ON COVERINGS OF EUCLIDEAN SPACE BY CONVEX SETS

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Let $\mathcal{K} = \{K_1, K_2, \cdots\}$ be an infinite countable class of compact convex subsets of euclidean n-dimensional space R^n. We shall say that $\mathcal{K}$ permits a space covering or, more precisely, a covering of R^n, if there are rigid motions $\sigma_1, \sigma_2, \cdots$ such that $R^n \subset \bigcup_{i=1}^{\infty} \sigma_i K_i$. In this paper we concern ourselves with necessary and sufficient conditions in order that a given class $\mathcal{K}$ permits a space covering.

If the set of diameters $\{d(K_i): K_i \in K\}$ is bounded, the problem has already been solved in [3] by showing that in this case $\mathcal{K}$ permits a covering of R^n if and only if the series $\sum_{i=1}^{\infty} v(K_i)$ of volumes $v(K_i)$ diverges. (The same result holds obviously without any restrictions on the diameters if $n = 1$.) On the other hand, if $\{d(K_i)\}$ is unbounded and $n > 1$, it is not difficult to see (cf. [1] and [2]) that the divergence of this series is no longer sufficient but only necessary. Only in the special case $n = 2$ are some necessary and sufficient conditions known [2].

Our principal results are stated in the following §2. Theorem 1 gives an inductive criterion that enables one to decide whether a given $\mathcal{K}$ permits a space covering. Theorems 2 and 3 serve the same purpose but are of a more explicit nature, involving the divergence of infinite series of geometric invariants associated with the members of $\mathcal{K}$. Other results, regarding coverings by balls, boxes (i.e. isometric images of n-dimensional intervals), and 2-dimensional sets, are stated and discussed in the same section. This is followed by the proofs of three lemmas in §3. Lemma 1 appears to be of some independent interest. §4 contains the proofs of our theorems.

2. Theorems and corollaries.

A nonempty compact convex set will be called a convex body. If K is a convex body in R^n, and if p, q are two points of K such that the length of the segment $[p, q]$ is equal to the diameter $d(K)$, then we call the orthogonal projection of K onto a hyperplane perpendicular to $[p, q]$ a normal projection of K. Of course, a normal projection of K is not uniquely determined by K. However, if $\{K_i\}$ is given we shall always assume that for each K_i a definite normal projection $N(K_i)$ has been selected and is kept fixed. Since each $N(K_i)$ is at most $(n-1)$-dimensional it is clear (using a self-explanatory extension of our original definition) what is meant by saying that the class

$\{N(K_i)\}$ permits a covering of R^{n-1}.   When a class $\{K_i\}$ is mentioned it is always assumed that $i = 1, 2, \cdots$.

THEOREM 1.   *Let* $\mathcal{K} = \{K_i\}$ *be a class of convex bodies* K_i *in* R^n ($n \geq 2$) *with diameters* $d(K_i)$, *volumes* $v(K_i)$, *and normal projections* $N(K_i)$. *Then,* $\mathcal{K}$ *permits a covering of* R^n *if and only if either* $\Sigma_{d(K_i)\leq1} v(K_i) = \infty$ *or* $\{N(K_i): d(K_i) > 1\}$ *permits a covering of* R^{n-1}.

If $n \geq 3$ it is clearly possible to apply the theorem again to the set $\{N(K_i): d(K_i) > 1\}$, and so on.   In this way one is lead to a criterion, expressing the property that $\mathcal{K}$ permits a space covering, entirely in terms of the convergence properties of infinite series.   To formulate a result of this kind we introduce the following notations.   For any convex body K in R^n ($n \geq 0$) we set

$$N^0(K) = K, \quad \text{and} \quad N^m(K) = N(N^{m-1}(K)) \qquad (m = 1, 2, \cdots, n).$$

Furthermore, if $m = 0, 1, \cdots, n$ we write

$$d^m(K) = d(N^m(K)) \quad \text{and} \quad v^m(K) = v(N^m(K)),$$

where v denotes the $(n - m)$-dimensional volume of $N^m(K)$, and define $v^n(K) = 1$, $d^n(K) = 0$.   Obviously, $d^0(K) \geq d^1(K) \geq \cdots \geq d^n(K) = 0$. Finally, if $n \geq 1$, we partition the class $\{K_i\}$ of the preceding theorem into $n + 1$ subclasses by the following definitions:

$$S^0 = \{K_i: d^0(K_i) \leq 1\},$$

$$S^m = \{K_i: d^m(K_i) \leq 1, d^{m-1}(K_i) > 1\} \ (m = 1, 2, \cdots, n).$$

Using these definitions we can now state:

THEOREM 2.   *A class* $\{K_i\}$ *of convex bodies* K_i *in* R^n *permits a covering of* R^n ($n \geq 1$) *if and only if one of the series* $\Sigma_{K_i \in S^m} v^m(K_i)$ ($m = 0, 1, 2, \cdots, n$) *diverges.*

The condition in the preceding theorem can be expressed in terms of the divergence of a single series. This in turn can be expressed in terms of series involving the Minkowskian cross-sectional measures (Quermassintegrale) $W_j(K_i)$ of the convex bodies K_i, as defined, for example, in[5, p. 209].

THEOREM 3.   *Let* $\{K_i\}$ *be a given class of convex bodies in* R^n ($n \geq 1$), *and let* d_i^j *be defined by* $d_i^j = d^j(K_i)$. *Then, each of the following*

three conditions is necessary and sufficient that $\{K_i\}$ *permits a covering of* R^n.

(a)
$$\sum_{i=1}^{\infty} \frac{d_i^0 d_i^1 \cdots d_i^{n-1}}{1 + d_i^0 + d_i^0 d_i^1 + d_i^0 d_i^1 d_i^2 + \cdots + d_i^0 d_i^1 d_i^2 \cdots d_i^{n-2}} = \infty,$$

(b)
$$\sum_{i=1}^{\infty} \left(\frac{1}{v^0(K_i)} + \frac{1}{v^1(K_i)} + \cdots + \frac{1}{v^{n-1}(K_i)} \right)^{-1} = \infty,$$

(c)
$$\sum_{i=1}^{\infty} \frac{W_0(K_i)}{W_1(K_i) + W_2(K_i) + \cdots + W_n(K_i)} = \infty.$$

To illustrate the applicability of these theorems consider first the case $n = 2$. We have $S^0 = \{K_i : d(K_i) \leqq 1\}$, $S^1 = \{K_i : d(K_i) > 1, d(N(K_i)) \leqq 1\}$, and $S^2 = \{K_i : d(N(K_i)) > 1\}$. Theorem 2 shows (combining S^1 and S^2) that $\{K_i\}$ permits a covering of R^2 if and only if either $\Sigma_{d(K_i) \leqq 1} v(K_i) = \infty$ or $\Sigma_{d(K_i) > 1} v(N(K_i)) = \infty$. The latter of these two conditions can also be written as $\Sigma_{d(K_i) > 1} w(K_i) = \infty$, where $w(K_i)$ is the (minimum) width of K_i. This follows from the easily established fact that there are constants c_1, c_2 such that $c_1 v(N(K_i)) \leqq w(K_i) \leqq c_2 v(N(K_i))$. These inequalities are, for example, a consequence of Lemma 1. Thus, we have the following corollary, which expresses one of the main results of [2].

COROLLARY 1. *Let* $\mathcal{K} = \{K_i\}$ *be a class of convex bodies in* R^2, *where* K_i *has content* $v(K_i)$ *and* (*minimum*) *width* $w(K_i)$. *Then* $\mathcal{K}$ *permits a covering of* R^2 *if and only if either* $\Sigma_{d(K_i) \leqq 1} v(K_i) = \infty$ *or* $\Sigma_{d(K_i) > 1} w(K_i) = \infty$.

It should also be noted that the conditions given in [2] in order that $\{K_i\}$ permit a covering of R^2, involving the divergence of series constructed from the areas, perimeters, and diameters of the K_i, are immediate special cases of Theorem 3.

As another application we consider a class $\{B_i\}$ of closed balls B_i in R^n. In this case all the sets S^m are seen to be empty, except S^0 or S^n. If d_i is the diameter of B_i the two resulting series in Theorem 2 are (disregarding constant factors) $\Sigma_{d_i \leqq 1} v(B_i)$ and $\Sigma_{d_i > 1} 1$. The latter of these series diverges if and only if $\Sigma_{d_i > 1} v(B_i)$ diverges (viz., if and only if there are infinitely many B_i with $d_i > 1$). Hence we find:

COROLLARY 2. *Let* $\mathcal{B} = \{B_i\}$ *be a collection of closed balls* B_i *in* R^n. *Then,* $\mathcal{B}$ *permits a covering of* R^n *if and only if* $\Sigma_{i=1}^{\infty} v(B_i) = \infty$.

This corollary shows also that the condition $\Sigma v(K_i) = \infty$, which is always necessary, and has been proved in [3] to be sufficient if the

diameters are uniformly bounded, is also sufficient if the sets K_i do not degenerate, in the sense that every K_i has a circumscribed sphere and an inscribed sphere whose ratio of radii is bounded by a constant independent of i.

For the case of ellipsoids the following corollary is another immediate consequence of Theorem 2.

COROLLARY 3. *Let* $\mathscr{E} = \{E_i\}$ *be a class of* n-*dimensional ellipsoids, where* E_i *has lengths of principal axes* $e_i^1 \geq e_i^2 \geq \cdots \geq e_i^n$. *Then* $\mathscr{E}$ *permits a covering of* R^n *if and only if one of the series*

$$\sum_{e_i^1 \leq 1} e_i^1 e_i^2 \cdots e_i^n, \quad \sum_{\substack{e_i^1 > 1 \\ e_i^2 \leq 1}} e_i^2 e_i^3 \cdots e_i^n, \cdots, \sum_{\substack{e_i^{n-2} > 1 \\ e_i^{n-1} \leq 1}} e_i^{n-1} e_i^n, \quad \sum_{e_i^{n-1} > 1} e_i^n,$$

diverges.

Finally we consider the case of boxes. If one wishes to apply Theorem 2 directly to this situation one meets the cumbersome problem of finding the normal projections (which are not projections in the directions of the edges). A more satisfactory solution is obtained by first inscribing in each box an ellipsoid of maximum volume and circumscribing an ellipsoid of minimal volume (considering first the case of a cube and then applying affine transformations). It is seen immediately that the boxes permit a covering if and only if the corresponding ellipsoids do. Furthermore, it is easily seen that the edges of the boxes and the corresponding lengths of the axes of the ellipsoids are proportional by a constant depending on n only. Hence, we obtain from Corollary 3:

COROLLARY 4. *Let* $\mathscr{I} = \{I_i\}$ *be a class of boxes, where* I_i *has edge lengths* $a_i^1 \geq a_i^2 \geq \cdots \geq a_i^n$. *Then* $\mathscr{I}$ *permits a covering of* R^n *if and only if one of the series*

$$\sum_{a_i^1 \leq 1} a_i^1 a_i^2 \cdots a_i^n, \quad \sum_{\substack{a_i^1 > 1 \\ a_i^2 \leq 1}} a_i^2 a_i^3 \cdots a_i^n, \cdots, \sum_{\substack{a_i^{n-2} > 1 \\ a_i^{n-1} \leq 1}} a_i^{n-1} a_i^n, \quad \sum_{a_i^{n-1} > 1} a_i^n,$$

diverges.

For example, the class of boxes with edge lengths $a_i^1 = i^2$, $a_i^2 = 1/\sqrt{i}$, $a_i^3 = 1/i$ does not permit a covering of R^3 although $\Sigma v(K_i) = \infty$, but the class with $a_i^1 = \sqrt{i}$, $a_i^2 = \sqrt{i}$, $a_i^3 = 1/i$ does.

3. Three lemmas.

LEMMA 1. *Let K be a convex body in R^n that is contained in a right cylinder Z of height $d(K)$ and base $N(K)$. Then K contains a translate of $\frac{1}{3}Z$.*

Proof. There is no loss in generality by assuming that the points $a = (0, 0, \cdots, 0, \alpha)$ $(\alpha \geqq 0)$ and $-a$ are in K and have the property that $2\alpha = d(K)$. We may also assume that $N(K)$ lies in the plane $\{(x^1, x^2, \cdots, x^n): x^n = 0\}$. The cylinder Z can then be defined by $Z = N(K) + [-a, a]$, and an arbitrary point $z \in Z$ can be represented in the form

$$(1) \qquad z = sa + p \qquad (p \in N(K), -1 \leqq s \leqq 1).$$

Since the line through z orthogonal to $N(K)$ contains a point of K, say k, we have

$$(2) \qquad k = ra + p \qquad (-1 \leqq r \leqq 1)$$

with the same p as in (1). From (1) and (2) we obtain

$$z = sa + p = sa + k - ra = k + \left(1 + \frac{s-r}{2}\right)a + \left(1 - \frac{s-r}{2}\right)(-a).$$

Because of $|s| \leqq 1$, $|r| \leqq 1$ we have $1 + (s-r)/2 \geqq 0$, $1 - (s-r)/2 \geqq 0$. Furthermore, we find $1 + (1 + (s-r)/2) + (1 - (s-r)/2) = 3$. Hence, the point $\frac{1}{3}z$ is a convex combination of the points $k, a, -a$ of K, and consequently $\frac{1}{3}z \in K$. Thus $\frac{1}{3}Z \subset K$ and the proof of the lemma is complete.

To formulate our second lemma we define a generalized k-cylinder as the vector sum $F + L$, where F is a nonempty convex body of dimension not greater than k $(1 \leqq k \leqq n)$, and L is an $(n - k)$-dimensional linear subspace that is orthogonal to F. The set F will be called a base of the cylinder.

LEMMA 2. *Let C be a generalized k-cylinder $(1 \leqq k \leqq n)$ with base F of k-dimensional volume $v_k(F)$, and let B_r denote the ball in R^n centered at the origin and of radius r. Then,*

$$(3) \qquad v(C \cap B_r) \leqq \omega_{n-k} v_k(F) r^{n-k},$$

where ω_{n-k} is the volume of the $(n - k)$-dimensional unit ball in R^{n-k} $(\omega_0 = 1)$.

Proof. If the dimension of F is less than k then $v(C) = 0$ and consequently $v(C \cap B_r) = 0$. Since (3) is obviously satisfied in this case, we may now assume that F has dimension equal to k. Then, for each point $x \in F$ there exists a unique $(n - k)$-dimensional affine subspace L_x that contains x and is orthogonal to F. Furthermore,

(4)
$$v(C \cap B_r) = \int_{x \in F} v_{n-k}(L_x \cap B_r)\,dx,$$

where v_{n-k} denotes the $(n-k)$-dimensional volume. Since $L_x \cap B_r$ is an $(n-k)$-dimensional sphere (or a single point, or empty) of radius at most r, we have $v_{n-k}(L_x \cap B_r) \leqq \omega_{n-k} r^{n-k}$, and (3) follows immediately from this inequality and (4).

For the formulation and the proof of our third lemma it is advantageous to introduce the following notation. Let ϕ and ψ be two real valued functions on the class of all convex bodies of R^n. Then we write $\phi \approx \psi$ or $\phi(K) \approx \psi(K)$ if there are positive constants (depending on ϕ and ψ only) such that $a\phi(K) \leqq \psi(K) \leqq b\phi(K)$ for all convex bodies K of R^n.

LEMMA 3. *Let* m *be an integer with* $0 \leqq m \leqq n$. *Then,*

(5)
$$v^m \approx d^m d^{m+1} \cdots d^{n-1}$$

and

(6)
$$W_m \approx d^0 d^1 \cdots d^{n-m-1},$$

where the right hand sides of these inequalities have to be set equal to 1 *if* $m = n$.

Proof. First we note that the case $m = n$ is completely trivial and can therefore be excluded from our further considerations. It is also clear that it suffices to prove (5) only in the case when $m = 0$, since $v^m(K) \approx d^m(K)d^{m+1}(K) \cdots d^{n-1}(K)$ in R^n expresses the same as $v^0(N^m(K)) \approx d^0(N^m(K)) \cdots d^{n-m-1}(N^m(K))$ in R^m. If $n = 1$ we have $v^0 = d^0$ and $W_0 = d^0$ which shows that both (5) and (6) are true in R^1. We make now the induction assumption that (5) (for $m = 0$) and (6) be also true in spaces of dimensions less than n. Let K be an arbitrary convex body of R^n. From Lemma 1 and the fact that $v(Z) = d^0(K)v^1(K)$ we obtain

$$v^0 \approx d^0 v^1.$$

(5) follows now from this relation and the fact that the induction assumption yields

$$v^1(K) = v^0(N(K)) \approx d^0(N(K))d^1(N(K)) \cdots d^{n-2}(N(K))$$
$$= d^1(K)d^2(K) \cdots d^{n-1}(K).$$

To prove (6) we denote again by Z (or $Z(K)$) the cylinder associated with K according to Lemma 1. Then we have $Z = I + N$, where I is a

line segment of length $d^0(K)$ and $N = N(K)$. It has been shown (see [5], p. 215) that there are positive constants a, b such that

$$(7) \qquad W_m(Z) = ad^0(K)W'_m(N) + bW'_{m-1}(N)$$

where W'_m, W'_{m-1} are cross-sectional measures with respect to R^{n-1}. The induction assumption implies

$$W'_m(N(K)) \approx d^0(N(K)) \cdots d^{n-m-2}(N(K)) = d^1(K) \cdots d^{n-m-1}(K),$$

$$W'_{m-1}(N(K)) \approx d^0(N(K)) \cdots d^{n-m-1}(N(K)) = d^1(K) \cdots d^{n-m}(K).$$

From these two relations and (7) we find (noting also that $d^{k+1}(K) \leqq d^k(K)$)

$$(8) \qquad W_m(Z(K)) \approx d^0(K) \cdots d^{n-m-1}(K).$$

(6) is now an immediate consequence of (8) since Lemma 1 and the monotonicity, translation invariance and homogeneity of W_m imply

$$W_m(K) \approx W_m(Z(K)).$$

4. Proofs of the theorems. It is convenient to first prove Theorem 2 and then Theorem 1.

Proof of Theorem 2. First we wish to show that the condition

$$(9) \qquad \sum_{K_i \in S^m} v^m(K_i) = \infty, \text{ for some } m = 0, 1, \cdots, n$$

implies that $\{K_i\}$ permits a covering of R^n. If $n = 1$ this is obvious, and we make the induction assumption that it be also true in R^k if $k < n$. If $\sum_{K_i \in S^0} v^0(K_i) = \sum_{d(K_i) \leqq 1} v(K_i) = \infty$ the desired conclusion has been shown to be true in [3]. For this reason we now assume that $n > 1$ and that $\sum_{K_i \in S^0} v^0(K_i) < \infty$. Then, there is no loss in generality by assuming that actually $S^0 = \varnothing$, i.e. $d(K_i) > 1$ for all K_i. There is an integer g such that $1 \leqq g \leqq n$ and $\sum_{K_i \in S^g} v^g(K_i) = \infty$. We now remark that for any divergent infinite series $\sum_{i=1}^\infty a_i$ with $a_i \geqq 0$ it is always possible to partition the indices into infinitely many disjoint subsets such that each of the corresponding subsequences $\{a_{1j}\}$ $\{a_{2j}\} \cdots$ satisfies $\sum_{j=1}^\infty a_{ij} = \infty$ $(i = 1, 2, \cdots)$ (First, let us define successively sums $s_1 = a_1 + a_2 + \cdots + a_{q_1}$, $s_2 = a_{q_1+1} + \cdots + a_{q_2}$, $s_3 = a_{q_2+1} + \cdots + a_{q_3}, \cdots$ so that $s_r \geqq 1$. The series $\sum_{h=0}^\infty s_{2h+1}$, $\sum_{h=0}^\infty s_{2(2h+1)}$, $\sum_{h=0}^\infty s_{4(2h+1)}, \cdots$ are then divergent and determine obviously such a partitioning.) Applied to the above series this implies that $\{K_i\}$ can be partitioned into pairwise disjoint subclasses $\{K_j^k\}$ so that for $k = 1, 2, \cdots$

$$\sum_{K_j^k \in S^g} v^g(K_j^k) = \infty.$$

Since $g \geqq 1$ this can also be written as

$$\sum_{N(K_j^k) \in S^{g-1}} v^{g-1}(N(K_j^k)) = \infty,$$

and according to the induction assumption it follows that $\{N(K_j^k)\}$ permits a covering of R^{n-1} (for every fixed k).

To complete the proof we circumscribe about each K_j^k a cylinder, say $Z(K_j^k)$, as indicated by Lemma 1. Then, since $d(K_j^k) \geqq 1$ and thus the height of $Z(K_j^k)$ is not less than 1, $\{Z(K_j^k)\}$ permits obviously a covering of a strip of width 1 (i.e. of the closed set between two parallel hyperplanes unit distance apart). The class $\{\frac{1}{3}Z(K_j^k)\}$ permits then a covering of a strip of width 1/3. But since every K_j^k contains a translate of $\frac{1}{3}Z(K_j^k)$ the class $\{K_j^k\}$ permits (for every fixed k) also a covering of a strip of width 1/3. Since R^n can be written as a union of strips of width 1/3, the total class $\{K_i\} = \{K_j^1\} \cup \{K_j^2\} \cup \cdots$ clearly permits a covering of R^n.

To prove the necessity of the conditions (9) we assume now that all the series $\Sigma_{K_i \in S^m} v^m(K_i)$ converge and $\{K_i\}$ covers R^n. The class S^n consists then of only finitely many bodies K_i, and therefore $\Sigma_{K_i \in S^n} v(K_i) \leqq c_0$ for some constant c_0. If $K_i \in S^m$, where $0 \leqq m < n$, let $Z_i = Z_i^{n-m}$ be a generalized $(n-m)$-cylinder with bases $N^m(K_i)$. Then $K_i \subset Z_i$, and if $\{K_i\}$ permits a covering of R^n the class $\{Z_i\}$ must have the same property. In fact, we may assume that the cylinders Z_i are already so arranged that

$$(10) \qquad\qquad R^n \subset \bigcup_{i=1}^{\infty} Z_i.$$

Now, let r be a number that is greater than 1. According to Lemma 2

$$v(B_r \cap Z_i^{n-m}) \leqq \omega_m v^m(K_i) r^m$$

and therefore

$$v\left(B_r \cap \bigcup_{i=1}^{\infty} Z_i\right) \leqq c_0 + \omega_0 \sum_{K_i \in S^0} v^0(K_i) + \omega_1 r \sum_{K_i \in S^1} v^1(K_i) + \cdots$$
$$+ \omega_{n-1} r^{n-1} \sum_{K_i \in S^{n-1}} v^{n-1}(K_i).$$

Since all the series $\Sigma_{K_i \in S^m} v^m(K_i)$ are assumed to converge and $r > 1$ there is a positive constant c such that $v(B_r \cap \bigcup_{i=1}^{\infty} Z_i) \leqq cr^{n-1}$. However, (10) would imply $v(B_r \cap \bigcup_{i=1}^{\infty} Z_i) = \omega_n r^n$, giving a contradiction if r is large enough. This completes the proof of Theorem 2.

Proof of Theorem 1.   If the assumptions of Theorem 1 are satisfied then either

$$(11) \qquad\qquad \sum_{d(K_i)\leq 1} v(K_i) = \infty$$

or one can apply Theorem 2 (for R^{n-1}) to the class $\{N(K_i): d(K_i) > 1\}$. This latter possibility leads immediately to the fact that

$$(12) \qquad\qquad \sum_{K_i \in S^m} v^m(K_i) = \infty, \text{ for some } m = 1, 2, \cdots, n - 1.$$

But if either (11) or (12) is satisfied, an application of Theorem 2 (for R^n) shows that $\{K_i\}$ permits a covering of R^n.

Conversely, if $\{K_i\}$ permits a covering of R^n, then the assumptions of Theorem 2 are satisfied, which means that either (11) holds or $\sum_{N(K_i) \in S^m} v^m(N(K_i)) = \infty$ for some $m = 0, 1, \cdots, n - 1$. But, again by Theorem 2 (for R^{n-1}) the latter condition can be interpreted as stating that $\{N(K_i): d(K_i) > 1\}$ permits a covering of R^{n-1}.

Proof of Theorem 3.   First we concérn ourselves with condition (a). It follows immediately from Lemma 3 that for $m = 0, 1, \cdots, n$ any of the series $\sum_{K_i \in S^m} v^m(K_i)$ which appear in Theorem 2 diverges if and only if the corresponding series $\sum_{K_i \in S^m} d_i^m d_i^{m+1} \cdots d_i^{n-1}$ diverges. If we write

$$D_i = 1 + d_i^0 + d_i^0 d_i^1 + \cdots + d_i^0 d_i^1 \cdots d_i^{n-2}$$

we have

$$\sum_{i=1}^{\infty} \frac{d_i^0 \cdots d_i^{n-1}}{D_i} = \sum_{K_i \in S^0} \frac{d_i^0 \cdots d_i^{n-1}}{D_i} + \sum_{K_i \in S^1} \frac{d_i^0 \cdots d_i^{n-1}}{D_i} + \cdots + \sum_{K_i \in S^n} \frac{d_i^0 \cdots d_i^{n-1}}{D_i}$$

and we see that it suffices to prove that $\sum_{K_i \in S^m} d_1^0 \cdots d_i^{n-1}/D_i$ diverges if and only if $\sum_{K_i \in S^m} d_i^m d_i^{m+1} \cdots d_i^{n-1}$ diverges.   To show this equivalence we note that the definition of S^m implies that for $K_i \in S^m$ and $m = 0, 1, \cdots, n$

$$(13) \qquad\qquad d_i^{n-1} \leq \cdots \leq d_i^m \leq 1 < d_i^{m-1} \leq \cdots \leq d_i^0,$$

where we have to omit for $m = 0$ the terms $d_i^{m-1}, \cdots, d_i^0$ and for $m = n$ the terms $d_i^{n-1}, \cdots, d_i^m$. As a consequence of (13) we obtain for $m = 0, 1, 2, \cdots, n$

$$(14) \qquad\qquad d_i^0 d_i^1 \cdots d_i^{m-1} \leq D_i \leq n d_i^0 d_i^1 \cdots d_i^{m-1}$$

(with $d_i^0 d_i^1 \cdots d_i^{m-1} = 1$ if $m = 0$). The equivalence of the conditions $\Sigma_{K_i \in S^m} d_i^0 \cdots d_i^{n-1}/D_i = \infty$ and $\Sigma_{K_i \in S^m} d_i^m \cdots d_i^{n-1} = \infty$ is now an immediate consequence of (14).

Regarding the condition (b) of Theorem 3 we remark that because of Lemma 3

$$\frac{1}{v^m(K)} \approx \frac{1}{d^m(K) \cdots d^{n-1}(K)}$$

and that

$$\frac{d_i^0 \cdots d_i^{n-1}}{D_i} = \left(\frac{1}{d_i^0 \cdots d_i^{n-1}} + \frac{1}{d_i^1 \cdots d_i^{n-1}} + \cdots + \frac{1}{d_i^{n-1}} \right)^{-1}.$$

These two relations show that (a) and (b) are equivalent.

To prove that (c) is necessary and sufficient we have only to point out that (a) and (c) are clearly equivalent because of (6).

It might be worth mentioning that the denominators in the series of condition (c) can be replaced by any other linear combination of the W_m ($m = 1, 2, \cdots, n$), provided that all coefficients of such a combination are positive. For example, if B denotes the unit ball of R^n it follows from Steiner's formula that $v(K + B) - v(K)$ can be used for this purpose. Hence, we have the following result:

$\{K_i\}$ *permits a covering of* R^n *if and only if*

$$\sum_{i=1}^{\infty} v(K_i)/(v(K_i + B) - v(K_i)) = \infty.$$

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