_M(\cdot,\zeta)-H_{M_\iota}(\cdot,\zeta)\|=0$ and, in view of the continuity of $\|H_M(\cdot,\zeta)-H_{M_\iota}(\cdot,\zeta)\|^2=\beta_M(\zeta,\zeta)-\beta_{M_\iota}(\zeta,\zeta)$ on $M$, obtain the consistency relation. Taking the directed set $\{\Omega\}$ of smooth plates $\Omega$ in $M$ as $\{M_\iota\}$ and observing $\Delta^2\beta_\Omega(\cdot,\zeta)=\delta_\zeta$ on $\Omega$ we see that

$$\begin{aligned}(\beta_M(\cdot,\zeta),\Delta^2\varphi)&=\lim_{\Omega\to M}(\beta_\Omega(\cdot,\zeta),\Delta^2\varphi)\\&=\lim_{\Omega\to M}(\Delta^2\beta_\Omega(\cdot,\zeta),\varphi)\\&=\varphi(\zeta)\end{aligned}$$

for every $\varphi\in C_0^\infty(M)$, and therefore $\Delta^2\beta_M(\cdot,\zeta)=\delta_\zeta$ on $M$.

6. An important special case is a plate $M$ for which the *iteration* $g^{(2)}(z,\zeta)$ of the harmonic Green kernel $g(z,\zeta)$ on $M$ can be defined:

$$g^{(2)}(z,\zeta)=\int_M g(s,z)g(s,\zeta)dpdq. \tag{8}$$

This is the case if and only if $g(\cdot,\zeta)\in L_2(M)$ for some and hence for every $\zeta\in M$. The function $g^{(2)}$ is continuous on $M\times M$, $\Delta^2g^{(2)}(\cdot,\zeta)=\Delta g(\cdot,\zeta)=\delta_\zeta$ on $M$, and if a part $\gamma$ of $\partial M$ is an open smooth arc, then $g^{(2)}(\cdot,\zeta)\in C^2(M\cup\gamma-\zeta)$ and $g^{(2)}(\cdot,\zeta)=0$ on $\gamma$. In this case $g(z,\zeta)$ is a fundamental kernel on $M$ and the result in No. 5 applies. Since $g(\cdot,\zeta)-H_M(\cdot,\zeta)\in H_2(M)$,

$$\begin{cases}\beta_M(z,\zeta)=\int_M g(s,z)H_M(s,\zeta)dpdq\\ \beta_M(\zeta,\zeta)=\|H_M(\cdot,\zeta)\|^2\leqq\|g(\cdot,\zeta)\|^2=g^{(2)}(\zeta,\zeta).\end{cases} \tag{9}$$

In view of $|\beta_M(z,\zeta)|\leqq(g^{(2)}(z,z))^{1/2}(\beta_M(\zeta,\zeta))^{1/2}$, $\beta_M(\cdot,\zeta)$ is continuous on $M\cup\gamma$ and $\beta_M(\cdot,\zeta)=0$ on $\gamma$. We remark that in the case in which $g^{(2)}$ exists, the following *sharpened form of the consistency relation* is valid. Suppose $\{M_\iota\}$ is a directed set exhausting $M$ such that $\partial M_\iota$ contains an open smooth arc $\gamma$ on $\partial M$. Then by

$$\begin{aligned}|\beta_M(z,\zeta)-\beta_{M_\iota}(z,\zeta)|&=\left|\int_M g(s,z)(H_M(s,\zeta)-H_{M_\iota}(s,\zeta))dpdq\right|\\&\leqq(g^{(2)}(z,z))^{1/2}\|H_M(\cdot,\zeta)-H_{M_\iota}(\cdot,\zeta)\|,\end{aligned}$$

$\beta_{M_\iota}(z,\zeta)$ converges to $\beta_M(z,\zeta)$ uniformly on $F_1\times F_2$, with $F_1$ any compact subset of $M\cup\gamma$, and $F_2$ any compact subset of $M$.

**Infinite strip.**

7. Having completed the preparatory part we proceed to our main discussion. We consider, as our basic plate, the infinite strip

$$S = \{z = x + iy\,;\ -\infty < x < \infty,\ -1 < y < 1\}.$$

The relative boundary $\partial S$ consists of the lines $y = \pm 1$. We denote by $g(z,\zeta)$ the harmonic Green's kernel on $S$. Let $S_m = \{z \in S\,; |x| < m\}$ and denote by $g_m(z,\zeta)$ the harmonic Green's kernel on $S_m (m = 1,2,\cdots)$. Fix an arbitrary $\zeta \in S$, an $n = 1,2,\cdots$, and then an $m = 1,2,\cdots$ such that $\zeta \in S_m$ and $|\operatorname{Re} z^{-n}| \in H(S - \bar{S}_m)$. Let $c_0$ ($c_1$, resp.) be the supremum (infimum, resp.) of $g(\cdot,\zeta)(|\operatorname{Re} z^{-n}|$, resp.) on $S \cap \partial S_m$, and set $c = c_0/c_1$. Comparing boundary values of $g_{m+k}(\cdot,\zeta)$ and $c|\operatorname{Re} z^{-n}|$ on $\partial(S_{m+k} - \bar{S}_m)$, we have $g_{m+k}(\cdot,\zeta) \leqq c|\operatorname{Re} z^{-n}|$ on $S_{m+k} - \bar{S}_m$. On letting $k \to \infty$ we see that $g(\cdot,\zeta) \leqq c|\operatorname{Re} z^{-n}|$ on $S - \bar{S}_m$, and conclude that

$$\lim_{x \to \pm\infty} g(z,\zeta)/|\operatorname{Re} z^{-n}| = 0 \qquad (n = 1,2,\cdots), \tag{10}$$

where $x = \operatorname{Re} z$ and $z \in S$. In particular, $g(\cdot,\zeta) \in L_2(S)$, and the result in No. 6 applies to $S$. We denote simply by $H(\cdot,\zeta)$ the beta density on $S$ and by $\beta(z,\zeta)$ the Green's kernel of the clamped plate $S$.

8. We study the class $H_2(S)$ and consider two subspaces $H_2(S)_k (k = 1,2)$ as follows. First let $H_2(S)_1$ be the subspace of $H_2(S)$ consisting of the functions $u \in H_2(S)$ with $u \in C^\infty(\bar{S})$, $\bar{S} = S \cup \partial S$, and $u(\cdot, \pm 1) \in L_2(-\infty,\infty)$. We maintain that $H_2(S)_1$ is dense in $H_2(S)$ in the $L_2$ norm, i.e.,

$$\overline{H_2(S)_1} = H_2(S). \tag{11}$$

To see this, let $h$ be an arbitrary element in $H_2(S)$ and consider $h_\lambda(z) = h(z/\lambda)$ on $S$ with $\lambda \in (1,\infty)$. By the Fubini theorem, since

$$\int_{-1}^{1} \psi(y)dy = \|h\|^2 < \infty, \quad \psi(y) = \int_{-\infty}^{\infty} h(x,y)^2 dx,$$

we see that $\psi(y) < \infty$ for almost every $y \in (-1,1)$ and a fortiori $h_\lambda(\cdot, \pm 1) \in L_2(-\infty,\infty)$ for almost every $\lambda \in (1,\infty)$. Thus we can choose a decreasing sequence $\{\lambda_n\}$ converging to 1 such that $h_n(\cdot, \pm 1) \equiv h_{\lambda_n}(\cdot, \pm 1) \in L_2(-\infty,\infty)$ for $n = 1,2,\cdots$. Since $\lim_{n\to\infty}\|h_n - h\| = 0$, as can be easily seen, we conclude that $h \in \overline{H_2(S)_1}$.

9. We next prove that the ideal boundary $x = \pm\infty$ is negligible for the class $H_2(S)$ in the sense that

$$\{h \in H_2(S)_1;\, h \mid \partial S = 0\} = \{0\}. \tag{12}$$

In the notation of the classification theory (e.g., **[8]**) this fact may be expressed as $S \in SO_{H_2}$. To prove (12) we choose an arbitrary $h$ in $H_2(S)_1$ with $h \mid \partial S = 0$ and consider

$$f(x) = \int_{-1}^{1} h(x, y)^2 dy$$

on $(-\infty, \infty)$. Keeping $\Delta h = 0$ in mind, we have

$$\frac{\partial^2}{\partial x^2} h(x, y)^2 = 2\left(\frac{\partial}{\partial x} h(x, y)\right) - 2h(x, y)\frac{\partial^2}{\partial y^2} h(x, y).$$

Since $h(x, \pm 1) = 0$, integration by parts gives

$$\int_{-1}^{1} h(x, y)\frac{\partial^2}{\partial y^2} h(x, y) dy = -\int_{-1}^{1} \left(\frac{\partial}{\partial y} h(x, y)\right)^2 dy.$$

Therefore,

$$\frac{d^2}{dx^2} f(x) = 2\int_{-1}^{1} |\nabla h(x, y)|^2 dy \geqq 0,$$

so that $f(x)$ is a nonnegative convex function on $(-\infty, \infty)$. On the other hand, the relation

$$\int_{-\infty}^{\infty} f(x) dx = \|h\|^2 < \infty$$

implies the existence of an increasing (decreasing, resp.) sequence $\{r_n^+\}$ ($\{r_n^-\}$, resp.) converging to $+\infty$ ($-\infty$, resp.) such that $\lim_{n \to \infty} f\{r_n^\pm\} = 0$. By the convexity of $f$,

$$0 \leqq \sup_{r_n^- \leqq x \leqq r_n^+} f(x) = \max(f(r_n^+), f(r_n^-))$$

for every $n$ and hence $f(x) \equiv 0$ on $(-\infty, \infty)$. Therefore, $\|h\| = 0$ and $h \equiv 0$ on $S$.

10. We now prove a simple lemma which will play a decisive role in our discussion. To state the lemma, it will be convenient to use the notation

$$\begin{cases} [h] = \limsup_{|x| \to \infty} [h](x) \\ [h](x) = \sup_{|y|<1} |h(x, y)| + \sup_{|y|<1} \left| \frac{\partial}{\partial x} h(x, y) \right| \end{cases}$$

for each $h \in H_2(S)$. We designate by $H_2(S)_2$ the subclass of $H_2(S)_1$ consisting of those $h \in H_2(S)_1$ for which $[h] < \infty$. In view of (12), it would seem reasonable to expect that $[h] < \infty$ for all $h \in H_2(S)$ or at least for the majority of $h$ in $H_2(S)$. This expectation is justified in the following form:

FUNDAMENTAL LEMMA. *The subspace $H_2(S)_2$ is dense in $H_2(S)_1$ and a fortiori in $H_2(S)$, i.e.,*

$$\overline{H_2(S)_2} = \overline{H_2(S)_1} = H_2(S). \tag{13}$$

The proof will be given in Nos. 11–12.

11. For any given $h \in H_2(S)$ we have to find a sequence $\{h_n\}$ in $H_2(S)_2$ converging to $h$ in the $L_2$ norm. By (11) we may assume that $h \in H_2(S)_1$. We choose two sequences $\{\varphi_j\}$ and $\{\psi_j\}$ $(j = 1, 2, \cdots)$ in $C^\infty(-\infty, \infty)$ such that $\varphi_j(x) = h(x, 1)$ and $\psi_j(x) = h(x, -1)$ on $|x| \leqq j$; $\varphi_j(x) = \psi_j(x) = 0$ on $|x| \geqq j + 1$; and

$$\lim_{j \to \infty} \left( \int_{-\infty}^{\infty} (\varphi_j(x) - h(x, 1))^2 dx + \int_{-\infty}^{\infty} (\psi_j(x) - h(x, -1))^2 dx \right) = 0. \tag{14}$$

We denote by $\hat{\varphi}_j = \mathcal{F}\varphi_j$ and $\hat{\psi}_j = \mathcal{F}\psi_j$ the Fourier transforms of $\varphi_j$ and $\psi_j$,

$$\hat{\varphi}_j(p) = (\mathcal{F}\varphi_j)(p) = \int_{-\infty}^{\infty} e^{-ipx} \varphi_j(x) dx,$$

with $p \in (-\infty, \infty)$. Since $\varphi_j$ and $\psi_j$ are in the subspace $C_0(-\infty, \infty)$ of the space $\mathcal{S}(-\infty, \infty)$ of rapidly decreasing functions on $(-\infty, \infty)$, $\hat{\varphi}_j$ and $\hat{\psi}_j$ are again in $\mathcal{S}(-\infty, \infty)$.

Consider the function

$$u_j(p, y) = \frac{\hat{\varphi}_j(p)e^p - \hat{\psi}_j(p)e^{-p}}{e^{2p} - e^{-2p}} \cdot e^{py} + \frac{\hat{\psi}_j(p)e^p - \hat{\varphi}_j(p)e^{-p}}{e^{2p} - e^{-2p}} \cdot e^{-py}$$
$$= \frac{e^p e^{py} - e^{-p} e^{-py}}{e^{2p} - e^{-2p}} \cdot \hat{\varphi}_j(p) + \frac{e^p e^{-py} - e^{-p} e^{py}}{e^{2p} - e^{-2p}} \cdot \hat{\psi}_j(p).$$

It is easy to see that $u_j \in C^\infty(\bar{S})$, $u_j(\cdot, y) \in \mathcal{S}(-\infty, \infty)$, and

$$(15)\qquad \begin{cases} |u_j(p,y)| \leqq c(|\hat{\varphi}_j(p)| + |\hat{\psi}_j(p)|) \ ((p,y) \in S) \\ \lim_{y\to 1} u_j(p,y) = \hat{\varphi}_j(p), \lim_{y\to -1} u_j(p,y) = \hat{\psi}_j(p) \quad (p \in (-\infty,\infty)), \end{cases}$$

where $c$ is a universal constant. Take the inverse Fourier transform

$$h_j(x,y) = (\bar{\mathscr{F}}u_j(\cdot,y))(x) = \int_{-\infty}^{\infty} e^{ixp}u_j(p,y)dp$$

of $u_j(\cdot,y)$. By the definition of $u_j$ and (15), we have $h_j \in H_2(S)_1$ with boundary values

$$\begin{cases} h_j(x,1) = (\bar{\mathscr{F}}\hat{\varphi}_j)(x) = (\bar{\mathscr{F}}\mathscr{F}\varphi_j)(x) = \varphi_j(x) \\ h_j(x,-1) = (\bar{\mathscr{F}}\hat{\psi}_j)(x) = (\bar{\mathscr{F}}\mathscr{F}\psi_j)(x) = \psi_j(x) \end{cases}$$

on $(-\infty,\infty)$. From the Plancherel theorem, the definition of $u_j$, and (15), we obtain

$$\int_{-\infty}^{\infty} |h_j(x,y) - h_{j+k}(x,y)|^2 dx = \int_{-\infty}^{\infty} |u_j(p,y) - u_{j+k}(p,y)|^2 dp \equiv a_{j,k}(y)$$

and

$$a_{j,k}(y)^{\frac{1}{2}} \leqq c\Big(\int_{-\infty}^{\infty} |\varphi_j(p) - \varphi_{j+k}(p)|^2 dp\Big)^{\frac{1}{2}} + c\Big(\int_{-\infty}^{\infty} |\psi_j(p) - \psi_{j+k}(p)|^2 dp\Big)^{\frac{1}{2}} \equiv b_{j,k}.$$

Therefore, $\|h_j - h_{j+k}\|^2 = \int_{-1}^{1} a_{j,k}(y)dy \leqq 2b_{j,k}^2$, and by (14),

$$\lim_{j\to\infty} \|h_j - h_{j+k}\| = 0.$$

In view of the completeness of $H_2(S)$, there exists an $h_\infty \in H_2(S)$ such that $\{h_j\}$ converges to $h_\infty$ in $L_2$ norm. By the local boundedness of $H_2(S)$ and the fact that $h_j(x,\pm 1) - h_{j+k}(x,\pm 1) = 0$ on $|x| \leqq j$, the convergence of $\{h_j\}$ to $h_\infty$ is also pointwise and uniform on each compact subset of $\bar{S}$. In particular, $h_\infty(x,\pm 1) = h(x,\pm 1)$ on $(-\infty,\infty)$ and $h_\infty \in H_2(S) \cap C^\infty(\bar{S})$. The function $v = h - h_\infty \in H_2(S)$ has vanishing boundary values on $\partial S$ and $v \in H_2(S)_1$. By (12), we have $v \equiv 0$ on $S$ and

$$\lim_{j\to\infty} \|h_j - h\| = 0 \quad (h_j \in H_2(S)_1).$$

12. It remains to show that $\{h_j\} \subset H_2(S)_2$, i.e., $[h_j] < \infty$ for every $j = 1,2,\cdots$. By (15) and the fact that $\hat{\varphi}_j$ and $\hat{\psi}_j$ belong to $\mathscr{S}(-\infty,\infty)$,

$$|h_j(x,y)| \leqq \int_{-\infty}^{\infty} |u_j(p,y)|dp \leqq c\int_{-\infty}^{\infty} (|\hat{\varphi}_j(p)| + |\hat{\psi}_j(p)|)dp \equiv c_j < \infty.$$

Similarly,

$$\left|\frac{\partial}{\partial x} h_j(x,y)\right| = \left|\int_{-\infty}^{\infty} e^{ixp} ipu_j(p,y)dp\right|$$
$$\leqq \int_{-\infty}^{\infty} |pu_j(p,y)|\,dp$$
$$\leqq c\int_{-\infty}^{\infty} (|p\hat{\varphi}_j(p)| + |p\hat{\psi}_j(p)|)dp \equiv c'_j < \infty,$$

since $p\hat{\varphi}_j(p)$ and $p\hat{\psi}_j(p)$ belong to $\mathscr{S}(-\infty,\infty)$ along with $\hat{\varphi}_j$ and $\hat{\psi}_j$. We conclude that $[h_j] \leqq c_j + c'_j < \infty$. The proof of the Fundamental Lemma is complete.

**Duffin's function.**

13. Consider the function

$$D(s,y) = \frac{1}{s^4} + \frac{sy \sinh s \sinh sy - (\sinh s + s\cosh s)\cosh sy}{s^4(s + \cosh s \sinh s)} \tag{16}$$

with $(s,y) \in \mathbf{C} \times [-1,1]$. Observe that $s = 0$ is a removable singularity and $D(p,y)$ is a real-valued $C^\infty$ function of $(p,y) \in S$. Take an arbitrary *nonnegative* function $\rho(x)$ belonging to the class $C_0^\infty(-\infty,\infty)$ and denote by $\hat{\rho}(p)$ the Fourier transform of $\rho(x)$. Since $\rho$ has compact support, $\hat{\rho}$ can be continued analytically to $\mathbf{C}$. In view of $\hat{\rho} \in \mathscr{S}(-\infty,\infty)$, the function

$$w(x,y) = w_\rho(x,y) = \int_{-\infty}^{\infty} e^{ixp} D(p,y)\hat{\rho}(p)dp, \tag{17}$$

to be referred to as *Duffin's function* with density $\rho(x)$, is well defined on $S$. We extend $\rho$ to $S$ by $\rho(z) \equiv \rho(x)$, and readily obtain the following properties of $w$:

$$\begin{cases} w \in C^\infty(S) & \\ \Delta_z^2 w(z) = \rho(z) & (z \in S) \\ w(z) = \dfrac{\partial}{\partial n} w(z) = 0 & (z \in \partial S) \\ [w] = 0. & \end{cases} \tag{18}$$

Less obvious is the following result: If $\rho \not\equiv 0$, then

$$\inf_{z \in S} w(z) < 0. \tag{19}$$

Definitions (16) and (17) as well as properties (18) and (19) are due to Duffin [1].

For the convenience of the reader we sketch Duffin's proof of (19). In the $(p, q)$-plane, consider the strip $T: |p| < \infty, 0 < q < c \equiv 3\pi/4$. The function $e^{ixs}D(s, y)\hat{\rho}(s)$, as a function of the complex variable $s = p + iq$, is holomorphic on $\bar{T}$ except for two simple poles $\alpha = a + ib$ $(a, b > 0)$ and $-\bar{\alpha} = -a + ib$ on $T$ which are nonzero roots of $s + \cosh s \sinh s = 0$ on $T$. We denote by $T_n$ the finite strip $|p| < n, 0 < q < c$, for $n = 1, 2, \cdots$. By the residue theorem,

$$\int_{\partial T_n} e^{ixs} D(s, y)\hat{\rho}(s) ds = R,$$

where $n > a$ and $R$ is the $2\pi i$-fold sum of the residues of $e^{ixs}D(s, y)\hat{\rho}(s)$ at $\alpha$ and $-\bar{\alpha}$. Since $\hat{\rho} \in \mathcal{S}(-\infty, \infty)$ and $D(s, y)$ is bounded on $T - T_n$,

$$\lim_{n \to \infty} \int_{T \cap \partial T_n} e^{ixs} D(s, y)\hat{\rho}(s) ds = 0.$$

Therefore,

$$w(z) = R + \int_{\operatorname{Im} s = c} e^{ixs} D(s, y)\hat{\rho}(s) ds.$$

Here the last term is dominated by $e^{-cx}\int_{-\infty}^{\infty} |D(p + ic, y)\hat{\rho}(p + ic)| dp$, with the integral bounded for $|y| < 1$. Computing $R$ explicitly we obtain

$$w(z) = A(y)e^{-bx} \cos(ax + B(y)) + O(e^{-cx}),$$

where $A(y)$ and $B(y)$ are functions of $y$ only, and $A(y) \neq 0$ for some $|y| < 1$. In view of $0 < b < c$, we conclude on letting $x \to \infty$ that (19) is valid.

14. In addition to (18) and (19), Duffin's function has the following properties, important from our point of view:

$$\begin{cases} \Delta w \in L_2(S) \\ \Delta w \perp H_2(S). \end{cases} \tag{20}$$

For the proof, observe that $\hat{\rho} \in \mathcal{S}(-\infty, \infty)$ implies the existence of a $\tau \in \mathcal{S}(-\infty, \infty)$ such that $|(p^2 D(p, y) - \partial^2 D(p, y)/\partial y^2)\hat{\rho}(p)| \leqq \tau(p)$ on $(-\infty, \infty)$. By the Plancherel theorem,

$$\int_{-\infty}^{\infty} |\Delta w(x,y)|^2 dx = \int_{-\infty}^{\infty} |(p^2 D(p,y) - \frac{\partial^2}{\partial y^2} D(p,y))\hat{\rho}(p)|^2 dp$$
$$\leqq \int_{-\infty}^{\infty} \tau(p)^2 dp \equiv k < \infty.$$

Therefore, $\|\Delta w\|^2 = \int_{-1}^{1}\int_{-\infty}^{\infty} |\Delta w(x,y)|^2 dxdy \leqq k\int_{-1}^{1} dy < \infty$, i.e., the first relation (20) is valid.

To prove the second relation (20), we have to show that $(h, \Delta w) = 0$ for every $h \in H_2(S)$. By (13), it suffices to establish this for every $h \in H_2(S)_2$. Let $S_n = \{z = x + iy; |x| < n, |y| < 1\}$ $(n = 1, 2, \cdots)$. Since $h$ and $w$ are in the class $C^\infty(\bar{S})$, the Green's formula can be applied to $h$ and $w$ on $\bar{S}_n$:

$$\int_{S_n} (h(z)\Delta w(z) - w(z)\Delta h(z))dxdy$$
$$= -\int_{\partial S_n} (h(z)\frac{\partial}{\partial n} w(z) - w(z)\frac{\partial}{\partial n} h(z))|dz|.$$

By (18), we have in the notation in No. 10,

$$|(h, \Delta w)_{S_n}| = \left|\int_{S_n \cap \partial S_n} \left(h(z)\frac{\partial}{\partial x} w(z) - w(z)\frac{\partial}{\partial x} h(z)\right)dy\right|$$
$$\leqq 2\max([h](n)\cdot[w](n), [h](-n)\cdot[w](-n)).$$

Since $h$ and $\Delta w$ belong to $L_2(S)$, $|(h, \Delta w)| = \lim_{n\to\infty}|(h, \Delta w)_{S_n}|$ and therefore,

$$|(h, \Delta w)| \leqq 4[h]\cdot[w].$$

From this and (18), we conclude that $(h, \Delta w) = 0$.

15. We recall the notation $H(z,\zeta)$ and $\beta(z,\zeta)$ for the beta density and the biharmonic Green's kernel of the clamped plate $S$ in No. 7. Let $\rho$ be as in No. 13 and denote by $S_\rho$ the support of $\rho$ in $S$. By (9), $|\beta(z,\zeta)|$ is dominated by $\beta(z,z)^{1/2} g^{(2)}(\zeta,\zeta)^{1/2} \leqq k\beta(z,z)^{1/2}$ on $S \times S_\rho$, with $k = \sup_{S_\rho} g^{(2)}(\zeta,\zeta)^{1/2} < \infty$. Therefore, the biharmonic Green's potential

(21) $$\beta(z;\rho) = \int_S \beta(z,\zeta)\rho(\zeta)d\xi d\eta, \quad \zeta = \xi + i\eta,$$

is well defined on $S$ and $|\beta(z;\rho)| \leqq k\cdot\beta(z,z)^{1/2}\cdot\sup_{S_\rho}\rho\cdot\text{meas}(S_\rho)$. We claim:

$$(22)\qquad \begin{cases} \Delta^2\beta(z;\rho)=\rho(z) & (z\in S)\\ \Delta\beta(\cdot\,;\rho)\in L_2(S)\\ \Delta\beta(\cdot\,;\rho)\perp H_2(S). \end{cases}$$

For the proof, consider the auxiliary function

$$(23)\qquad v(z)=v_\rho(z)=\int_S H(z,\zeta)\rho(\zeta)d\xi d\eta.$$

By (9) and the Fubini theorem,

$$\int_S\left(\int_S |H(z,\zeta)|\rho(\zeta)d\xi d\eta\right)^2 dxdy \leqq \|\rho\|^2\int_{S_\rho}\left(\int_S H(z,\zeta)^2dxdy\right)d\xi d\eta$$
$$=\|\rho\|^2\int_{S_\rho}\beta(\zeta,\zeta)d\xi d\eta$$
$$\leqq k_-^2\|\rho\|^2\operatorname{meas}(S_\rho)<\infty.$$

Similarly, $(\Delta\varphi, v)=((\Delta\varphi, H(\cdot\,,\zeta)),\rho)_\zeta=(\varphi,\rho)$ for any $\varphi\in C_0^\infty(S)$, i.e., $\Delta v=\rho$ in the sense of distributions, and by $\rho\in C_0^\infty(\bar S)$ and $v\in L_2(S)$, in the genuine sense on $S$:

$$(24)\qquad \begin{cases} v\in L_2(S)\cap C^\infty(S)\\ \Delta v(z)=\rho(z) \quad (z\in S). \end{cases}$$

By (21), the relation $\beta(z,\zeta)=(g(\cdot\,,z),H(\cdot\,,\zeta))$, and the Fubini theorem, we have

$$(25)\qquad \beta(z;\rho)=\int_S g(s,z)v(s)dpdq$$

on $S$. Hence $\Delta\beta(z;\rho)=v(z)$ on $S$, and (24) implies the first two relations (22). To prove the third, take an arbitrary $h$ in $H_2(S)$ and observe that

$$(h,\Delta\beta(\cdot\,;\rho))=(h,v)=((h,H(\cdot\,,\zeta)),\rho)_\zeta=0.$$

16. A comparison of properties (18) and (20) of Duffin's function $w=w_\rho$ with properties (22) of $\beta(\cdot\,;\rho)$ suggests that $w\equiv\beta(\cdot\,,\rho)$ on $S$. We will prove that this is indeed the case. Observe that $\Delta(\Delta w-\Delta\beta(\cdot\,;\rho))=0$, that is, $\Delta w-\Delta\beta(\cdot\,;\rho)$ belongs to $H(S)$ and, in fact, to $H_2(S)$ since both $\Delta w$ and $\Delta\beta(\cdot\,;\rho)$ belong to $L_2(S)$. On the other hand, both $\Delta w$ and $\Delta\beta(\cdot\,;\rho)$ are orthogonal to $H_2(S)$ and a fortiori $\Delta w-\Delta\beta(\cdot\,;\rho)$ is orthogonal to $H_2(S)$ and at the same time belongs to $H_2(S)$. Therefore,

$$\Delta w(z) \equiv \Delta\beta(z\,;\rho) \qquad (z \in S). \tag{26}$$

Denote by $g_n(z, \zeta)$ the harmonic Green's kernel on $S_n = \{z\,; |x| < n, |y| < 1\}$ $(n = 1, 2, \cdots)$. Let $h_n \in H(S_n) \cap C(\bar{S}_n)$ such that $h_n | \bar{S}_n \cap \partial S = 0$ and $h_n | S \cap \partial S_n = w$. Note that $|h_n| \leqq \max([w](n), [w](-n))$ on $\partial S_n$ and, therefore, on $S_n$. By (18),

$$\lim_{n\to\infty} \sup_{z\in S_n} |h_n(z)| = 0. \tag{27}$$

Since $w(z) - (g_n(\cdot, z), \Delta w)_{S_n}$ is harmonic on $S_n$ with boundary values $w = h_n$ on $\partial S_n$, we have $w(z) - (g_n(\cdot, z), \Delta w)_{S_n} = h_n$ on $S_n$. In view of (27), we conclude on letting $n \to \infty$ that

$$w(z) = \int_S g(\zeta, z)\Delta w(\zeta)d\xi d\eta$$

on $S$. Using (23), (25), (26), and the Fubini theorem, we obtain

$$\begin{aligned} w(z) &= (g(\cdot, z), \Delta\beta(\cdot\,;\rho)) = (g(\cdot, z), (H(\zeta, \cdot), \rho))_\zeta \\ &= ((g(\cdot, z), H(\cdot, s)), \rho)_s = (\beta(z, \cdot), \rho) = \beta(z\,;\rho). \end{aligned}$$

We have established the following

MAIN THEOREM. *Duffin's function $w$ with the density $\rho$ is a biharmonic Green's potential of the density $\rho$:*

$$w(z) = \int_S \beta(z, \zeta)\rho(\zeta)d\xi d\eta. \tag{28}$$

**Hadamard's conjecture.**

17. Consider a plate $M$ with a continuous and consistent Green's kernel $\beta_M(z, \zeta) = (H_M(\cdot, z), H_M(\cdot, \zeta))$ (cf. No. 5), which satisfies the clamping conditions $\beta_M(\cdot, \zeta) = \partial\beta_M(\cdot, \zeta)/\partial n = 0$ on $\partial M$ if $M$ is a smooth plate (cf. No. 2). Let $\mu$ and $\nu$ be any (signed) Radon measures on $M$ and set

$$(H_M\mu)(s) = \int_M H_M(s, z)d\mu(z).$$

The beta mutual energy $\beta_M[\mu, \nu]$ is given by

$$\beta_M[\mu, \nu] = \int_{M\times M} \beta_M(z, \zeta)d\mu(z)d\nu(\zeta) = (H_M\mu, H_M\nu). \tag{29}$$

Therefore, the biharmonic Green's kernel $\beta_M$ satisfies the *energy principle* (strict definiteness):

$$\beta_M[\mu, \mu] \geqq 0 \tag{30}$$

and the equality holds if and only if $\mu = 0$. The mere positiveness is clear from (29). Suppose $\beta_M[\mu, \mu] = 0$. Then $H_M\mu \equiv 0$ on $M$, and the distribution identity $\Delta H_M\mu = \mu$ implies that $\mu = 0$. As a special case of (30), we obtain the relation

$$\beta_M(z, z) = \beta_M[\delta_z, \delta_z] = \|H_M\delta_z\|^2 = \|H_M(\cdot, z)\|^2 > 0, \tag{31}$$

which, in fact, we have repeatedly used.

18. The biharmonic Green's kernel $\beta_M(z, \zeta)$ certainly takes on positive values on $M$: $\beta_M(z, z) > 0$. That $\beta_M(z, \zeta)$ *cannot take on any negative values* is known as Hadamard's conjecture **[3]**. By (19) and (28), the relation $\rho \geqq 0$ implies that $\beta_S(z, \zeta)$ takes on negative values on $S \times S$. Thus we have the following counterexample to Hadamard's conjecture:

EXAMPLE (Duffin). The biharmonic Green's function $\beta_S(\cdot, \zeta)$ of the clamped infinite strip $S$: $|x| < \infty, |y| < 1$ takes on both positive and negative values for a suitable choice of the pole $\zeta$ in $S$.

19. Let $\{\Omega_\iota\}$ be a directed set of subregions of $S$ such that $\bigcup_\iota \Omega_\iota = S$. By the consistency relation (cf. No. 5), $\{\beta_{\Omega_\iota}\}$ converges to $\beta_S$ uniformly on each compact subset of $S \times S$. Therefore, $\inf \beta_{\Omega_\iota} < 0$ along with $\beta_S$ if $\Omega_\iota$ is sufficiently close to $S$. We have here a good example of the importance and effectiveness of discussing potential theory on noncompact carriers even for the study of compact carriers. As an example, consider in $S$ the ellipse

$$E_n = \left\{ z = x + iy\,; \frac{x^2}{n^2} + y^2 < 1 \right\}$$

whose eccentricity tends to $\infty$ with $n$. Since $\{E_n\}$ is increasing and exhausts $S, \{\beta_{E_n}\}$ converges to $\beta_S$ uniformly on each compact subset and hence $\inf \beta_{E_n} < 0$ for all sufficiently large $n$. Thus we have a new noncomputational proof for the following

EXAMPLE. (Garabedian). The biharmonic Green's function $\beta_E(\cdot, \zeta)$ of a clamped sufficiently eccentric ellipse $E$ takes on both positive and negative values on $E$ for a suitable choice of the pole $\zeta$ in $S$.

20. Actually, we can produce as many regions as we wish as counterexamples to Hadamard's conjecture by the above method of exhausting Duffin's infinite strip $S$. We add only one more example, the incentive of which was Duffin's **[1]** suggestion made without proof, that a quadrilateral close to a rectangle be a counterexample. Let $S_n = \{z; |x| < n, |y| < 1\}$. Then $\{\beta_{S_n}\}$ converges to $\beta_S$ as $n \to \infty$ uniformly on each compact subset of $S$ (cf. No. 6). We thus obtain the following "new" counterexample:

EXAMPLE. The biharmonic Green's function $\beta_R(\cdot, \zeta)$ of a clamped sufficiently elongated rectangle $R$ takes on both positive and negative values on $R$ for a suitable choice of the pole $\zeta$ in $R$.

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