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ON PRIME GAMMA RINGS

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The notion of a Γ-ring was introduced by N. Nobusawa. The class of Γ-rings contains not only all rings but also Hestenes ternary rings. Recently, W. E. Barnes, J. Luh, W. E. Coppage and the author studied the structure of Γ-rings and obtained various generalizations analogous of corresponding parts in ring theory. The object of this paper is to study the properties of prime Γ-rings. Main results are the following theorems: (1) A Γ-ring M is a subdirect sum of prime Γ-rings if and only if $\mathscr{P}(M)=0$, where $\mathscr{P}(M)$ denotes the prime radical of M. (2) For the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ we have $\mathscr{P}(M_{m,n})=(\mathscr{P}(M))_{m,n}$, where M is a ring such that $x \in M\Gamma x\Gamma M$ for every $x \epsilon M$.

2. Preliminaries. Let M and Γ be two abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$, the conditions (1) $x\alpha y \in M$ (2) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$, (3) $(x\alpha y)\beta z = x\alpha(y\beta z)$ are satisfied, then we call M a Γ-ring.

If A and B are subsets of a Γ-ring M and $\Theta \subseteq \Gamma$, we denote $A\Theta B$, the subset of M consisting of all finite sums of the form $\Sigma a_i\gamma_i b_i$ where $a_i \in A$, $b_i \in B$ and $\gamma_i \in \Theta$. For singleton subsets we abbreviate this notation for example, $\{a\}\Theta B = a\Theta B$. A right ideal (left ideal) of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right and a left ideal, then we say that I is an ideal, or two-sided ideal of M. For each a of a Γ-ring M, the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a\rangle$. Similarly we define $\langle a|$ and $\langle a\rangle$, the principal left and two-sided (respectively) ideals generated by a.

Let I be an ideal of a Γ-ring M. If for each $a+I$, $b+I$ in the factor group M/I, and each $\gamma \in \Gamma$, we define $(a+I)\gamma(b+I) = a\gamma b + I$, then M/I is a Γ-ring which we shall call the Γ-residue class ring of M with respect to I.

If M_i is a Γ_i-ring for $i=1,2$ then an ordered pair (θ, ϕ) of mappings is called a homomorphism of M_1 onto M_2 if it satisfies the following properties: (1) θ is a group homomorphism from M_1 onto M_2 (2) ϕ is a group isomorphism from Γ_1 onto Γ_2 (3) For every $x, y \in M_1, \gamma \in \Gamma_1$, $(x\gamma y)\theta = (x\theta)(\gamma\phi)(y\theta)$. The kernel of the homomorphism (θ, ϕ) is defined to be $K = \{x \in M \mid x\theta = 0\}$. Clearly K is an ideal of M. If θ is a group isomorphism, that is, if $K=0$, then (θ, ϕ) is called an isomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2.

Let I be an ideal of the Γ-ring M. Then the ordered pair (ρ, ι) of mappings, where $\rho: M \to M/I$ is defined by $x\rho = x + I$ and ι is the identity mapping of Γ, is a homomorphism called the natural homomorphism from M onto M/I.

For all other notions relevant to Γ-rings we refer to **[4]**.

3. Semi-primeness.

DEFINITIONS. An ideal P of a Γ-ring M is prime if for any ideals $A, B \subseteq M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A subset S of M is an m-system in M if $S = \varnothing$ or if $a, b \in S$ implies $\langle a \rangle \Gamma \langle b \rangle \cap S \neq \varnothing$. The prime radical $\mathcal{P}(A)$ is the set of x in M such that every m-system containing x meets A. The prime radical of the zero ideal in a Γ-ring M is called the prime radical of the Γ-ring M which we denote by $\mathcal{P}(M)$. An ideal Q of M is semiprime if, for any ideal U, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ-ring M is semi-prime if the zero ideal is semi-prime.

The following theorem characterizes semi-primeness for ideals in Γ-rings. The proof is a minor modification of the proof of the corresponding theorem in ring theory, and we omit it.

THEOREM 1. *If Q is an ideal in a Γ-ring M, all the following conditions are equivalent.*

(1) *Q is a semi-prime ideal.*

(2) *If $a \in Q$ such that $a\Gamma M\Gamma a \subseteq Q$, then $a \in Q$.*

(3) *If $\langle a \rangle$ is a principal ideal in M such that $\langle a \rangle \Gamma \langle a \rangle \subseteq Q$, then $a \in Q$.*

(4) *If U is a right ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$.*

(5) *If V is a left ideal in M such that $V\Gamma V \subseteq Q$, then $V \subseteq Q$.*

COROLLARY 1. *A Γ-ring M is semi-prime if and only if $a\Gamma M\Gamma a = 0$ implies $a = 0$.*

DEFINITION. A subset S of M is strongly nilpotent if there exists a positive integer n such that $(S\Gamma)^n S = (0)$.

It follows easily by induction that if Q is a semi-prime ideal and A is an ideal such that $(A\Gamma)^n A \subseteq Q$ for an arbitrary positive integer n, then $A \subseteq Q$. Hence, (0) is a semi-prime ideal if and only if M contains no nonzero strongly nilpotent ideal. By Theorem 1 (4) and (5), we have also that (0) is a semi-prime ideal if and only if M contains no nonzero strongly nilpotent right (left ideal).

The author **[3]** showed the following result.

THEOREM 2. *An ideal Q in a Γ-ring M is a semi-prime ideal in M if and only if $\mathcal{P}(Q) = Q$.*

By Theorem 2, (0) is a semi-prime ideal if and only if $\mathcal{P}(M) = (0)$.

Thus we have the following theorem.

THEOREM 3. *A Γ-ring M has zero prime radical if and only if it contains no strongly nilpotent ideal (right ideal, left ideal).*

4. Prime Γ-rings. In this section we shall be concerned with the concept introduced in the following definition.

DEFINITION. A Γ-ring M is said to be prime if the zero ideal is prime.

The following theorem is analogous to the corresponding theorem in ring theory, and we omit its proof.

THEOREM 4. *If M is a Γ-ring, the following conditions are equivalent:*

(1) *M is a prime Γ-ring.*

(2) *If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.*

(3) *If $\langle a \rangle$ and $\langle b \rangle$ are principal ideals in M such that $\langle a \rangle \Gamma \langle b \rangle = (0)$, then $a = 0$ or $b = 0$.*

(4) *If A and B are right ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

(5) *If A and B are left ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

The importance of the concept of prime Γ-rings stems primarily from the following fact.

THEOREM 5. *If P is an ideal in the Γ-ring M, then the Γ-residue class ring M/P is a prime Γ-ring if and only if P is a prime ideal in M.*

We prepare the following lemma which is fairly easy to prove, and we omit the proof.

LEMMA 1. *Let (θ, ι) be a homomorphism of Γ-ring M onto the Γ-ring N, with kernel K. Then each of the following is true:*

(1) *If I is an ideal (right ideal) in M, then $I\theta$ is an ideal (right ideal) in N.*

(2) *If J is an ideal (right ideal) in N, then $J\theta^{-1}$ is an ideal (right ideal) in M which contains K.*

(3) *If I is an ideal (right ideal) in M which contains K, then $I=(I\theta)\theta^{-1}$.*

(4) *The mapping $I\to I\theta$ defines a one to one mapping of the set of ideals (right ideals) in M which contain K onto the set of all ideals (right ideals) in N.*

Proof of Theorem 5. Let M/P be prime and A, B be ideals of M such that $A\Gamma B\subseteq P$. Let (ρ, ι) be the natural homomorphism from M onto M/P. Then by Lemma 1 $A\theta$ and $B\theta$ are ideals of M/P such that $A\theta\Gamma B\theta=(0)$. Since M/P is prime, it follows that $A\theta=(0)$ or $B\theta=(0)$, that is, $A\subseteq P$ or $B\subseteq P$. Thus P is a prime ideal in M.

Conversely, let P be a prime ideal in M. Lemma 1 shows that each ideal in M/P is of the form A/P, where A is an ideal in M which contains P. Thus we may assume that $A/P, B/P$ be ideals of M/P such that $(A/P)\Gamma(B/P)=(0)$, which implies $A\Gamma B\subseteq P$. Then by the primeness of P we have $A\subseteq P$ or $B\subseteq P$. Hence $A=P$ or $B=P$ and so $A/P=(0)$ or $B/P=(0)$. This completes the proof.

Barnes **[1]** has characterized $\mathscr{P}(M)$ as the intersection of all prime ideals of M.

The author **[4]** has shown the following lemma.

LEMMA 2. *A Γ-ring M is a subdirect sum of Γ-rings S_i, $i\in\mathfrak{A}$, if and only if for each $i\in\mathfrak{A}$ there exists in M a two-sided ideal K_i such that* $M/K_i\cong S_i$, moreover $\bigcap_{i\in\mathfrak{A}} K_i=(0)$.

Thus, these facts and Theorem 5 yield the following theorem which is analogous to Theorem 4.3 in **[4]**.

THEOREM 6. *A Γ-ring M is a subdirect sum of prime Γ-rings if and only if $\mathscr{P}(M)=(0)$.*

Following Luh **[2]**, we introduce the matrix ring $M_{m,n}$.

Let G be an additive group. We shall denote by $G_{m,n}$ the additive group of all $m\times n$ matrices over the group G. For $1\leqq i\leqq m, 1\leqq j\leqq n$, and $a\in G$, let aE_{ij} denote the matrix having a at the ith row and jth column, and 0 elsewhere.

Let M be a Γ-ring. Consider the group $M_{m,n}$ and $\Gamma_{n,m}$. For $(a_{ij}), (b_{ij})\in M_{m,n}$ and $(\gamma_{ij})\in\Gamma_{n,m}$, define $(a_{ij})(\gamma_{ij})(b_{ij})=(c_{ij})$, where $c_{ij}=\sum_{k=1}^{m}\sum_{h=1}^{n} a_{ih}\gamma_{hk}b_{kj}$. Then $M_{m,n}$ forms a $\Gamma_{n,m}$-ring.

We now prove the next theorem which will indicate one way to construct new prime Γ-rings from given ones.

THEOREM 7. *If M is a Γ-ring, the matrix ring $M_{m,n}$ is a prime $\Gamma_{n,m}$-ring if and only if M is a prime Γ-ring.*

Proof. Let us prove that if M is not prime, then $M_{m,n}$ is not prime. If M is not prime, there exist nonzero elements a and b of M such that $a\Gamma M\Gamma b=0$. Then, we have, for example, $aE_{11}\Gamma_{n,m}M_{m,n}\Gamma_{n,m}bE_{11}=0$ with aE_{11} and bE_{11} nonzero elements of $M_{m,n}$. Hence, $M_{m,n}$ is not prime. Conversely, suppose that $M_{m,n}$ is not prime, and hence that there exist nonzero matrices $\Sigma_{i,j}a_{ij}E_{ij}$ and $\Sigma_{i,j}b_{ij}E_{ij}$ such that $(\Sigma_{i,j}a_{ij}E_{ij})\Gamma_{n,m}M_{m,n}\Gamma_{n,m}(\Sigma_{i,j}b_{ij}E_{ij})=0$. Let p, q, r and s be fixed positive integers such that $a_{p,q}\neq 0$ and $b_{rs}\neq 0$. As a special case of the preceding equation, we find that for each $x\in M$, each $\gamma,\eta\in\Gamma$,

$$(\Sigma a_{ij}E_{ij})(\gamma E_{qp})(xE_{ps})(\eta E_{sr})(\Sigma b_{ij}E_{ij})=\Sigma a_{iq}\gamma x\eta b_{rj}E_{ij}=0.$$

In particular, the (p,s) element must be zero, that is, $a_{pq}\gamma x\eta b_{rs}=0$. Since this is true for every x in M and every γ,η in Γ, we have $a_{pq}\Gamma M\Gamma b_{rs}=0$, and M is not prime. This completes the proof.

Luh [2] has obtained the following lemma.

LEMMA 3. *Let M be a Γ-ring such that $x\in M\Gamma x\Gamma M$ for every $x\in M$. Then the ideals of the $\Gamma_{n,m}$-ring $M_{m,n}$ are the form $U_{m,n}$, where U is an ideal of M.*

We prepare the following lemma.

LEMMA 4. *If I is an ideal in the Γ-ring M, then the matrix $\Gamma_{n,m}$-ring $(M/I)_{m,n}$ is isomorphic to the $\Gamma_{n,m}$-ring $M_{m,n}/I_{m,n}$.*

Proof. Let θ be a mapping of the $\Gamma_{n,m}$-ring $(M/I)_{m,n}$ to the $\Gamma_{n,m}$-ring $M_{m,n}/I_{m,n}$ such that $(x_{ij}+I)\theta=(x_{ij})+I_{m,n}$. Clearly, θ is a group isomorphism from $(M/I)_{m,n}$ onto $M_{m,n}/I_{m,n}$. Let ι be an identity mapping from $\Gamma_{n,m}$ onto $\Gamma_{n,m}$. By the definition of multiplications of the Γ-residue class ring, we have that

$$\begin{aligned}[(x_{ij}+I)(\gamma_{ij})(y_{ij}+I)]\theta &= (z_{ij}+I)\theta,\text{ where }(z_{ij})=(x_{ij})(\gamma_{ij})(y_{ij})\\ &= (x_{ij})(\gamma_{ij})(y_{ij})+I_{m,n}\\ &= [(x_{ij})+I_{m,n}](\gamma_{ij})[(y_{ij})+I_{m,n}]\\ &= (x_{ij}+I)\theta(\gamma_{ij})\iota(y_{ij}+I)\theta.\end{aligned}$$

This shows that (θ,ι) is an isomorphism of $(M/I)_{m,n}$ onto $M_{m,n}/I_{m,n}$.

We now prove the following result.

THEOREM 8. *Let M be a Γ-ring such that $x \in M\Gamma x\Gamma M$ for every $x \in M$. If $\mathcal{P}(M)$ is the prime radical of the Γ-ring M, then $\mathcal{P}(M_{m,n}) = (\mathcal{P}(M))_{m,n}$.*

Proof. From Lemma 3 it follows easily that $I \to I_{m,n}$ (I an ideal in M) is a one to one mapping of the set of all ideals in M onto the set of all ideals in $M_{m,n}$. Moreover, by Lemma 4, $(M/I)_{m,n} \cong M_{m,n}/I_{m,n}$. Hence, by Theorem 7, $M_{m,n}/I_{m,n}$ is a prime $\Gamma_{n,m}$-ring if and only if M/I is a prime Γ-ring. From Theorem 5 it follows that $I_{m,n}$ is a prime ideal of $M_{m,n}$ if and only if I is a prime ideal of M. Thus, if $\{P_i \mid i \in \mathfrak{A}\}$ is the set of all prime ideals in M, we have

$$\mathcal{P}(M_{m,n}) = \bigcap_{i\in\mathfrak{A}} (P_i)_{m,n} = (\bigcap_{i\in\mathfrak{A}} P_i)_{m,n} = (\mathcal{P}(M))_{m,n}.$$

REMARKS. A Γ-ring M is said to be simple if (1) $M\Gamma M \neq 0$ and (2) M has no ideals other than 0 and M itself. If M is simple, $M\Gamma x\Gamma M = M$ for each nonzero element x in M. Hence $x \in M\Gamma x\Gamma M$. Thus, for a simple Γ-ring M, $\mathcal{P}(M_{m,n}) = (\mathcal{P}(M))_{m,n} = 0$.

If there exists an element ϵ in M and an element δ in Γ such that $x\partial\epsilon = \epsilon\partial x = x$ for every element $x \in M$, ϵ is called an unity of M. If M has an unity, for every x in M $x \in M\Gamma x\Gamma M$, and then $\mathcal{P}(M_{m,n}) = (\mathcal{P}(M))_{m,n}$.

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