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COALLOCATION BETWEEN LATTICES WITH APPLICATIONS TO MEASURE EXTENSIONS

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It is well known that in a locally compact Hausdorff space every countably additive measure on $R_\sigma(\mathscr{K}_\delta)$, the σ-ring generated by the compact G_δ sets, can be extended to a countably additive measure on $\sigma(\mathscr{F})$, the σ-algebra generated by the closed sets. In a locally compact Hausdorff space $\mathscr{F}$, the lattice of closed sets, countably coallocates (Definition 4.7) the lattice of compact G_δ sets. Our purpose is to show that coallocation and countable coallocation are properties basic to many extension theorems.

Dubins [5] considered the following situation. $K \subseteq L$ are two lattices containing the null set (a lattice is a collection of subsets of some set closed under finite unions and intersections). u is a bounded measure defined on K. Dubins asked when u_*, defined by $u_*(b) = \sup\{u(k)/k \subseteq b, k \in K\}$, is a measure on L. A necessary and sufficient condition is for L to allocate K. L allocates K if the following is true. For any $k \in K$ contained in the union of two sets l and h from L there exist sets p and q from the lattice K such that $k = p \cup q$ and $p \subseteq l$, $q \subseteq h$.

With two lattices $K \subseteq L$ and u a measure on K, we show that a sufficient condition for u^{**}, defined by $u^{**}(b) = \inf\{u_*(l')/b \subseteq l', l' \in L'\}$, to be a measure on the algebra generated by L is for u_* to be modular on L'. l' is the complement of the set l and $L' = \{l'/l \in L\}$. It follows that if u is a K inner regular measure on $R(K)$, the ring generated by K, then u^{**} is a L inner regular extension of u to $A(L)$, the algebra generated by L.

Thus when L coallocates K (i.e. L' allocates K) Dubin's result shows that for every K regular bounded measure u on $R(K)$, u^{**} is a L regular extension of u to $A(L)$. If L countably coallocates K then u^{**} is countably additive when u is countably additive. From this we obtain the stated result on locally compact Hausdorff spaces [Halmos 7] as well as a related result by Levin and Stiles [8]. Countable coallocation also yields an extension theorem by Marik [9] on countably paracompact normal spaces and a theorem by Berberian [2]. In most instances we can and do prove our results for measures that are not bounded.

We also look at measures that are τ-smooth. A measure u on K is τ-smooth if for any net $\{k_\alpha\}$ decreasing to $\varnothing$, $k_\alpha \in K$, $\lim_\alpha u(k_\alpha) = 0$. We show that any bounded K regular measure u on $R(K)$ that is τ-smooth on K can be extended to a bounded measure on $A(\tau(K))$ that is τ-smooth on $\tau(K)$. $\tau(K)$ is the smallest lattice containing K that is closed under arbitrary intersections. We prove u_* is modular on $\tau(K)'$ and obtain u^{**}, defined with respect to $\tau(K)$, as the desired extension.

2. Definitions and notation. All lattices are collections of subsets of an abstract set X that are closed under finite unions and intersections. The fact that X contains points has no importance in this paper — the boolean algebra of all subsets of X can be replaced by any complete boolean algebra. Subsets of X will be denoted by lower case letters. If we are considering a lattice L and a set l, it will usually be assumed that l belongs to L.

l' denotes the complement of the set l in X and $L' = \{l'/l \in L\}$. $R(L)$ is the ring generated by L; $A(L)$ the algebra generated by L. $R_\sigma(L)$ is the σ-ring generated by L and $\sigma(L)$ is the σ-algebra generated by L.

A measure u on a lattice A is an extended real valued set function such that for $a, b \in A$

(i) $u(a) + u(b) = u(a \cup b) + u(a \cap b)$.

(ii) $u(a) + u(b) = u(a \cup b)$ whenever $a \cap b = \varnothing$.

(iii) $a \subseteq b$ implies $u(a) \leqq u(b)$.

Let K be a lattice contained in A. A measure u on A is K inner regular if for any $a \in A$, $u(a) = \sup\{u(k)/k \subseteq a, k \in K\}$.

A measure u on a lattice A is σ-smooth if for any sequence $\{a_n\}$ decreasing to $\varnothing$, $\lim_n u(a_n) = 0$. u is countably additive on A if $\Sigma_1^\infty u(a_n) = u(\bigcup_1^\infty a_n)$ whenever $\{a_n\}$ is a disjoint sequence of sets from A such that $\bigcup_1^\infty a_n \in A$. For a ring A any finite valued measure u which is σ-smooth on $K \subseteq A$ and K inner regular is countably additive on A.

A measure u on a lattice A is σ-finite if for every $a \in A$, a is contained in $\bigcup_1^\infty a_n$ where $a_n \in A$ and $u(a_n)$ is finite for all n. If A is a ring then by the Caratheodory extension theorem any countably additive, σ-finite measure u on A can be uniquely extended to a countably additive measure on $R_\sigma(A)$. The extension is the outer measure defined by $\hat{u}(b) = \inf\{\Sigma_1^\infty u(a_n)/b \subseteq \bigcup_1^\infty a_n, a_n \in A\}$.

The bounded measures on the algebra $A(L)$ are denoted by $M(L)$. It is easy to verify that if u is bounded and L inner regular then $u(a) = \inf\{u(l')/a \subseteq l', l \in L\}$ for $a \in A(L)$. A measure satisfying the last equality is called L' outer regular. If a measure is both L inner regular and L' outer regular then it is L regular. The L regular, bounded measures on $A(L)$ are denoted by $M_r(L)$. Those measures belonging to $M_r(L)$ which are σ-smooth are denoted by $M_r^s(L)$. These measures are countably additive and hence can be uniquely extended to a countably additive measure on $\sigma(L)$.

For a measure u on a lattice K which contains $\varnothing$, u_* is defined as in the introduction. The definition of u^{**} as given in the introduction depends on the lattice L used (L must also contain $\varnothing$).

3. The modularity of u_*. Let $K \subseteq L$ be two lattices containing $\varnothing$ and u a measure on K. u_* is modular on L' if $u_*(l_1') + u_*(l_2') = u_*(l_1' \cup l_2') + u_*(l_1' \cap l_2')$. We now show that if u_* is modular on L' then u^{**} is an L' outer regular measure on $A(L)$ where u^{**} is defined with respect to L. Furthermore u^{**} is a complete measure on $\mathscr{E}(u, L') = \{e / u^{**}(l') = u^{**}(e \cap l') + u^{**}(e' \cap l')$ for all $l \in L\}$.

The easy proofs of the following lemmas are omitted.

LEMMA 3.1. *Let u be a measure on K. If u_* is modular on L' then for a, b subsets of X,*

$$u^{**}(a \cup b) + u^{**}(a \cap b) \leqq u^{**}(a) + u^{**}(b).$$

LEMMA 3.2. *Let u be a measure on K and u_* be modular on L'. Suppose $l' \cap a = \varnothing$, where a is any subset of X. Then $u^{**}(l') + u^{**}(a) = u^{**}(a \cup l')$.*

u^* is σ-smooth on L' if $\lim_n u_*(l_n') = u_*(\bigcup_1^\infty l_n')$ *whenever $\{l_n'\}$ is an increasing sequence such that $\bigcup_1^\infty l_n' \in L'$.*

THEOREM 3.3. *Let u be a measure on K.*

(i) The modularity of u_* on L' is equivalent to u^{**} being an L' outer regular measure on $A(L)$.

(ii) If u_* is modular on L' then $\mathscr{E}(u, L')$ is an algebra containing $A(L)$ and u^{**} is a complete measure on $\mathscr{E}(u, L')$.

(iii) Suppose L is closed under countable intersections. If u_* is modular and σ-smooth on L' then $\mathscr{E}(u, L')$ is a σ-algebra containing $\sigma(L)$ and u^{**} is countably additive on $\mathscr{E}(u, L')$.

Proof. (i), (ii). That modularity is necessary is obvious. The sufficiency of (i) and (ii) will be proved. If u_* is modular on L' then $\mathscr{E}(u, L')$ is closed under complementation and by Lemma 3.2 it contains L'.

Fix $l' \in L'$. It is sufficient to assume $u_*(l')$ is finite. Let e_1, e_2 belong to $\mathscr{E}(u, L')$. By Lemma 3.1,

$$u_*(l') \leqq u^{**}((e_1 \cup e_2) \cap l') + u^{**}((e_1 \cup e_2)' \cap l'). \tag{1}$$

For the reverse inequality choose l'_j, h'_j from L' such that $l'_j \supseteq e_j \cap l'$, $h'_j \supseteq e'_j \cap l'$ and

$$u_*(l') \geqq u_*(l'_j) + u_*(h'_j) - \epsilon/3 \qquad j = 1, 2. \tag{2}$$

We claim that

$$u_*(l') \geqq u_*(l'_1 \cup l'_2) + u_*(h'_1 \cap h'_2) - \epsilon. \tag{3}$$

This inequality is implied by

$$u_*(l'_1 \cup l'_2) + u_*(h'_1 \cap h'_2) \leqq u_*(l'_1) + u_*(h'_1) + \tfrac{2}{3}\epsilon$$

which is equivalent to

$$u_*(l'_2) + u_*(h'_2) \leqq u_*(l'_1 \cap l'_2) + u_*(h'_1 \cup h'_2) + \tfrac{2}{3}\epsilon$$

by the modularity of u_*. The last inequality is true by (2) and the modularity of u_*.

(3) implies the reverse direction of (1) and hence $e_1 \cup e_2$ belongs to $\mathscr{E}(u, L')$. Hence $\mathscr{E}(u, L')$ is an algebra containing L.

To show u^{**} is a measure suppose l' contains $e_1 \cup e_2$ and that $u(l') - u(e_1 \cup e_2) < \epsilon$. Then by (3)

$$u_*(l'_1) + u_*(l'_2) \leqq u_*(l') + u_*(l'_1 \cap l'_2) + \epsilon.$$

Therefore

$$u^{**}(e_1) + u^{**}(e_2) \leqq u^{**}(e_1 \cup e_2) + u^{**}(e_1 \cap e_2) + 2\epsilon.$$

By Lemma 3.1 u^{**} is modular on $\mathscr{E}(u, L')$ and by Lemma 3.2 $u^{**}(\varnothing) = 0$. It is easy to verify that $\mathscr{E}(u, L')$ contains all e such that $u^{**}(e) = 0$.

(iii) Let $\{e_n\}$ be a sequence from $\mathscr{E}(u, L')$. Choose $l'_1 \supseteq e_n \cap l'$, $b'_n \supseteq e_n \cap l'$ such that

$$u_*(l') \geqq u_*(l'_n) + u_*(h'_n) - \epsilon/2^n. \tag{4}$$

We can show using (4) and the modularity of u_* that

$$u_*\left(\bigcup_1^n l'_j\right) + u_*\left(\bigcap_1^n h'_j\right) \leqq u_*(l') + \sum_1^n \frac{\epsilon}{2^j}. \tag{5}$$

Since u_* is σ-smooth on L' there exists an n large enough such that

$$u_*(l') \geqq u_*\left(\bigcup_1^\infty l'_j\right) + u_*\left(\bigcap_1^n h'_j\right) - 2\epsilon. \tag{6}$$

It follows that

$$u_*(l') \geqq u^{**}\left(l' \cap \left(\bigcup_1^\infty e_j\right)\right) + u^{**}\left(l' \setminus \left(\bigcup_1^\infty e_j\right)\right) - 2\epsilon. \tag{7}$$

Therefore by Lemma 3.1, $\bigcup_1^\infty e_j \in \mathscr{E}(u, L')$.

To show u^{**} is countably additive, we can assume $\Sigma_1^\infty u^{**}(e_j)$ is finite. Choose $f'_j \supseteq e_j$, $f'_j \in L'$ such that $u_*(f'_j) - u^{**}(e_j) \leqq \epsilon/2^j$. Let $l' = \bigcup_1^\infty f'_j$. Then since u_* is σ-smooth and modular on L',

$$u_*(l') \leqq \sum_1^\infty u_*(f'_j) < +\infty.$$

Inequality (7) holds for l' and since $l' \supseteq \bigcup_1^\infty e_j$,

$$\begin{aligned} u^{**}\left(\bigcup_1^\infty e_j\right) &\leqq u_*(l') \\ &\leqq \sum_1^\infty u_*(f'_j) \\ &\leqq \sum_1^\infty u^{**}(e_j) + 2\epsilon. \end{aligned}$$

Thus u^{**} is countably additive on $\mathscr{E}(u, L')$.

We now give sufficient conditions for u^{**} to extend u.

THEOREM 3.4. *Let u be a K inner regular measure on $S(K)$ which represents either $A(K)$ or $R(K)$. If u_* is modular on L' and u^{**} is finite on K then $u = u^{**}$ on $S(K)$.*

Proof. $u(b) = u^{**}(b)$ when $u(b) = +\infty$. If $u(b)$ is finite then $u^{**}(b)$ is finite. This follows because every $b \in S(K)$ is of the form $\bigcup_{j=1}^n k_j \cap h'_j$ where for all $j, h_j \in K$ and either $k_j \in K$ or $k_j = X$.

Choose $l' \supseteq b$ such that $u_*(l') - u^{**}(b) \leqq \epsilon/3$. Choose $k_0 \subseteq b$ such that $u(b) - u(k_0) \leqq \epsilon/3$ and choose $k_1 \subseteq l'$ such that $u_*(l') - u(k_1) \leqq \epsilon/3$. Let $k = k_0 \cup k_1$. Then since u is K inner regular, $|u(k) - u(b)| \leqq \frac{2}{3}\epsilon$. Then $u_*(l') - u(b) < \epsilon$. Hence $u^{**}(b) = u(b)$.

A set function v on a collection of subsets $\mathscr{H}$ is σ-finite with respect to $\mathscr{S} \subset \mathscr{H}$ if for every $h \in \mathscr{H}$, $h \subseteq \bigcup_1^\infty s_j$ where $s_j \in \mathscr{S}$ and $v(s_j)$ is finite for all j. Note that since our measure u is finite on K, u is σ-finite with respect to $R(K)$ when u is defined on $R_\sigma(K)$.

THEOREM 3.5. *Suppose L is closed under countable intersections. Suppose u_* is modular and σ-smooth on L' where u is a countably additive measure defined on*

(i) *$\sigma(K)$, σ-finite with respect to $A(K)$. If u^{**} is finite on K then it is a countably additive extension of u to $\sigma(L)$.*

(ii) *$R_\sigma(K)$. If u^{**} is finite valued on K then u^{**} is a countably additive extension of u to $\sigma(L)$.*

Proof. If u is a countably additive, σ-finite measure on a σ-ring generated by a ring S then for all b in the σ-ring,

$$u(b) = \inf\left\{\sum_1^\infty u(s_j)/b \subset \bigcup_1^\infty s_j, s_j \in S\right\}.$$

That $u = u^{**}$ follows in both cases from this and Theorem 3.4.

Define $K \cap L = \{k \cap l/k \in K, l \in L\}$.

COROLLARY 3.6. *Suppose $u \in M_r(K)$ and u_* modular on L'. Then u^{**} is a $K \cap L$ regular extension of u to $A(L)$. If u is countably additive, u_* σ-smooth on L', and L closed under countable intersections then u^{**} is a countably additive measure on $\sigma(L)$.*

4. Coallocation and the extension of K inner regular measures. We will assume throughout this section that $K \subseteq L$, and that any measure on K (or $A(K)$, $R(K)$, $\sigma(K)$, $R_\sigma(K)$) is finite valued on K. In most examples we consider there should be no confusion as to which lattice is used for L in the definition of u^{**}. We specify this lattice only occasionally.

Allocation is defined as in the introduction. A lattice L coallocates K if L' allocates K. Though Dubin's paper deals with bounded measures on a lattice, we state his theorem for any extended real valued measure. His proof remains valid despite the change.

THEOREM 4.1. *Let $\emptyset \in H$; and let J be any other lattice (J need not contain $\emptyset$). The following two statements are equivalent.*

(i) *For every measure u on H, u_* is a measure on J.*

(ii) *J allocates H.*

Proof. Assume J allocates H. Choose from J any j_1, j_2 and choose from H $h \subseteq j_1 \cup j_2$, $l \subseteq j_1 \cap j_2$. Then since J allocates H there exists

$p_1, p_2 \in H$ such that

(1) $p_1 \subseteq j_1$, $p_2 \subseteq j_2$

(2) $p_1 \cup p_2 = h \cup l$, $p_1 \cap p_2 \supseteq l$.

Therefore $u_*(j_1 \cup j_2) + u_*(j_1 \cap j_2) \leqq u_*(j_1) + u_*(j_2)$. The reverse inequality is always true. Thus (ii) implies (i).

Assuming J does not allocate H, Dubins constructed a measure u on H for which u_* is not a measure on J. Thus (i) implies (ii).

COROLLARY 4.2. *Suppose L coallocates K and $u \in M(K)$. Define u^{**} with respect to L.*

(i) *u^{**} is a complete measure on $\mathscr{E}(u, L') \supseteq A(L)$ and is the smallest L' outer regular measure on $\mathscr{E}(u, L')$ such that $u^{**} \geqq u$ on K.*

(ii) *If $u \in M_r(K)$ then $u^{**} \in M_r(L)$ and $u^{**} = u$ on $A(K)$.*

(iii) *If $u \in M_r^s(K)$, u_* σ-smooth on L' and L closed under countable intersections then $u^{**} \in M_r^s(L)$ and $u^{**} = u$ on $A(K)$.*

For any lattice K, $R(K)$ is an ideal in $A(K)$, i.e. $r \cap a$ belongs to $R(K)$ whenever $r \in R(K)$, $a \in A(K)$. Thus $A(K)$ coallocates $R(K)$. Hence for any K inner regular measure u on $R(K)$, u^{**} defined with respect to $A(K)$ is an extension of u to $A(K)$. Since $u^{**} = u_*$, the extension is K inner regular.

In many instances the lattice K' separates the lattice L. A lattice H separates L if whenever $l_1 \cap l_2 = \varnothing$, there exists disjoint sets h_1, h_2 such that $h_1 \supseteq l_1$, $h_2 \supseteq l_2$. H coseparates L if H' separates L.

THEOREM 4.3. *Suppose K coseparates L and $K \subseteq L$. Then L coallocates K.*

Proof. Suppose $l_1' \cup l_2' \supseteq k$. Then $l_1 \cap k$, $l_2 \cap k$ are disjoint members of L. Since K coseparates L there exist disjoint sets k_1', k_2' containing $l_1 \cap k$ and $l_2 \cap k$ respectively. Since $k_1 \subseteq k' \cup l_1'$, $k_1 \cap k \subseteq l_1'$. Similarly $k_2 \cap k \subseteq l_2'$. Now $(k_2 \cap k) \cup (k_1 \cap k) = k$.

Let X be a topological space. We give the following notation for some natural lattices occuring in X. $\mathscr{F}$ is the lattice of closed sets, $\mathscr{Z}$ is the lattice of zero sets, $\mathscr{K}$ is the lattice of compact sets, and $\mathscr{K}_\delta$ is the lattice of compact G_δ sets. If X is a normal space then $\mathscr{Z}$ coseparates $\mathscr{F}$ by Urysohn's lemma. Hence every $u \in M_r(\mathscr{Z})$ extends to $u^{**} \in M_r(\mathscr{F})$.

$\mathscr{F}$ coseparates itself in a normal space and $\mathscr{Z}$ coseparates itself in an arbitrary topological space. Consequently for any $u \in M(\mathscr{Z})$, in any space X, u^{**} is the smallest outer regular measure on $A(\mathscr{Z})$ such that $u^{**} \geqq u$. Here u^{**} is defined with respect to $\mathscr{Z}$.

It will follow from the next theorem that $\mathcal{F}$ coallocates $\mathcal{K}_\delta$ in any completely regular Hausdorff space.

DEFINITION 4.4. A lattice K is an L-ideal if $K \cap L \subseteq K$. $K \cap L = \{k \cap l/k \in K, l \in L\}$.

THEOREM 4.5. *Let K be an H-ideal where $K \subseteq H \subseteq L$. If H coseparates $K \cap L$ then L coallocates K.*

Proof. Let $l_1' \cup l_2' \supseteq k$. Then $(k \cap l_1) \cap (k \cap l_2) = \varnothing$. There exists h_1' and h_2' which are disjoint and contain $k \cap l_1$, and $k \cap l_2$ respectively. Then $h_1 \cap k \subseteq l_1'$, $h_2 \cap k \subseteq l_2'$ and $(h_1 \cup h_2) \cap k = k$. Since K is an H ideal, L coallocates K.

In a completely regular Hausdorff space $\mathcal{K}_\delta$ is a $\mathcal{Z}$-ideal. $\mathcal{Z}$ coseparates the compact sets and therefore $\mathcal{Z}$ certainly coseparates $\mathcal{K}_\delta \cap \mathcal{F}$. Hence $\mathcal{F}$ coallocates $\mathcal{K}_\delta$. Therefore we have the following.

THEOREM 4.6. *Let X be a completely regular Hausdorff space. Suppose $u \in M_r^s(\mathcal{K}_\delta)$. Then $u^{**} \in M_r^s(\mathcal{F})$ and is a $\mathcal{K}_\delta \cap \mathcal{F}$-regular extension of u to $\sigma(\mathcal{F})$.*

Proof. That u^{**} is σ-smooth follows from the fact that $\mathcal{K}_\delta \cap \mathcal{F}$ is a compact lattice (any collection $\{f_\alpha\}$ from the lattice has a nonempty intersection whenever every finite subcollection has a nonempty intersection). The rest of the theorem follows from Corollary 4.2.

The following definition is useful in determining when u^{**} is countably additive.

DEFINITION 4.7. L countably allocates K if whenever $k \subseteq \bigcup_1^\infty l_j$ then there exist $k_i \in K$ such that each k_i is contained in a finite union of the l_j and $\bigcup_1^\infty k_i = k$. If L' countably allocates K then L countably coallocates K.

THEOREM 4.8. *Suppose L countably coallocates K. Consider a countably additive measure u on $\sigma(K)$ (or $R_\sigma(K)$). Then u_* is σ-smooth on L'.*

Proof. Suppose $l' = \bigcup_1^\infty l_j'$ and $\bigcup_1^\infty l_j' \in L'$. Choose $k \subseteq l'$. There exist $k_i \in K$ such that $k_i \subset \bigcup_1^n l_j'$ for some n and $\bigcup_1^\infty k_i = k$. Since u is countably additive, $\lim_i u(k_i) = u(k)$. Thus $u_*(l') \leqq \lim_n u_*(\bigcup_1^n l_j')$. The reverse inequality is always true.

In a locally compact Hausdorff space if $k \subseteq \bigcup_1^\infty o_j$ where the o_j are open, then $k = \bigcup_1^n k_i$, $k_i \in \mathcal{K}_\delta$, and $k_i \subseteq o_j$ for some j. Thus $\mathcal{F}$ countably coallocates $\mathcal{K}_\delta$. Also for every $k \in \mathcal{K}_\delta$ $k \subseteq z_1' \subseteq k_1$ where z_1 is a zero set and $k \in \mathcal{K}_\delta$. Applying Theorems 4.8, 3.3 and 3.5 we obtain the following.

THEOREM 4.9. *Let X be a locally compact Hausdorff space. Every countably additive measure u on $R_\sigma(\mathcal{K}_\delta)$, is $\mathcal{K}_\delta$-inner regular. u^{**} is a countably additive extension of u to $\sigma(\mathcal{F})$.*

Proof. All that has to be shown is that u is $\mathcal{K}_\delta$-inner regular. This follows from the fact that for each $b \in R(\mathcal{K}_\delta)$, $b = \bigcup_1^\infty k_j$, $k_j \in \mathcal{K}_\delta$.

Levin and Stiles **[8]** showed that the conclusions of Theorem 4.9 no longer are true if $R_\sigma(\mathcal{K}_\delta)$ is replaced by $\sigma(\mathcal{K}_\delta)$ even if X is locally compact and Hausdorff. Suppose X is locally compact, paracompact and Hausdorff. Levin and Stiles prove that for any countably additive measure u on $\sigma(\mathcal{K}_\delta)u(b) = \inf\{u(o)/b \subseteq o,\ o \text{ open and } o \in \sigma(\mathcal{K}_\delta)\}$. Thus if u is also $\mathcal{K}_\delta$-inner regular then u^{**} must be a countably additive extension of u to $\sigma(\mathcal{F})$ according to Theorem 3.3. This result is found in the paper of Levin and Stiles.

In a countably paracompact, normal space the lattice $\mathcal{F}$ countably coallocates $\mathcal{Z}$. In any topological space, for every zero set z, $z \subseteq z_1' \subseteq z_2$ where z_1, z_2 are zero sets. Thus we obtain Marik's **[9]** result.

THEOREM 4.10. *Every countably additive measure u on $\sigma(\mathcal{Z})$ is $\mathcal{Z}$-inner regular. If X is countably paracompact and normal then u^{**} is a countably additve extension of u to $\sigma(\mathcal{F})$.*

Let X be a countable product, $\Pi_1^\infty X_k$, of discrete topological spaces. Define for $x = (x_1, \cdots)$, $y = (y_1, \cdots)$ $y = x(\text{mod}\, n)$ if $x_i = y_i$, $i = 1, \cdots, n$. For any subset A of X define $t_A(x)$ to be the least positive integer n, if any, such that $y \in A$ whenever $y = x(\text{mod}\, n)$. If there exists no such n then let $t_A(x) = +\infty$. Suppose $C \subseteq \bigcup_1^\infty O_k$ where C is a clopen set (both closed and open in X) and each O_k is open. Define inductively

$$C_1 = \{c \in C/t_{O_1}(c) \leqq t_{O_k}(c), k \neq 1\},$$
$$C_n = \{c \in C\backslash(\textstyle\bigcup_1^{n-1} C_i)/t_{O_n}(c) \leqq t_{O_k}(c), k \neq n\}.$$

Then $C = \bigcup_1^\infty C_k$, $C_k \subseteq O_k$ for all k and each C_k is clopen. Thus $\mathcal{F}$ countably coallocates $\mathcal{C}\ell$, the lattice of clopen sets. Dubins is interested in measures defined on $\mathscr{C}\ell = A(\mathscr{C}\ell)$. These measures are called strategic measures. Strategic measures are always $\mathscr{C}\ell$-inner regular.

THEOREM 4.11. *Let X be a countable product of discrete topological spaces. For every countably additive strategic measure u, u^{**} is a countably additive extension of u to $\sigma(\mathscr{F})$.*

Let R be a ring of subsets in X. Define $\mathscr{L}(R)$ to be those subsets b such that $b \cap r \in R$ for every $r \in R$. $\mathscr{L}(R)$ is an algebra containing R. $\mathscr{L}(R)$ certainly coallocates R and if R is a σ-ring then $\mathscr{L}(R)$ is an σ-algebra that countably coallocates R. For a measure (not necessarily finite valued on R) define $u_*(b) = \sup\{u(r)/r \subseteq b, r \in R\}$, and u^{**} with respect to $\mathscr{L}(R)$. It is easy to see that $u^{**} = u_*$ on $\mathscr{L}(R)$. By Theorem 3.3 u_* is an extension of u to $\mathscr{L}(R)$. By Theorems 4.8 and 3.3 if R is a σ-ring and u is countably additive then u_* is countably additive on $\mathscr{L}(R)$. $\mathscr{L}(R)$ is called the class of sets locally measurable with respect to R. The result for countably additive measures on a σ-ring is found in a paper by Berberian **[2]**.

If $K \subseteq L$ is an L-ideal, then $A(L) \subseteq \mathscr{L}(R(K))$. Clearly $l \cap r$ belongs to $R(K)$ for all $l \in L$ and $r \in R(K)$. Suppose $b \cap r$ and $c \cap r$ belong to $R(K)$ for all $r \in R(K)$. Then $(b \cup c) \cap r$ belongs to $R(K)$ for all $r \in R(K)$. If $b \cap r \in R(K)$ then $b' \cup r'$ is in $A(K)$. Therefore $r \cap b' = r \cap (b' \cup r')$ belongs to $R(K)$. Thus $A(L)$ is contained in $\mathscr{L}(R(K))$. Also $\sigma(L)$ is contained in $\mathscr{L}(R_\sigma(K))$. Thus in a Hausdorff space $\sigma(\mathscr{F})$ is contained in the locally measurable sets of $R_\sigma(\mathscr{K})$ where $\mathscr{K}$ is the lattice of compact sets [Berberian and Jakobsen **3**]. In a completely regular Hausdorff space $\sigma(\mathscr{Z})$ is contained in the locally measurable sets of $R(\mathscr{K}_\delta)$. We also have, for any lattice K, $A(K) \subseteq \mathscr{L}(R(K))$ and $\sigma(K) \subseteq \mathscr{L}(R_\sigma(K))$. In the following theorems the measures need not be finite on any particular set.

THEOREM 4.12. *Any measure on $R(K)$ extends to a $R(K)$ inner regular measure on $A(K)$. Any countably additive measure on $R_\sigma(K)$ extends to a $R_\sigma(K)$ inner regular, countably additive measure on $\sigma(K)$.*

THEOREM 4.13. *In a Hausdorff space any countably additive measure on $R_\sigma(\mathscr{K})$ has a countably additive, $R_\sigma(\mathscr{K})$ inner regular extension to $\sigma(\mathscr{F})$. In a completely regular Hausdorff space any countably additive measure on $R_\sigma(\mathscr{K}_\delta)$ can be extended to a countably additive, $R_\sigma(\mathscr{K}_\delta)$ inner regular measure on $\sigma(\mathscr{Z})$.*

THEOREM 4.14. *Let $K \subseteq L$ be a L-ideal. Then for every $R(K)$ inner regular measure on $A(K)$ has a $R(K)$ inner regular extension to*

$A(L)$. *Every countably additive, $R_\sigma(K)$ inner regular measure on $\sigma(K)$ has a countably additive, $R_\sigma(K)$ inner regular extension to $\sigma(L)$.*

In view of Theorem 4.14 the next example shows that coallocation is not necessary for every K inner regular measure u on $A(K)$ to have u_* modular on L'. Also countable coallocation is not implied if u_* is σ-smooth on L' for every countably additive K inner regular measure on $R_\sigma(K)$.

Topologize the set of real numbers as follows. For $x \neq 0$ or 2 a neighborhood basis for x is the collection of open intervals containing x. A neighborhood basis for 0 is the collection of open intervals containing 0 and 1. Likewise a neighborhood basis for 2 is the collection of open intervals containing 1 and 2. The interval $[0,2]$ is a compact closed set and the intervals $I_1 = (-1, 3/2)$ and $I_2 = (1/2, 3)$ are open sets. There does not exist a sequence $\{C_n\}$ of closed, compact sets such that $\bigcup_{n=1}^{\infty} C_n = [0,2]$ and each C_n is contained in either I_1 or I_2. Therefore the closed sets $\mathscr{F}$ do not coallocate or countably coallocate the lattice of compact closed sets though this lattice is an $\mathscr{F}$-ideal.

5. The extension of τ-smooth measures. A measure on a lattice L is τ-smooth if for any net $\{l_\alpha\}$ decreasing to $\varnothing$, $\lim_\alpha u(l_\alpha) = 0$. We will study the measures on $A(L)$ which are L inner regular, finite valued on L and τ-smooth on L. Denote these measures by $\mathscr{M}_\tau^i(L)$. $M_\tau^i(L)$ are those measures in $\mathscr{M}_\tau^i(L)$ which are bounded.

For a lattice L, $\tau(L)$ is the smallest lattice containing L that is closed under arbitrary intersections. We now show that every $u \in \mathscr{M}_\tau^i(L)$ extends to u^{**}, defined with respect to $\tau(L)$ on $A(L)$, and τ-smooth on $\tau(L)$.

LEMMA 5.1. *Let u be a measure on $A(L)$, τ-smooth on L. For any t in $\tau(L)$,*

$$u_*(t') = \lim_\alpha u(l'_\alpha)$$

where $t' = \bigcup_\alpha l'_\alpha$ and $\{l'_\alpha\}$ is an increasing net of sets from L'.

Proof. Choose $l \subseteq t'$. Since $t \in \tau(L)$ there exists a net $\{l'_\alpha\}$ from L' which is increasing and $\bigcup_\alpha l'_\alpha = t'$. Since u is τ-smooth, $\lim_\alpha u(l'_\alpha) = u(l) + \lim_\alpha u(l'_\alpha \cap l')$. Therefore $u_*(t') = \lim_\alpha u(l'_\alpha)$.

THEOREM 5.2. ***Suppose u is a measure on $A(L)$, τ-smooth on L. Then u_* is modular on $\tau(L)'$.***

Proof. Let $s, t \in \tau(L)$. Then $s' = \bigcup_\alpha h'_\alpha$, $t' = \bigcup_\beta l'_\beta$ where $\{h'_\alpha\}$, $\{l'_\beta\}$ are increasing nets from L'.

Form the net $\{k'_\gamma\}$ of unions $k'_\gamma = h'_\alpha \cup l'_\beta$. For the same γ, α, and β define $p'_\gamma = h'_\alpha \cap l'_\beta$. $\{k'_\gamma\}$ is a net increasing to $t' \cup s'$ and $\{p'_\gamma\}$ is a net increasing to $t' \cap s'$. Thus

$$\begin{aligned} u_*(t' \cup s') + u_*(t' \cap s') &= \lim_\gamma \, (u(k'_\gamma) + u(p'_\gamma)) \\ &= \lim_\gamma \, (u(h'_\alpha) + u(l'_\beta)) \\ &\leqq u_*(t') + u_*(s'). \end{aligned}$$

THEOREM 5.3. *Let $u \in \mathcal{M}^s_r(L)$. If u^{**} is finite on L then it extends u to $A(\tau(L))$ and belongs to $\mathcal{M}^s_r(\tau(L))$.*

Proof. u^{**} extends u according to Theorems 5.2 and 3.4. u^{**} is τ-smooth and finite on $\tau(L)$ since each $t \in \tau(L)$ is the intersection of sets from L. Consider t, s from $\tau(L)$. Choose v from $\tau(L)$ such that $s \subseteq v'$ and $u^{**}(v') - u^{**}(s) < \epsilon$. Then $u^{**}(t \cap s') - u^{**}(t \cap v) < \epsilon$. Every set in $A(\tau(L))$ is of the form $\bigcup_1^n t_k \cap s'_k$ where s_k belongs to $\tau(L)$ and either $t_k \in \tau(L)$ or $t_k = X$. Therefore u^{**} is $\tau(L)$ inner regular.

COROLLARY 5.4. *Let u be a L inner regular, countably additive measure on $R_\sigma(L)$, τ-smooth and finite on L. If u^{**} is finite on L then u^{**} is a countably additive extension of u to $\sigma(\tau(L))$ and u^{**} is τ-smooth and finite on $\tau(L)$.*

COROLLARY 5.5. *Suppose X is a completely regular space. Suppose u is a L inner regular, countably additive measure on $\sigma(\mathcal{Z})$ that is τ-smooth and finite on $\mathcal{Z}$. Then u^{**} is a countably additive extension of u to $\sigma(\mathcal{F})$ and u^{**} is τ-smooth and finite on $\mathcal{F}$.*

A collection of sets has the finite (countable) intersection property if every finite (countable) subcollection has a nonempty intersection. A lattice L is compact if every collection with the finite intersection property has a nonempty intersection. L is Lindelof if every collection with the countable intersection property has a nonempty intersection. A measure on a compact lattice is always τ-smooth and any σ-smooth measure on a Lindelof lattice is τ-smooth. $\mathcal{M}_r(L)$ are the L inner regular measures on $A(L)$ that are finite on L and $\mathcal{M}^s_r(L)$ are those that are also σ-smooth on L.

COROLLARY 5.6. *If L is compact then every $u \in \mathcal{M}_r(L)$ for which u^{**} is finite on L extends to $u^{**} \in \mathcal{M}_r^i(\tau(L))$. If L is Lindelof then for every $u \in \mathcal{M}_r^s(L)$ such that u^{**} is finite on L, $u^{**} \in \mathcal{M}_r^i(\tau(L))$ and extends u.*

The result concerning compact lattices has been proved by using Zorn's lemma to show that u^{**} on $A(\tau(L))$ is, in an appropriate sense, a maximal extension of u [P. A. Meyer **10**].

Suppose u is a L' outer regular measure on $A(L)$. Then for any decreasing net $\{l_\alpha\}$ from L such that $\bigcap_\alpha l_\alpha \in A(L)$, $\lim_\alpha u(l_\alpha) = u(\bigcap_\alpha l_\alpha)$. If L is a regular lattice then this property is a sufficient condition for a measure u to be L' outer regular.

DEFINITION 5.7. L is K regular if for any $l \in L$ there exists $\{h_\alpha\}$ from L such that $l = \bigcap_\alpha h_\alpha$ and for each α there exists k_α from K such that $l_\alpha \subseteq k'_\alpha \subseteq h_\alpha$. If $L = K$ then L is a regular lattice.

THEOREM 5.8. *Assume L is K regular and that $K \subseteq A(L)$. If for any net $\{l_\alpha\}$ decreasing to $\bigcap_\alpha l_\alpha \in A(L)$, $\lim_\alpha u(l_\alpha) = u(\bigcap l_\alpha)$ then u is K' outer regular on L. If $K = L$ then u is L' outer regular on $A(L)$. In addition, if u is finite on L and L is regular, then u is L inner regular on $A(L)$.*

Proof. The collection $\{l_\alpha\} \subseteq L$ such that $l_\alpha \supseteq k'_\alpha \supseteq l$, is a net decreasing to l. Therefore

$$\begin{aligned} u(l) &\leqq \inf\{u(k'_\alpha)/l \subseteq k'_\alpha \subseteq l_\alpha\} \\ &\leqq \inf\{u(l_\alpha)/l \subseteq k'_\alpha \subseteq l_\alpha\} \\ &= u(l). \end{aligned}$$

To give a similar result for measures on $\sigma(L)$ we need the following theorem. $\delta(L)$ is the smallest lattice containing L closed under countable intersections.

THEOREM 5.9. *Let u be a countably additive, σ-finite measure on a ring R containing L. If u is L inner regular then the countably additive extension of u to $R_\sigma(R)$ is $\delta(L)$ inner regular.*

Proof. Let S be the collection of sets s in $R_\sigma(R)$ such that $u(s) = \sup\{u(k)/l \subseteq s, \ l \in \delta(L)\}$. Then $R \subseteq S$. Let $\{s_k\}$ be any sequence from S such that $u(s_k)$ is finite for all k. Then since u is countably additive, $\bigcup_1^\infty s_k$ and $\bigcap_1^\infty s_k$ belong to S.

Take any set b in $R_\sigma(R)$ such that $u(b)$ is finite. There exists a sequence $\{r_k\}$ from R such that $r = \bigcup_1^\infty r_k$ contains b and $u(r) - u(b) < \epsilon$. There exists $\{t_k\}$ from R such that $t = \bigcup_1^\infty t_k$ contains $r \backslash b$ and $u(t) < \epsilon$. Then $r \backslash t \subseteq b$ and $u(b) - u(r \backslash t) < \epsilon$. For each k, $r \backslash t_k = \bigcup_{j=1}^\infty r_j \backslash t_k$ belongs to S. Since $r \backslash t = \bigcap_1^\infty r \backslash t_k$, $r \backslash t$ belongs to S. This implies that b belongs to S.

Every $b \in R_\sigma(R)$ is the countable union of sets b_k such that $u(b_k)$ is finite. Therefore $R_\sigma(R) = S$. A similar proof shows the extension of u is $\delta(L)'$ outer regular when u is L' outer regular.

THEOREM 5.10. *Suppose L is a regular lattice. Let u be a countably additive, σ-finite measure on $\sigma(L)$, finite on L. If for any net $\{l_\alpha\}$ decreasing to $\bigcap_\alpha l_\alpha \in A(L)$, $\lim_\alpha u(l_\alpha) = u(\bigcap_\alpha l_\alpha)$, then u is $\delta(L)$ regular on $\sigma(L)$.*

COROLLARY 5.11. *Let X be a topological space and u a countably additive, finite measure defined on $\sigma(\mathcal{F})$ such that for any decreasing net of closed sets $\{f_\alpha\}$*

$$\lim_\alpha u(f_\alpha) = u\left(\bigcap_\alpha f_\alpha\right).$$

(i) *If X is a regular space then u is $\mathcal{F}$ regular.*

(ii) *If X is completely regular then u is $\mathcal{F} \cap \mathcal{Z}$-regular and for every closed set f*

$$u(f) = \inf\{u(z')/f \subseteq z', z \in \mathcal{Z}\}.$$

(iii) *If X is 0-dimensional then u is $\mathcal{F} \cap \mathcal{C}\ell$ regular where $\mathcal{C}\ell$ is the lattice of clopen sets and for every closed set f*

$$u(f) = \inf\{u(c)/f \subseteq c, c \text{ clopen}\}.$$

(iv) *If X is a locally compact Hausdorff space then u is $\mathcal{K}_\delta \cap \mathcal{F}$ regular and for every closed set f*

$$u(f) = \inf\{u(k')/f \subseteq k', k \in \mathcal{K}_\delta\}.$$

COROLLARY 5.12. *Suppose X is a locally compact Hausdorff space and u a countably additive, finite measure on $\sigma(\mathcal{Z})$ such that for any decreasing net $\{z_\alpha\}$ of zero sets, where $\bigcap_\alpha z_\alpha \in A(\mathcal{Z})$,*

$$\lim_{\alpha} u(z_\alpha) = u\left(\bigcap_{\alpha} z_\alpha\right).$$

Then u is $\mathcal{K}_\delta$ regular.

Part (i) of 5.11 was proven by Gardner **[6]**.

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