PACIFIC JOURNAL OF MATHEMATICS
Vol 75, No. 1, 1978

A CHARACTERIZATION OF NON-NEGATIVE MATRIX OPERATORS ON l^p TO l^q WITH $\infty > p \geqq q > 1$

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Ladyženskiĭ's characterization of nonnegative matrix operators on l^p to l^q ($\infty > p = q > 1$) is extended to the case $\infty > p \geqq q > 1$. A solution is also given to a conjecture of Vere-Jones concerning nonnegative matrix operators on l^p.

1. Introduction. A scalar matrix $A = (a_{ij})_{i,j=1}^{\infty}$ is called nonnegative if $a_{ij} \geqq 0$ for all i, j. If A determines a matrix operator on l^p $(1 \leqq p < \infty)$ to l^q $(1 \leqq q < \infty)$, we denote the operator norm of A by

$$\|A\|_{p,q} = \sup\{\|Ax\|_q : \|x\|_p = 1\}.$$

The infinite unit matrix is denoted by E.

In [4] Ladyženskiĭ proved the following theorem: An infinite nonnegative matrix $A = (a_{ij})$ maps l^p $(1 < p < \infty)$ into itself if and only if there exist positive numbers C and $s_1, s_2, \cdots$ such that

$$\sum_{i=1}^{\infty} a_{ij}\Big(\sum_{k=1}^{\infty} a_{ik}s_k\Big)^{p-1} \leqq C s_j^{p-1} \quad (j \geqq 1),$$

and then $\|A\|_{p,p} \leqq C^{1/p}$.

We shall extend the above result to the case where A maps l^p into l^q, $\infty > p \geqq q > 1$ (Theorem 1). Finally, a simple application of Ladyženskiĭ's theorem gives an affirmative answer to a conjecture of Vere–Jones (§3).

2. The main result. Our aim is to prove the following

THEOREM 1. *Let $\infty > p \geqq q > 1$. Then an infinite nonnegative matrix $A = (a_{ij})$ maps l^p into l^q if and only if there exist a positive constant C and a sequence $u = (u_j)_{j=1}^{\infty}$ of nonnegative numbers with the following properties:*

(i) *$u_j = 0$ if and only if $a_{ij} = 0$ for every i;*

(ii) *$\|u\|_p \leqq 1$ if $p > q$;*

(iii) *for each $j = 1, 2, \cdots$,*

$$\sum_{i=1}^{\infty} a_{ij}\Big(\sum_{k=1}^{\infty} a_{ik}u_k\Big)^{q-1} \leqq C u_j^{p-1}. \tag{1}$$

The best value of C in (1) *for which such a sequence u can be found is* $(\|A\|_{p,q})^q$.

Proof. Sufficiency. Assume that C and u_j $(j \geqq 1)$ are positive numbers satisfying (ii) and (iii). We will show that

$$\|Ax\|_q \leqq C^{1/q}\|x\|_p, \qquad x \in l^p. \tag{2}$$

Let $x = (x_j) \in l^p$ be given. By Hölder's inequality,

$$\left|\sum_{j=1}^{\infty} a_{ij}x_j\right|^q \leqq \left(\sum_{j=1}^{\infty} a_{ij}u_j^{1-q}|x_j|^q\right)\left(\sum_{k=1}^{\infty} a_{ik}u_k\right)^{q-1},$$

$i = 1, 2, \cdots$. Combining this result with (1) we get

$$\|Ax\|_q^q \leqq C\sum_{j=1}^{\infty} u_j^{p-q}|x_j|^q.$$

If $p > q$, a second application of Hölder's inequality yields

$$\sum_{j=1}^{\infty} u_j^{p-q}|x_j|^q \leqq \|u\|_p^{p-q}\|x\|_p^q,$$

and thus (2) is established. (For $p = q$ see [**4**, p. 140].)

Necessity. Let $A = (a_{ij})$ be a nonnegative matrix taking l^p into l^q. Assume first that A is positive (i.e. $a_{ij} > 0$ for all i, j) and put $C = (\|A\|_{p,q})^q$. For each $n = 1, 2, \cdots$ we can then find a positive n-tuple $u^{(n)} = (u_j^{(n)})$ with $\|u^{(n)}\|_p = 1$ such that

$$\sum_{i=1}^{n} a_{ij}\left(\sum_{k=1}^{n} a_{ik}u_k^{(n)}\right)^{q-1} \leqq C(u_j^{(n)})^{p-1},$$

$j = 1, \cdots, n$ (see [**3**, §9] and [**6**, pp. 223–224]).

Define, for $j = 1, 2, \cdots$,

$$u_j = \underline{\lim}_n\,(u_j^{(n)}) \quad \text{or} \quad u_j = \underline{\lim}_n\,(u_j^{(n)}/u_1^{(n)})$$

according to whether $p > q$ or $p = q$. It is easy to see that $u = (u_j)_{j=1}^{\infty}$ is a sequence of positive numbers such that (ii) and (iii) are satisfied.

Finally, if some elements of A are zero, we can apply the above result to $A + \epsilon B$ $(\epsilon > 0)$, where B is a fixed positive matrix mapping l^p into l^q. A simple continuity argument $(\epsilon \to 0)$ completes the proof of the theorem.

COROLLARY 1. *Let* $1 < p < \infty$. *Then an infinite nonnegative matrix* $A = (a_{ij})$ *maps* l^p *into itself if and only if there exist positive numbers* C *and* $u_1, u_2, \cdots$ *such that*

$$\sum_{j=1}^{\infty} a_{ij} u_j \leqq C u_i, \qquad i = 1, 2, \cdots,$$

and

$$\sum_{i=1}^{\infty} a_{ij} u_i^{p-1} \leqq C u_j^{p-1}, \qquad j = 1, 2, \cdots.$$

If this is the case, then $\| A \|_{p,p} \leqq C$.

Proof. The "if" part is clear by Theorem 1, and the "only if" part follows by applying Theorem 1 to $A + E$.

REMARK 1. The statement of Theorem 1 does not hold for $1 \leqq p < q < \infty$. (Consider a diagonal matrix A with diagonal elements $a_j = u_j^{(p-q)/q}$, $u_j \downarrow 0$.)

REMARK 2. The case $p = 2$ of Corollary 1 is essentially due to Ladyženskiĭ [**4**, Remark 2]. The "if" part for $p = 2$ (with $u_j = 1$ for all j) is a result of Schur [**5**] (see also [**1**, p. 126], [**2**, Problem 37], [**6**, Theorem 6.12-A]).

3. Solution to a conjecture of Vere–Jones. In [**7**, p. 614], Vere–Jones formulated the following

Conjecture. (i) An infinite nonnegative matrix $A = (a_{ij})$ acts as a bounded linear operator on l^p $(1 < p < \infty)$ if and only if there exist a positive vector (μ_j) and a positive number ρ such that

$$\sum_{j=1}^{\infty} a_{ij} \mu_j^{1/p} \leqq \rho \mu_i^{1/p}, \qquad i = 1, 2, \cdots,$$

and

$$\sum_{i=1}^{\infty} a_{ij} \mu_i^{1/p'} \leqq \rho \mu_j^{1/p'}, \qquad j = 1, 2, \cdots,$$

where $p' = p/(p-1)$.

(ii) Moreover, the norm of the operator can be identified with the least number ρ for which such a vector (μ_j) can be found.

We note first that Part (i) of the conjecture is valid by Corollary 1. Part (ii), however, fails in general, as may be seen by means of the next two propositions. We denote the operator norm $\|\cdot\|_{p,p}$ by $\|\cdot\|_p$.

PROPOSITION 1. *Let* $1 < p < \infty$. *Assume that* $A = (a_{ij})$ *is an infinite nonnegative matrix operator on* l^p *such that Conjecture* (ii) *holds for* $A + E$. *Then*

$$\|A + E\|_p = \|A\|_p + 1.$$

Proof. Apply Corollary 1 to $A + E$.

PROPOSITION 2. *Let* $1 < p < \infty$ *and let* E_n *denote the unit* $n \times n$ *matrix. Given a nonnegative* $n \times n$ *matrix* A, *we have*

$$\|A + E_n\|_p = \|A\|_p + 1 \tag{3}$$

if and only if $\|A\|_p = \lambda_A$, *the greatest nonnegative eigenvalue of* A.

Proof. Choose a nonnegative n-tuple x such that $\|x\|_p = 1$ and $\|A + E_n\|_p = \|Ax + x\|_p$. Then (3) implies

$$\|Ax + x\|_p = \|Ax\|_p + 1 = \|A\|_p + 1,$$

whence $Ax = \lambda x$ for some $\lambda \geqq 0$. Now

$$\|A\|_p = \|Ax\|_p = \lambda \leqq \lambda_A \leqq \|A\|_p,$$

from which $\|A\|_p = \lambda_A$. The "if" part follows from

$$\lambda_A + 1 \leqq \|A + E_n\|_p \leqq \|A\|_p + 1.$$

In conclusion, we remark that Conjecture (ii) is true when $p = 2$ and A is symmetric. (Apply Theorem 1 to $n^{-1}A + E, n = 1, 2, \cdots$.)

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Received January 20, 1977 and in revised form May 27, 1977.

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