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POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS

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1. Introduction. Let Ω be a domain (open, connected, possibly unbounded) in $\mathbf{R}^n$ and, as usual, let $x = (x_1, \cdots, x_n)$ denote a point in $\mathbf{R}^n$ with norm $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Using the summation convention, let L denote the partial differential operator defined by

$$Lu = -a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u. \tag{1.1}$$

The coefficients of L are assumed to be real functions defined on Ω and $a_{ij} = a_{ji}$.

If the a_{ij} are continuously differentiable in Ω, L may be written in the form

$$Lu = -(a_{ij}(x)u_{x_j})x_i + \tilde{b}_i(x)u_{x_i} + c(x)u, \tag{1.2}$$

where $\tilde{b}_i = b_i + \Sigma_{j=1}^n \partial a_{ij}/\partial x_j$. In this form L is symmetric (formally self-adjoint) if $\tilde{b}_i(x) \equiv 0, i = 1, \cdots, n$.

In a recent paper [1] Allegretto established the following result. Here $[r]$ denotes the integer part of the number r.

THEOREM 1.1. *Suppose that*

(A) *L is symmetric and the a_{ij} are in class $C^{m+1}(\bar{\Omega})$ while c is in $C^m(\bar{\Omega})$, where*

$$m = 3\left[[3 + n/2]/2\right],$$

(B) *L is elliptic in $\bar{\Omega}$, that is, $(a_{ij}(x))$ is positive definite for each $x \in \bar{\Omega}$,*

(C) *there is a number $R > 0$ such that $\Omega \cap \{|x| > R\}$ is connected, and*

(D) *for every bounded domain D with $\bar{D} \subseteq \Omega \cap \{|x| > R\}$*

$$\inf_{\substack{\phi \in C_0^\infty(D) \\ \phi \neq 0}} \frac{(L\phi, \phi)}{(\phi, \phi)} > 0, \tag{1.3}$$

where $(\phi, \psi) = \int_{\mathbf{R}^n} \phi\psi dx$.

Then there exists a positive solution of $Lu = 0$ in $\Omega \cap \{|x| > R\}$.

If L is viewed as an operator on $L_2(D)$ with domain $C_0^\infty(D)$, then hypothesis (D) states that the smallest generalized eigenvalue of L is greater than zero, or equivalently, that the smallest eigenvalue of the Friedrichs extension of L is greater than zero. If the left hand side of (1.3) equals zero, D is called a nodal domain for L by I. Glazman, W. Allegretto and others.

Theorem 1.1 is interesting for two reasons. First, as Allegretto points out, it represents an extension of a property that is clearly valid for the corresponding ordinary differential operator, and at the same time clarifies the oscillation theory of symmetric, second order, elliptic operators. Second, it answers a question posed by one of the authors in **[11]**. There it was shown that a result such as Theorem 1.1 would imply that the finiteness of the number of negative eigenvalues of a self-adjoint realization of L in $L^2(\Omega)$, here $\Omega = \mathbf{R}^n$, is invariant under perturbation of the coefficients by smooth functions with compact support. Diagrammatically we have the implications finite negative spectrum $\Rightarrow$ hypothesis (D) holds for R sufficiently large, and
$Lu = 0$ has a positive solution in $\{|x| > R\}$ $\Rightarrow$ finite negative spectrum.
The first is found in Glazman's book **[5]**, while the second is proved in **[11]**. Theorem 1.1 supplies the link that makes for a closed chain of implications. It should be mentioned that the second implication was employed in **[12]** to unify and extend some older criteria on the potential $c(x)$ that ensured finite negative spectrum for $L = -\Delta + c(x)$.

The following result extends Theorem 1.1 to the general nonsymmetric case and at the same time weakens the smoothness required on the coefficients of L.

THEOREM 1.2. *Let L be defined by* (1.1) *and let the coefficients of L be defined in a domain* $G \subseteq \mathbf{R}^n$. *Assume that*

(a) *L is elliptic in G,*

(b) *the coefficients of L are locally Hölder continuous in G,*

(c) *for every bounded domain D with $\bar{D} \subseteq G$ the only solution in class $C^2(D) \cap C^0(\bar{D})$ to $Lu = 0$ in D, $u = 0$ on $\dot{D}$ (the boundary of D) is $u \equiv 0$.*

Then there is a positive solution v of the equation $Lv = 0$ defined on G.

Allegretto's proof of Theorem 1.1 leaned heavily on the theory of symmetric quadratic forms in Hilbert space. As such it required that L be symmetric. Our proof of Theorem 1.2 seems simpler and more direct. The method is a fairly straight forward application of Serrin's version of the Harnack inequality for positive solutions of second order, elliptic equations (see **[13]**). We also note that the smoothness required in Theorem 1.2 is mild and consonant with modern existence theories for elliptic boundary value problems. Thus it represents a distinct sharpening of Theorem 1.1 even in the symmetric case.

In Theorem 1.1 ∞ is the only possible point at which L may degenerate or the coefficients of L become unbounded. In Theorem 1.2 any boundary point may have this property. There is no significant difference here. Assuming L is symmetric and that (a) and (b) hold, (c) is equivalent to (D) with $\Omega \cap \{|x| > R\}$ replaced by G. This will be discussed further in §3. Hypothesis (D) is more easily used in the Hilbert space context, while (c) is best adapted for our proof of Theorem 1.2.

2. Preliminary lemmas. The proof of Theorem 1.2 uses the maximum principle, the Harnack inequality, and the Schauder existence theory. The first two of these are stated in the following lemmas. For the Schauder existence theory the reader is referred to Miranda **[10]**. In Lemmas 2.1 and 2.2 L is given by (1.1) and the coefficients of L are defined on a domain D in $\mathbf{R}^n$.

LEMMA 2.1. *Assume that*

(i) *there is a positive constant* μ *such that* $a_{ij}(x)\xi_i\xi_j \geqq \mu$, $x \in D$, $|\xi| = 1$,

(ii) a_{ii} *and* b_i, $i = 1, \cdots, n$, *are bounded in* D *and* $c \geqq 0$ *in* D. *If* $u \in C^2(D)$, $Lu \geqq 0$ *in* D, *and there exists* $x_0 \in D$ *such that* $\inf_G u = u(x_0) \leqq 0$, *then* $u \equiv$ *constant.*

COROLLARY. *If* $u \in C^2(G)$, $Lu \geqq 0$ *in* G, $u \geqq 0$ *in* G *and* $u \not\equiv 0$, *then* $u > 0$ *in* G.

With $u(x_0) \leqq 0$ replaced by $u(x_0) < 0$, this is a well known result due to E. Hopf **[6]**. In case $u(x_0) = 0$, the boundary point principle of G. Giraud **[4]** can be applied (see also **[10]**, pp. 6–7 and **[2]**, pp. 150–152).

The following extension of the classical Harnack inequality for positive harmonic functions is due to Serrin **[13]**.

LEMMA 2.2. *Suppose there exist positive constants* μ *and* M *and a continuous, nondecreasing function* ϕ *with*

$$\int_{0^+}^{\infty} \frac{\phi(s)}{s}\, ds < +\infty$$

such that for $x, y \in D$

(i) $\mu \leqq a_{ij}(x)\xi_i\xi_j \leqq \mu^{-1}, |\xi| = 1$,

(ii) $(\Sigma_{i=1}^n b_i^2(x))^{1/2} \leqq M$,

(iii) $0 \leqq c(x) \leqq M$,

(iv) $|a_{ij}(x) - a_{ij}(y)| \leqq \phi(|x - y|)$.

Then for any bounded domain D_0 $\bar{D}_0 \subset D$, *there is a constant* $K = K(\mu, M, \phi, D_0, D)$ *so that for each positive solution* u *of* $Lu = 0$ *in* D,

$$K^{-1}u(y) \leqq u(x) \leqq Ku(y), \quad x, y \in D_0. \tag{2.1}$$

It is interesting to note that Serrin's proof uses only the maximum principle and a suitable parametrix, and that Serrin showed in case $n = 2$ that hypothesis (iv) is unnecessary. We shall not be concerned with this refinement. What will be important here is the following.

COROLLARY. *Replace* (iii) *by*
(iii)′ *There is a* v *in* $C^2(\bar{D})$ *so that* $v > 0$ *and* $Lv \geqq 0$ *on* $\bar{D}$.
Then (2.1) *still holds, where* K *may depend on* v *as well as the other parameters.*

For the proof set $\hat{u} = u/v$ and apply Serrin's theorem with L and u replaced by $\hat{L}$ and $\hat{u}$. $\hat{L}$ is defined by

$$\hat{L}\hat{u} = -a_{ij}\hat{u}_{x_ix_j} + (b_i - 2a_{ij}v_{x_j}/v)\hat{u}_{x_i} + (Lv/v)\hat{u}.$$

Since $Lu = 0$, it follows that $\hat{L}\hat{u} = 0$ since $\hat{L}\hat{u} = Lu/v$. Thus there is a constant $\hat{K}$ such that

$$\hat{K}^{-1}\hat{u}(y) \leqq \hat{u}(x) \leqq \hat{K}\hat{u}(y).$$

From the definition of $\hat{u}$ we can easily see that (2.1) holds if we set

$$K = \frac{\max_{D_0} v}{\min_{D_0} v}\hat{K}.$$

We now apply the Schauder existence theory to prove

LEMMA 2.3. *Suppose the hypotheses of Theorem* 2.1 *hold. Then for every bounded domain* F *with* $\bar{F} \subseteq G$ *and* $\dot{F}$ *sufficiently smooth, there is a solution* $v \in C^2(\bar{F})$ *to* $Lu = 0$ *in* F *which is positive on* $\bar{F}$.

Proof. According to hypothesis (b) the coefficients of L are Hölder continuous in $\bar{F}$. Let $\alpha = \alpha(F)$ denote the minimum of the Hölder exponents in $\bar{F}$ of the coefficients of L. We assume that $\dot{F}$ is sufficiently smooth so that $\dot{F} \in C^{2+\alpha}$. Then the Schauder existence theory shows that there exists a function

$$\begin{aligned} &v \in C^{2+\alpha}(\bar{F}) \\ &Lv = 0 \text{ in } F \\ &v = 1 \text{ on } \dot{F}. \end{aligned} \tag{2.2}$$

More specifically, a result of Boboc and Mustata [3] shows that there is a positive constant $\gamma = \gamma(F)$ such that if $u \in C^2(F) \cap C^0(\bar{F})$, then

$$\| u \|_0^F \leqq \gamma [\| Lu \|_0^F + \| u \|_0^{\dot{F}}].$$

This estimate can be used in conjunction with the classical Schauder estimate up to the boundary to prove the existence of solutions to Dirichlet problems such as (2.2) (see, for example, **[10]**, p. 166, 36, II).

Now we claim that v is positive. First, $v \geqq 0$ on F for otherwise let O denote some connected component of $\{x \in F: v(x) < 0\}$. Then $v < 0$ in O, $Lv = 0$ in O, $v = 0$ on $\dot{O}$, a contradiction to (c). But since $v \geqq 0$ on F,

$$(2.3) \qquad - a_{ij} v_{x_i x_j} + b_i v_{x_i} + c^+ v = c^- v \geqq 0 \text{ on } F,$$

where $c^+(x) = \max\{c(x), 0\}$ and $c^-(x) \equiv c^+(x) - c(x)$, so that the corollary to Lemma 2.1 implies that $v > 0$ on F.

3. *Proof of Theorem* 2.2. There exists a sequence of bounded domains in $\mathbf{R}^n$ with analytic boundaries such that $\bar{G}_k \subseteq G_{k+1} \subseteq \bar{G}_{k+1} \subseteq G$, $k = 1, 2, \cdots$, and $G = \bigcap_{k=1}^{\infty} G_k$ (see, for example **[7]**, pp. 317–319). Let α_k denote the minimum of the Hölder exponents in $\bar{G}_k$ of the coefficients of L. Furthermore, let u_k denote the positive solution to

$$\begin{aligned} u_k &\in C^{2+\alpha_k}(\bar{G}_k) \\ Lu_k &= 0 \text{ in } G_k \\ u_k &= 1 \text{ on } \dot{G}_k \end{aligned}$$

which exists according to Lemma 2.3.

Now choose $x_0 \in G_1$ and let $v_k = u_k / u_k(x_0)$. Next, apply the corollary to Lemma 2.2 with $D_0 = G_k, D = G_{k+1}, v = v_{k+1}$ and $u = v_l, l \geqq k + 1$. Then there are positive constants m_k and M_k so that

$$m_k \leqq v_l(x) \leqq M_k, \quad x \in \bar{G}_k, \quad l \geqq k + 1.$$

According to the Schauder estimates plus Ascoli's theorem, $\{v_l\}_{l \geqq k+1}$ has a subsequence which converges in $C^{2+\alpha_k}(\bar{G}_k)$. By a diagonalization process a subsequence of $\{v_k\}_{k \geqq 1}$ is obtained which converges uniformly on every compact subset of G to a function $v \in C^{2+\alpha_k}(\bar{G}_k)$, $k = 1, 2, \cdots$, with $Lu = 0$ in G. Also $m_k \leqq v \leqq M_k$ on $\bar{G}_k$ and $v(x_0) = 1$. Thus the desired positive solution exists.

In §1 we stated that assuming L is symmetric and (a) and (b) hold, (c) is equivalent to (D) with $\Omega \cap \{|x| > R\}$ replaced by G. To see that (c) implies (D) let D be a domain as in (D) and let F be a bounded, smooth domain with $\bar{D} \subseteq F \subseteq \bar{F} \subseteq G$. By Lemma 2.3 there is a function $v \in C^2(\bar{F})$, $v > 0$ on $\bar{F}$ and $Lv = 0$ in F. Let $\phi \in C_0^{\infty}(D)$ and set $\tilde{\phi} =$

ϕ/v. Then $\tilde{\phi} \in C_0^2(D)$. Applying Green's theorem several times we obtain

$$\frac{(L\phi, \phi)}{(\phi, \phi)} = \frac{\int_D v^2 a_{ij} \tilde{\phi}_{x_i} \tilde{\phi}_{x_j} dx}{\int_D \tilde{\phi}^2 v^2 dx}.$$

Hence

$$\frac{(L\phi, \phi)}{(\phi, \phi)} \geqq M \frac{\int_D \sum_{i=1}^n \tilde{\phi}_{x_i}^2 dx}{\int_D \tilde{\phi}^2 dx} \geqq MC,$$

where

$$M = \frac{\min_D v^2}{\max_D v^2} \min_{\bar{D} \times \{|\xi|=1\}} a_{ij}(x) \xi_i \xi_j,$$

and C is the constant in Friedrichs' inequality which depends only on D and not on $\tilde{\phi}$ (see, for example, **[9]**, p. 290).

To see that (D) $\Rightarrow$ (c), again let D and F be domains as above. Consider L as an operator on $L_2(F)$ with domain $C_0^\infty(F)$, and choose a nonnegative function $f \not\equiv 0$ in $C_0^\infty(F)$. Then the equation $Lu = f$ has a unique generalized solution v in $H_0^1(F)$. Following Allegretto and others, it can be shown that $f \geqq 0$ implies $v \geqq 0$ a.e. in F. Since for smooth F,

$$\{u \in C^{2+\alpha}(\bar{F}) \colon u = 0 \text{ on } \dot{F}\} \subseteq H_0^1(F) \ (\alpha \text{ may depend on } F),$$

it follows that the Dirichlet problem

$$\begin{aligned} & u \in C^{2+\alpha}(\bar{F}) \\ & Lu = f \text{ in } F \qquad (3.1) \\ & u = 0 \text{ on } \dot{F} \end{aligned}$$

has at most one solution. But this implies existence according to the general Schauder theory (see **[10]**, p. 166, 36, II), so that v can be identified with the solution to (3.1). Since v satisfies (2.3), the corollary to Lemma 2.1 implies that $v > 0$ on F. But then the generalized maximum principle (see **[10]**, p. 163, 35, IX) implies that the problem

$Lu = 0$ in D, $u = 0$ on $\dot{D}$ has at most one solution in class $C^2(D) \cap C^0(\bar{D})$.

4. Conclusion. An example in Allegretto's paper ([1], p. 324) has led us to formulate the following corollary of Theorem 1.2.

COROLLARY. *Consider the operator L defined by*

$$-L = \Delta + p$$

where p is nonnegative and continuous on $\mathbf{R}^n$. If $p \not\equiv 0$, then L has a nodal domain in $\mathbf{R}^2$.

Proof. We can assume without loss of generality that p is locally Hölder continuous. Otherwise, choose such a $\tilde{p}$ with $0 \leqq \tilde{p} \leqq p$ and $\tilde{p} \not\equiv 0$. Then by comparison of quadratic forms, the existence of a nodal domain for $-\Delta - \tilde{p}$ implies the same for L.

Now by Theorem 1.2 if no nodal domain existed, we could find a positive $v \in C^2(\mathbf{R}^2)$ with $Lv = 0$. But then $\Delta v = -pv \leqq 0$ and $v > 0$ would imply by Liouville's theorem that $v \equiv$ constant. But then $p \equiv 0$ would necessarily follow.

There is a direct computational proof of this corollary. By translation of coordinates, we may assume that $p(x) \geqq p_0$ for $|x| \leqq r_0$, where p_0 and r_0 are positive constants. It suffices to produce a nodal domain for the equation

$$\Delta u + \tilde{p}(r)u = 0, \quad r = |x|, \tag{4.1}$$

where $\tilde{p}(r)$ is continuous and $0 \leqq \tilde{p}(|x|) \leqq p(x)$ for all x and $\tilde{p}(r) = p_0$ for $0 \leqq r \leqq r_0$. If $\phi(r)$ is the solution to the initial value problem

$$\phi''(r) + \frac{1}{r}\phi'(r) + \tilde{p}(r)\phi(r) = 0, \quad r > 0 \tag{4.2}$$

$$\phi(0) = 1, \quad \phi'(0) = 0,$$

then $v(x) = \phi(|x|)$ is a regular solution of (4.1). We prove that there is an $R > 0$ such that $\phi(R) = 0$. Thus $v(x) = \phi(|x|)$ satisfies (4.1) and $\{|x| < R\}$ is a nodal domain for (4.1). There will then be a nodal domain for L contained in $\{|x| < R\}$.

Since $r = 0$ is a regular singular point for (4.2), we can compute the series

$$\phi(r) = 1 - \frac{1}{4}p_0 r^2 + \cdots$$

which is valid at least for $0 \leqq r < r_0$. Clearly, we can find an $r_1, 0 < r_1 < r_0$ so that $\phi'(r_1) < 0$. Now if $\phi(r) > 0$ for all $r > 0$, ϕ would satisfy the inequality

$$(r\phi'(r))' \leqq 0, \quad r \geqq 0.$$

Integrating, we have

$$r\phi'(r) \leqq r_1'\phi'(r_1) \equiv -\alpha < 0, \quad r \geqq r_1.$$

Thus

$$0 < \phi(r) \leqq \phi(r_1) + \alpha \log \frac{r_1}{r}, \quad r \geqq r_1.$$

A contradiction arises for sufficiently large r.

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