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PRINCIPAL IDEAL AND NOETHERIAN GROUPS

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Let Π be a ring property. An additive group G is said to be an (associative) strongly Π-group if G is not nil, and if every (associative) ring R with additive group G such that R is not a zeroring has property Π. The (associative) strongly principal ideal groups, and the (associative) strongly Noetherian groups are classified for groups which are not torsion free. Some results are also obtained for the torsion free case.

(i) All groups considered here are abelian, with addition the group operation. Rings are not necessarily associative.

Let π be a ring property. A group G is said to be an (associative) π-group, denoted by $(A)\pi$-group, if there exists an (associative) ring R with additive group G such that R is not a zeroring, and R has property π. G is an (associative) strongly π-group, denoted by $(A)S\pi$-group. if G is an $(A)\pi$-group, and every (associative) ring with additive group G which is not the zeroring on G has property π.

If the only (associative) ring with additive group G is the zeroring, then G is said to be an (associative) nil group, denoted by (A) nil group.

The two ring properties π considered in this paper are:
1. every two-sided ideal is principal, denoted by *PI*,
2. every two-sided ideal is finitely generated, denoted by *N*.

In (ii) a complete characterization of the torsion $(A)SPI$ groups will be given. It will be shown that there are no mixed $(A)SPI$ groups. Some results concerning torsion free $(A)SPI$ groups will be obtained. In (iii), the torsion, and mixed *SN* groups will be completely characterized. Some results concerning torsion free *SN* groups will be given.

(ii) If X is a nonempty subset of a group or ring, (X) denotes the additive subgroup generated by X, and $\langle X\rangle$ denotes the ideal generated by X.

If $G = G_1 \oplus G_2$ is a group, π_{G_i} is the natural projection of G on G_i, for $i = 1, 2$.

LEMMA 1. *Let $G = H \oplus K, H \neq 0, K \neq 0$, be an ASPI-group. Then H and K are either both cyclic or both A nil.*

Proof. Suppose that H is not A nil. Let S be an associative ring on H which is not the zeroring on H, and let T be the zeroring on K. The ring direct sum $R = S \oplus T$ is an associative ring on G, which is not the zeroring on G. T is an ideal in R, and hence $T = \langle x \rangle$. Clearly $K = (x)$. Therefore K is not A nil. The above argument, interchanging the roles of H and K, yields that H is cyclic.

COROLLARY. *Let $G = H \oplus K, H \neq 0, K \neq 0$ be an SPI-group. Then H and K are cyclic.*

Proof. It suffices to negate that H and K are both A nil. Suppose this is so. Let R be a ring on G which is not the zeroring on G.

(1) Suppose that $R^2 \subseteq K$. There exist $h_0 \in H$, $k_0 \in K$ such that $R = \langle h_0 + k_0 \rangle$. Let $h \in H$. Since $h \in R$, there exists an integer n, and $x \in R^2$ such that $h = n(h_0 + k_0) + x$. However, $x \in K$. Hence $h = nh_0$, and H is cyclic, contradicting the fact that H is A nil.

(2) Suppose that $R^2 \not\subseteq K$. For all $g_1, g_2 \in G$, define $g_1 \times g_2 = \pi_H(g_1 g_2)$. Then $S = (G, \times)$ is a not necessarily associative ring on G, which is not the zeroring on G, satisfying $S^2 \subseteq H$. The argument employed in (1) yields that K is cyclic, contradicting the fact that K is A nil.

THEOREM 1. *Let G be a nonzero torsion group. The following are equivalent*:

(1) *Either G is cyclic, or $G \simeq Z(p) \oplus Z(p)$ for a prime p.*

(2) *G is SPI.*

(3) *G is ASPI.*

Proof. (1) $\Rightarrow$ (2): Nontrivial cyclic groups are clearly *SPI*. Suppose that $G = (x_1) \oplus (x_2)$ with $|x_i| = p$ a prime, $i = 1, 2$. Let R be a ring on G which is not the zeroring on G, and let l be a proper ideal in R. Then $|l| = 0$ or p, and hence l is generated by a single element. It therefore suffices to show that R is generated by a single element. We may assume that $R \neq \langle x_1 \rangle$, and that $R \neq \langle x_2 \rangle$. Hence $\langle x_i \rangle = (x_i)$ for $i = 1, 2$. This implies that

$$x_i x_j = \begin{cases} k_i x_i, 0 < k_i < p, & \text{if } i = j, i = 1, 2 \\ 0 & \text{if } i \neq j, i, j = 1, 2. \end{cases}$$

Put $l = \langle x_1 + x_2 \rangle$. Let r, s be integers such that $rk_1 + sp = 1$. Then $rx_1(x_1 + x_2) = rk_1 x_1 = (1 - sp)x_1 = x_1$. Hence $x_1 \in l$, and so $(x_1 + x_2) - x_1 = x_2 \in l$. Therefore $l = R$.

(2) $\Rightarrow$ (3): Let G be a torsion *SPI* group. It suffices to show that G

admits an associative, nonzero multiplication. If G is indecomposable then $G \simeq Z(p^n), p$ a prime, and n a positive integer or ∞ [**1**, Corollary 27.4]. If $n = \infty$ then G is nil. If n is a positive integer then G admits a ring structure is isomorphic to Z_{p^n} the ring of integers modulo p^n. If G is decomposable, then by the Corollary to Lemma 1, $G \simeq Z(n) \oplus Z(m)$. Hence G admits a ring structure isomorphic to the ring direct sum $Z_n \oplus Z_m$.

(3) $\Rightarrow$ (1): Suppose that G is *ASPI*. If G is indecomposable, then $G \simeq Z(p^k), p$ a prime, $1 \leqq k \leq \infty$, [**1**, Corollary 27.4]. However $Z(p\infty)$ is A nil [**3**, Satz 1, and Zusatz]. Hence G is cyclic. By Lemma 1 and [**1**, Theorem 120.3] we may assume $G = (x_1) \oplus (x_2)$ with $|x_i| = n_i, i = 1, 2$. If $(n_1, n_2) = 1, G$ is cyclic; otherwise, let p be a prime divisor of (n_1, n_2). Then $G = (y_1) \oplus (y_2) \oplus H$, with $|y_i| = p^{m_i}, i = 1, 2$, and $1 \leqq m_1 \leqq m_2$. Since $(y_1) \oplus (y_2)$ is neither cyclic nor A nil, $H = 0$ by Lemma 1.

Now let R be the ring on G with multiplication defined by $y_i y_j = p^{m_2 - 1} y_2$ for $i, j = 1, 2$. Then R is an associative ring on G which is not a zeroring, so $R = \langle s_1 y_1 + s_2 y_2 \rangle$ for some $s_1, s_2 \in Z$. Every $x \in R$ has the form $x = k_x s_1 y_1 + (k_x s_2 + m_x p^{m_2 - 1}) y_2$, for some $k_x, m_x \in Z$. In particular, $y_1 = k_{y_1} s_1 y_1$, and $y_2 = (k_{y_2} s_2 + m_{y_2} p^{m_2 - 1}) y_2$. Hence if $m_2 > 1, k_{y_1} s_1 \equiv 1 (\mathrm{mod}\, p)$, $k_{y_2} s_2 + m_{y_2} p^{m_2 - 1} \equiv 1 (\mathrm{mod}\, p)$, so $p \nmid k_{y_1}$ and $p \nmid s_2$. However $k_{y_1} s_2 + m_{y_1} p^{m_2 - 1} \equiv 0 (\mathrm{mod}\, p)$, so that either $p \mid k_{y_1}$, or $p \mid s_2$. This is a contradiction, so $m_2 = 1 = m_1$.

THEOREM 2. *There are no mixed ASPI-groups.*

Proof. Let G be a mixed *ASPI*-group. G is decomposable, [**1**, Corollary 27.4], so by Lemma 1, $G = H \oplus K, H \neq 0, K \neq 0$, with H and K either both cyclic or both A nil.

(1) Suppose that H and K are both A nil. There are no mixed A nil groups [**4**, hilfssatz 9] so we may assume that H is a torsion group, and that K is torsion free. Let R be an associative ring on G, such that R is not a zeroring. Clearly H is an ideal in R, and so $H = \langle h \rangle$. Let $m = |h|$, so $mh = 0$. There are no nontrivial, bounded A nil-groups [**3**, Satz 1 and Zusatz], a contradiction.

(2) Suppose that $H = (x)$, and $K = (e)$ with $|x| = n < \infty$, and $|e| = \infty$. Let R be the ring on G with multiplication induced by $x^2 = xe = ex = 0, e^2 = ne$. Clearly, R is an associative ring on G, and R is not a zeroring. Hence $R = \langle sx + te \rangle, s, t \in Z$. Every $y \in R$ is of the form $y = m_y sx + (m_y + u_y n)te, m_y, u_y \in Z$. In particular, $(m_e + u_e n)t = 1$. Hence $t = \pm 1$. Therefore $m_x + u_x n = 0$, so that $n \mid m_x$. However $x = m_x sx = 0$, a contradiction.

THEOREM 3. *Let G be a torsion free, ASPI-group. Then G is either indecomposable, or the direct sum of two A nil-groups.*

Proof. By Lemma 1, it suffices to negate that $G = (x_1) \oplus (x_2)$, $x_i \neq 0$, $i = 1, 2$. Suppose this is so. Let R be the ring on G with multiplication induced by

$$x_i x_j = \begin{cases} 3x_i & \text{if } i = j, i = 1, 2 \\ 0 & \text{if } i \neq j, i, j = 1, 2 \end{cases}$$

Then $R = \langle k_1 x_1 + k_2 x_2 \rangle$, $k_i \in Z$, $k_i \neq 0$, $i = 1, 2$. Every $x \in R$ is of the form

$$x = (r_x + 3s_x)k_1 x_1 + (r_x + 3t_x)k_2 x_2,\ r_x, s_x, t_x \in Z.$$

$r_{x_1} + 3s_{x_1} = \pm 1$, so that $r_{x_1} \equiv \pm 1 (\mathrm{mod}\, 3)$. However, $r_{x_1} + 3t_{x_1} = 0$, so that $r_{x_1} \equiv 0 (\mathrm{mod}\, 3)$, a contradiction.

COROLLARY. *Let G be a torsion free SPI-group. Then G is indecomposable.*

Proof. Theorem 3, and the Corollary to Lemma 1.

The above Corollary yields that if G is a torsion free *SPI*-group then either $G \simeq Q$, the group of rational numbers, or G is reduced, [**1**, Theorem 21.3].

THEOREM 4. *Let G be a nonzero torsion group. The following are equivalent*:

(1) *G is bounded.*
(2) *G is an API-group.*
(3) *G is a PI-group.*

Proof. (1) ⇒ (2): Suppose that $nG = 0$, n a positive integer. Then $G \simeq \bigoplus_{p|n} [\bigoplus_{\alpha_k} Z(p^k)]$, p a prime with $p^k \mid n$, and α_k a cardinal number, [**1**, Theorem 17.3, and Theorem 8.4]. For each $p^k | n$, put $H_{p^k} = \bigoplus_{\alpha_k} Z(p^k)$. Then $G \simeq \bigoplus_{p^k | n} H_{p^k}$: There exists an associative unital *PI*-ring R_{p^k} on H_{p^k}, for all $p^k \mid n$, [**1**, Lemma 122.3]. The ring direct sum $R = \bigoplus_{p^k | n} R_{p^k}$ is an *API*-ring on G which is not the zeroring on G [**5**, Chapt. 4, Theorem 33].

(2) ⇒ (3): Obvious.

(3) ⇒ (1): Let R be a *PI*-ring on G which is not the zeroring on G. Then $R = \langle x \rangle$. Let $n = |x|$. Clearly, $nG = 0$.

COROLLARY. *Let G be a mixed group. If G is an (A)PI-group, then $T(G)$ (the torsion part of G) is bounded, and $T/T(G)$ is an (A)PI-group. Conversely, if $T(G)$ is bounded, and if there exists a unital (A)PI-ring on $G/T(G)$, then G is (A)PI.*

Proof. Let G be an $(A)PI$-group, and let R be an $(A)PI$-ring on G which is not a zeroring. Since $T(G)$ is an ideal in R, $T(G) = \langle x \rangle$, and $nT(G) = 0, n = |x|$. Clearly $R/T(G)$ is an $(A)PI$-ring with identity on $G/T(G)$.

Suppose that $T(G)$ is bounded, and that there exists an $(A)PI$-ring with identity on $G/T(G)$. Then $G \simeq T(G) \oplus G/T(G)$, [**1**, Theorem 100.1]. There exists an API-ring R_1 with identity on $T(G)$, [**1**, Lemma 122.3]. Let R_2 be a unital $(A)PI$-ring on $G/T(G)$. Let R be the ring direct sum $R = R_1 \oplus R_2$, with e_i the identity of R_i, $i = 1, 2$. Let I be an ideal in R. Then $I = (I \cap R_1) \oplus (I \cap R_2)$. Since $I \cap R_i$ is an ideal in R_i, $I \cap R_i = \langle x_i \rangle, i = 1, 2$. Clearly, $\langle x_1 + x_2 \rangle \subseteq I$. However $x_i = e_i(x_1 + x_2) \in \langle x_1 + x_2 \rangle$ for $i = 1, 2$. Hence $I = \langle x_1 + x_2 \rangle$.

Additional information concerning PI-groups and the classification of π-groups for other ring properties π may be found in [**2**].

LEMMA 2. *If a group G is finitely generated, then G is SN.*

Proof. Obvious.

LEMMA 3. *Let a group $G = H \oplus K, H \neq 0, K \neq 0$, be SN. Then either G is finitely generated, or H and K are both nil.*

Proof. Suppose that H is not nil. Let S be a nonzeroring on H, and let T be the zeroring on K. The ring direct sum $R = S \oplus T$ is a ring on G which is not the zeroring, with ideal T. Let $t_1, \cdots, t_n$ be a finite set of generators for T. Then $K = (t_1, \cdots, t_n)$. This implies that K is not nil. The same argument, interchanging the roles of H and K, yields that H is finitely generated. Hence G is finitely generated.

THEOREM 5. *Let G be a non torsion free group. G is SN if and only if G is finitely generated.*

Proof. By Lemma 2, it suffices to show that if G is non torsion free and SN, then G is finitely generated.

(1) Suppose that G is a torsion group. If G is indecomposable then G is cyclic [**1**, Corollary 27.4 and Theorem 120.3]. We may assume, by Lemma 3, that $G = H \oplus K, H \neq 0, K \neq 0$, with H and K both nil. This implies that G is nil [**1**, Theorem 120.3]. A contradiction.

(2) Suppose that G is a mixed group. Then G is decomposable [**1**, Corollary 27.3]. By Lemma 3, it suffices to negate that $G = H \oplus K, H \neq 0, K \neq 0$, with H and K both nil. Suppose that this is so. By [**1**, Theorem 120.3] we may assume that H is a torsion group and that K is torsion free. However, by [**1**, Proposition 126.2] H is bounded and hence not nil, a contradiction.

COROLLARY. *Let G be an SN group. Then $T(G)$ and $G/T(G)$ are SN.*

Proof. If G is torsion free then the statement is trivial. Otherwise, G is finitely generated by Theorem 5, and so $T(G)$ and $G/T(G)$ are SN by Lemma 2.

A torsion-free SN-group need not be finitely generated; e.g. Q the group of rational numbers. However, we have the following:

THEOREM 6. *Let G be an SN-group. Then G is either indecomposable, or finitely generated.*

Proof. By Lemma 3, it suffices to negate that $G = H \oplus K, H \neq 0, K \neq 0$, with H and K both nil. Suppose this is so. Let R be a ring on G which is not a zeroring. Then $R = \langle x_1, \cdots, x_n \rangle$, n a positive integer. Put $x_i = h_i + k_i$, $h_i \in H$, $k_i \in K$, $1 \leqq i \leqq n$.

(1) Suppose that $R^2 \subseteq K$. Let $h \in H$. Since $h \in R$, $h = \sum_{i=1}^n r_i(h_i + k_i) + x$, $r_i \in Z$, $1 \leqq i \leqq n$, $x \in R^2$. However $R^2 \subseteq K$. Hence $h = \sum_{i=1}^n r_i h_i$, and H is finitely generated. This contradicts the fact that H is nil.

(2) Suppose that $R^2 \not\subseteq K$. For $g_1, g_2 \in G$ define $g_1 \times g_2 = \pi_H(g_1 g_2)$. Then $S = (G, \times)$ is a ring on G which is not a zeroring satisfying $S^2 \subseteq H$. The above argument yields that K is finitely generated, contradicting the fact that K is nil.

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