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ANALYTIC DISCS IN THE MAXIMAL IDEAL SPACE OF $M(G)$

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Let $M(G)$ denote the convolution algebra of finite regular Borel measures on a locally compact abelian group G, and let Δ denote the maximal ideal space of $M(G)$. It is well-known that on certain subsets of Δ the Gelfand transforms $\mu^\wedge$ of members μ of $M(G)$ behave like holomorphic functions. The simplest way to exhibit this is to use Taylor's description of Δ as the semigroup of all continuous semicharacters of a compact semigroup S — the structure semigroup of $M(G)$ (see [10]). If $f \in \Delta(=S^\wedge)$ and $f(s) \geqq 0$ for all $s \in S$, then $f^z \in \Delta$ for $\mathrm{Re}(z) > 0$. Thus, provided $f^2 \neq f$, there is an analytic disc around f in the sense that $\mu^\wedge(f^z)$ is holomorphic on $\mathrm{Re}(z) > 0$ for all $\mu \in M(G)$. Using this fact, Taylor (*loc. cit.*) has shown that if f is a strong boundary point of $M(G)$, then $|f|^2 = |f|$.

We have already shown ([2]) that there is a point derivation at the idempotent h which corresponds to the direct sum decomposition of $M(G)$ into discrete and continuous measures. It was also possible to prove that this point derivation is continuous in the spectral radius norm so that we were able to deduce that h is not a strong boundary point. Here we strengthen the main result of that earlier paper to show that there is an analytic disc around h, and that this disc remains analytic for the completion of $M(G)$ in the spectral radius norm.

Here we strengthen the main result of that earlier paper to show that there is an analytic disc around h, and that this disc remains analytic for the completion of $M(G)$ in the spectral radius norm.

In fact, our methods here are in some ways more straightforward than those we used in that paper, and can be extended to encompass the case when h is an idempotent corresponding to the direct sum decomposition of $M(G)$ induced by a single generator symmetric Raikov system.

The proofs rely heavily on refinements and modifications of techniques given by Williamson in [13] and Varopoulos in [11] in connection with independent subsets of locally compact abelian groups. Indeed our results in §3 are of interest in producing yet another direct sum decomposition of $M(G)$ associated with an independent set. This one lies between the Raikov construction and that of Varopoulos (*loc. cit.*).

In §2 we prove the existence of the disc subject to having a certain decomposition of the measure algebra. Later sections are devoted to the proof of the existence of such a decomposition.

2. Definitions and statement of theorem. Fix a locally compact abelian group G. Let $M(G)$ denote its measure algebra and Δ the maximal ideal space of the latter. It will be convenient to adopt Šreider's description of members of Δ as generalized characters (cf. e.g. **[1]**). By an *analytic disc* in Δ we mean an injection φ of the open unit disc in $\mathbf{C}$ into Δ such that $\mu^\wedge \circ \varphi$ is holomorphic for each $\mu \in M(G)$. The *centre* of such a disc is defined to be $\varphi(0)$.

A *Raikov system* in G is a nonempty collection $\mathcal{R}$ of nonempty $\mathcal{F}_\sigma$ subsets (countable unions of compact sets) of G satisfying

(i) if $F_1 \in \mathcal{R}$ and F_2 is an $\mathcal{F}_\sigma$ subset of F_1 then $F_2 \in \mathcal{R}$;

(ii) the union of a countable collection of members of $\mathcal{R}$ is in $\mathcal{R}$;

(iii) if $F \in \mathcal{R}$, then $F + x \in \mathcal{R}$ for all $x \in G$;

(iv) if $F \in \mathcal{R}$, then $F + F \in \mathcal{R}$.

Raikov systems are discussed in more detail in **[12]** and **[13]**. $\mathcal{R}$ is said to be *symmetric* if

(v) $F \in \mathcal{R}$ implies $-F \in \mathcal{R}$.

The smallest Raikov system in G consisting of all countable subsets of G is, of course, symmetric. $\mathcal{R}$ is said to be a *proper* Raikov system if it does not contain all $\mathcal{F}_\sigma$ subsets of G. The intersection of any family of Raikov systems is again a Raikov system, so that it is meaningful to talk about the Raikov system *generated* by a given collection of $\mathcal{F}_\sigma$-subsets of G. We shall be interested only in singly generated (or equivalently countably generated) Raikov systems — the smallest Raikov system is a trivial example of such an object.

Any Raikov system $\mathcal{R}$ leads to a direct sum decomposition of $M(G)$. Specifically, let

$$A = \{\mu : |\mu|(F) = \|\mu\| \text{ for some } F \in \mathcal{R}\}$$

and

$$I = \{\mu : |\mu|(F) = 0 \text{ for all } F \in \mathcal{R}\}.$$

Then A is an L-subalgebra and I an L-ideal in $M(G)$ and $M(G) = A \oplus I$. (Recall that B is an L-subspace if it is a closed subspace such that whenever $\mu \in B$ and ν is absolutely continuous with respect to μ then $\nu \in B$. L-ideals and L-subalgebras are respectively ideals and subalgebras which are at the same time L-subspaces.) Thus we can define a generalized character h of $M(G)$ by

$$h_\mu = \begin{cases} 1 & \text{if } \mu \in A \\ 0 & \text{if } \mu \in I. \end{cases}$$

We refer to this as the idempotent (generalized character) *associated with* $\mathcal{R}$. Now we are able to state the theorem.

THEOREM. *Let $\mathcal{R}$ be a singly generated proper symmetric Raikov system and let h be the idempotent associated with it. Then there is an analytic disc φ with centre h. Moreover the functions $\mu^\wedge \circ \varphi$ on the unit disc are exactly those holomorphic functions with absolutely convergent Taylor series.*

COROLLARY 1. *There is a nonzero uniformly continuous point derivation at h.*

COROLLARY 2. *The idempotent h is not a strong boundary point of $M(G)$.*

Proofs of corollaries. Define, for f in the uniform closure of $M(G)^\wedge$,

$$d(f) = (f \circ \varphi)'(0),$$

and Corollary 1 is proved. Corollary 2 now follows from ([3] Ch. II, Ex. 12 (e)).

The reader is referred to §5 for a more concrete description of d as we have constructed it. We conclude this section with the first step in the proof of the theorem.

LEMMA 1. *Let $M(G) = A \oplus I$ be a direct sum decomposition of $M(G)$ into an L-algebra A and an L-ideal I, and let k be the corresponding idempotent. Suppose that there exist mutually orthogonal L-subspaces $A = B_0, B_1, B_2, \cdots$ of $M(G)$ such that $B_1 \neq (0)$;*

(i) *$\mu \in B_n$, $\nu \in B_m$ implies $\mu * \nu \in B_{n+m}$ for all positive integers n and m;*

(ii) *$(\bigoplus_{n=0}^{\infty} B_n)^\perp$ is an L-ideal of $M(G)$.*

Then there is an analytic disc φ with centre k, and $\{\hat{\mu} \circ \varphi : \mu \in M(G)\}$ consists of those holomorphic functions on the unit disc with absolutely convergent Taylor series.

Proof. We define, for each z in the open unit disc D, a generalized character $\varphi(z)$ by writing

$$\varphi(z)_\mu = \begin{cases} z^n \ (\mu\text{.a.e.}) & \text{if} \quad \mu \in B_n \qquad (n = 0, 1, 2, \cdots) \\ \\ 0 \quad (\mu\text{.a.e.}) & \text{if} \quad \mu \perp \bigoplus\limits_{n=0}^{\infty} B_n. \end{cases}$$

As every member λ of $M(G)$ may be written uniquely in the form

$$\lambda = \sum_{n=0}^{\infty} \mu_n + \nu \tag{1}$$

where $\mu_n \in B_n$ $(n = 0, 1, 2, 3, \cdots)$ and $\nu \perp \bigoplus_{n=0}^{\infty} B_n$, $\varphi(z)$ is completely defined for such $z \in D$. It is a straightforward matter to check that $\varphi(z)$ is a generalized character.

Using (1), we have, for $\lambda \in M(G)$,

$$\lambda^{\wedge}(\varphi(z)) = \sum_{n=0}^{\infty} z^n \int_G d\mu_n$$

which, since $\Sigma_{n=0}^{\infty} \| \mu_n \| \leqq \| \lambda \|$, shows that φ is an analytic disc with centre k, and that $\lambda^{\wedge} \circ \varphi$ has an absolutely convergent Taylor series. Since $B_n \neq (0)$ $(n = 0, 1, 2, \cdots)$ every such function arises in this way. This completes the proof.

It may be helpful to the reader if we now indicate how Varopoulos's direct sum decompositon of $M(G)$ may be used to complete the proof of the Theorem in the case when h corresponds to the smallest Raikov system on G, and G is metrizable.

Let K be a strongly independent compact perfect subset of G. Let T_1 consist of the continuous measures on K, and, for $n > 1$, let T_n be the L-subspace of $M(G)$ generated by products of n elements of T_1. If we write $B_n (n \geqq 1)$ for the translation invariant L-subspace of $M(G)$ generated by T_n, and B_0 for the L-algebra of discrete measures on G, then the conditions of the preceding lemma are satisfied. The proof of this fact is contained in **[11]**.

While extending the result to more general idempotents, we have tried to simplify the part of the proof that corresponds to Varopoulos's arguments. The result is a somewhat different direct sum decomposition associated with an independent set.

3. The direct sum decomposition. In this section, we focus on the problem of finding the L-subspaces B_n $(n = 1, 2, 3, \cdots)$ of Lemma 1 in the case when the direct sum decomposition $M(G) = A \oplus I$ is induced by the singly generated symmetric Raikov system $\mathscr{R}$ with associated idempotent h.

It is easily seen that we may choose the generator of $\mathscr{R}$ to be an $\mathscr{F}_\sigma$-subgroup H of G and then $\mathscr{R}$ consists of all countable unions of cosets of H. The fact that $\mathscr{R}$ is proper implies that H is a first category set (and, equivalently of zero Haar measure).

We now introduce a concept which is an extension to general locally compact abelian groups of a definition due to Williamson **[13]** in the special case of the real line. Williamson's definition has to be modified to cater for the existence of torsion in the group.

DEFINITION 1. A subset K of G is *H-independent* if, whenever $x_1, \cdots, x_r$ are distinct elements of K and $n_1, \cdots, n_r$ are integers such that $\sum_{i=1}^{r} n_i x_i \in H$ then $n_i x_i \in H$ $(i = 1, 2, \cdots, r)$.

Of course this definition is equivalent to asking that the set of cosets $\{k + H \colon k \in K\}$ is independent in G/H in the sense of (**[9]** 97). Write $o_H(x)$ for the order of $x + H$ in G/H.

In the next section we shall show that there always exists a perfect H-independent subset of G. To be precise, we shall prove the following result.

PROPOSITION 1. *Let H be a first category $\mathscr{F}_\sigma$ subgroup of G. Then there exists $p \in \{2, 3, 4, \cdots, \infty\}$ and an H-independent perfect subset K of G such that $o_H(x) = p$ for all $x \in K$.*

Assuming this for the moment, we proceed to define the L-spaces B_n $(n > 0)$. Notice, first, that each element of the group Q generated by K and H has a representation in the form

$$\sum_{i=1}^{r} n_i x_i + h \tag{2}$$

where $0 < |n_i| < p$, $h \in H$ and $x_1, x_2, \cdots, x_r$ are distinct. This representation is clearly unique except for the possible replacement of n_i by $n_i \pm p$ for some i's and the corresponding change in h.

Let S_n consist of all sums of the form

$$x_1 + x_2 + \cdots + x_n + h \tag{3}$$

where $x_1, \cdots, x_n$ are distinct elements of K and $h \in H$. Evidently S_n is a Borel subset of G.

Now we define B_n to consist of all measures $\mu \in M(G)$ satisfying:

(α) μ is concentrated on $\bigcup_{m=1}^{\infty} y_m + S_n$ for some $y_1, y_2, \cdots$ in G.

(β) $|\mu|(y + S_r) = 0$ for all $y \in G$ and $r < n$.

Evidently each B_n is an L-subspace of $M(G)$, and putting $n = 0$ in the definitions of S_n and B_n, we have $S_0 = H$, $B_0 = A$. Furthermore the definition forces the L-subspaces B_n to be mutually orthogonal. Since K is perfect and $|K \cap x + H| \leqq 1$ for all $x \in G$, $B_1 \neq (0)$. We proceed to the proofs of the remaining hypotheses of Lemma 1.

LEMMA 2. *Let $\mu \in B_n$, $\nu \in B_m$. Then*

$$\mu * \nu \in B_{n+m} \qquad (n, m = 0, 1, 2, 3, \cdots).$$

Proof. First we show that $\mu * \nu$ is concentrated on a countable union of translates of S_{n+m}. It will simplify matters if we assume that μ and ν are probability measures. There is no harm in assuming further that μ is concentrated on S_m and ν is concentrated on S_n. We shall, in these circumstances, show that $\mu * \nu$ is concentrated on S_{n+m}.

Evidently $\mu * \nu$ is concentrated on $S_n + S_m$, and the complement W of S_{n+m} in this set consists of sums $x + y$ $(x \in S_n, y \in S_m)$ such that the sums of the form (3) for x and y have an element of K in common. The inverse image of this set under the map $\xi: (x, y) \leadsto x + y$ from $S_n \times S_m$ to $S_n + S_m$ is the subset $\bar{W}$ of $S_n \times S_m$ consisting of those ordered pairs (x, y) whose sums (3) have a common component in K. Fix $y \in S_m$ and consider the section

$$\bar{W}_y = \{x : (x, y) \in \bar{W}\}.$$

This is contained in a finite union of sets of the form $x + S_{n-1}$ $(x \in K)$; we obtain one such set for each member of K in the sum for y. Since $\mu \in B_n$, this implies that $\mu(\bar{W}_y) = 0$. By Fubini's theorem $\mu \times \nu$ $(\bar{W}) = 0$ and so $\mu * \nu$ $(W) = 0$. This proves that $\mu * \nu$ is concentrated on S_{n+m}.

It only remains to show that $\mu * \nu$ annihilates all sets of the form $x + S_r$ for $r < n + m$. Consider $(x + S_r) \cap S_{n+m}$. Obviously we need only consider the case where this is nonempty, and in this case x belongs to Q. Thus write

$$x = \sum_{i=1}^{s} n_i u_i + h$$

as in (2). If $w_1 + w_2 + \cdots + w_n + t_1 + t_2 + \cdots + t_m + h' \in S_{n+m}$ is in $x + S_r$ then one of the elements w_i $(i = 1, 2, \cdots, n)$, t_j $(j = 1, 2, \cdots, m)$ is equal to an element of the form u_k $(k = 1, 2, \cdots, s)$. Thus the inverse image of $(x + S_r) \cap S_{n+m}$ in $S_n \times S_m$ under the map ξ is contained in a finite union of sets of the form $S_n \times (u_k + S_{m-1})$ and $(u_l + S_{n-1}) \times S_m$. All of these sets are $\mu \times \nu$-null so that $\mu * \nu(x + S_r) = 0$.

To complete the proof that the L-subspaces B_n $(n = 0, 1, 2, \cdots)$ satisfy the hypotheses of Lemma 1, we have to show that $(\bigoplus_{n=0}^{\infty} B_n)^{\perp}$ is an L-ideal. This will be done once we have established the following lemma.

LEMMA 3. *Let μ be orthogonal to $\bigoplus_{n=0}^{\infty} B_n$ and $\nu \in M(G)$. Then $\mu * \nu$ is orthogonal to $\bigoplus_{n=0}^{\infty} B_n$.*

Proof. Let $B = \bigoplus_{n=0}^{\infty} B_n$ and $J = B^{\perp}$. Evidently B is translation-invariant and hence so is J. Thus we may assume, without loss of generality, that μ and ν are continuous probability measures. Let $\mathcal{R}'$ be the Raikov system generated by H and K, and let $M(G) = A' \oplus I'$ be the corresponding direct sum decomposition of $M(G)$ into an L-algebra A' and an L-ideal I'. Clearly $B \subset A'$, so that if μ or $\nu \in I'$, $\mu * \nu \in I'$ and hence is orthogonal to B. In other words, we may make the assumption that both μ and ν belong to A'. This means that μ and ν are sums of measures each concentrated on a set of the form $x + (n)K + H$ where $x \in G$, n is a nonnegative integer and

$$(n)K = \{x_1 + x_2 + \cdots + x_n : x_i \in K \qquad (i = 1, 2, 3, \cdots, n)\}.$$

It will further simplify matters if we consider each of μ and ν to be concentrated on just one of these sets and, by translating the measures in question, we may take μ concentrated on $(n)K + H$ and ν concentrated on $(m)K + H$. Let n, m be the least positive integers for which this argument is valid. Thus μ annihilates every translate of $(r)K + H$ for $0 \leqq r \leqq n-1$ and ν annihilates every translate of $(r)K + H$ for $0 \leqq r \leqq m-1$.

Suppose now that $\mu * \nu$ is not orthogonal to B. Then $\mu * \nu (x + S_r) > 0$ for some $x \in G$ and some nonnegative integer r; again we choose r to be the least such integer. Since $\mu * \nu$ is concentrated on $(n+m)K + H$, x must belong to the group Q so that we may write

$$x = \sum_{i=1}^{s} n_i u_i + h,$$

where $u_1, u_2, \cdots, u_s$ are distinct elements of K.

Suppose first that $r > m + n$. If $w \in (x + S_r) \cap (n+m)K + H$

$$(4) \quad w = x + x_1 + x_2 + \cdots + x_r + h_1 = y_1 + y_2 + \cdots + y_n + z_1 + \cdots + z_m + h_2,$$

where $h_1, h_2 \in H$, $x_i, y_j, z_k \in K$ for all i, j, k and $x_1, \cdots, x_r$ are distinct. It follows that some x_i must equal some u_j.
Thus $x + S_r \cap (n+m)K + H$ is contained in a finite union of translates of S_{r-1} and this contradicts the fact that r is the least integer for which $\mu(x + S_r) > 0$ for some $x \in G$.

Next assume that $r < m + n$. Again we have an equation of the form (4) for any $w \in (x + S_r) \cap (n+m)K + H$. By the previous argument we may ignore those w's for which some x_i is equal to some u_j. The remaining w's satisfy at least one of the following statements:

(i) some y_i equals some u_j;

(ii) some z_i equals some u_j;
(iii) some y_i equals some z_j;
(iv) at least p of the y_i's are the same;
(v) at least p of the z_i's are the same.

We consider the subsets $E_1, E_2, \cdots, E_5$ of $((n)K + H) \times ((m)K + H)$ which are the inverse images under the map $\xi: (u, v) \sim\!\rightarrow u + v$ of the sets of w's satisfying (i), (ii), (iii), (iv), (v) respectively.

E_1 is contained in a finite union of sets of the form $(v + (n-1)K + H) \times ((m)K + H)$ and so, because n is the least integer with its defining property, $\mu \times \nu(E_1) = 0$. A similar argument with μ, n replaced by ν, m works for E_2. To cope with E_3, we use Fubini's theorem as in the proof of Lemma 2, and the defining property of n or m. This gives $\mu \times \nu(E_3) = 0$.

If p of the y_i's are the same, they sum to an element of H so that E_4 is contained in $((n-p)K + H) \times ((m)K + H)$ and this is $\mu \times \nu$–null. E_5 is dealt with in the same way.

We have shown that the only possible value of r is $n + m$. Now we prove that $x = 0$. To see this, put $r = n + m$ in (4). As before we may assume that each x_i is distinct from each u_j. It follows that each x_i must equal some y_j or some z_k. But then

$$x_1 + x_2 + \cdots + x_{n+m} = y_1 + \cdots + y_n + z_1 + \cdots + z_m$$

so that $x \in H$ and hence is 0.

If we look a little more closely at the preceding argument, we notice that $\xi^{-1}(S_{(n+m)}) \cap ((n)K + H) \times ((m)K + H)$ is $S_n \times S_m$. Since $\mu * \nu(S_{(n+m)}) > 0$, it follows that $\mu(S_n) > 0$, which contradicts the fact that μ is orthogonal to B. This completes the proof.

The direct sum decomposition $M(G) = B \oplus J$ defined here in the case when $\mathcal{R}$ is the smallest Raikov system lies between that given by Varopoulos in **[11]** and the one induced by the Raikov system generated by K. In the former decomposition the L-algebra is the smallest one containing all of the discrete measures together with all measures on K, whereas in the latter it consists of all translates of all measures on $\bigcup_{n=1}^{\infty}(n)K$. Evidently the L-algebra in our decomposition is contained in the L-algebra given by the Raikov system. On the other hand, since $K = S_1$, the proof of Lemma 2 shows that a convolution of n continuous measures on K is concentrated on S_n and this yields that the L-algebra in the Varopoulos decomposition is contained in ours. It is straightforward to see that both of the inclusions are proper.

4. The construction of a perfect H-independent set.

In this section we prove Proposition 1. Our methods are essentially those of Williamson ([13]) but we have to modify them to take account of torsion and to deal with the nonmetrizable case. As in §3, H is a first category F_σ-subgroup of G, so that we may write

$$H = \bigcup_{n=1}^{\infty} H_n$$

where each H_n is compact and nowhere dense in G and we assume $H_n \subseteq H_{n+1}$.

Let p be the supremum of all positive integers q with the property that every neighbourhood of 0 contains an element x with $o_H(x) \geqq q$. Because H has zero Haar measure $p \in \{2, 3, \cdots, \infty\}$.

Let us also write

$$L_\infty = G, \qquad L_q = \{x \in G : qx \in H\}, \qquad 1 \leqq q < \infty.$$

Evidently L_q is a closed subgroup of G, $1 \leqq q \leqq \infty$.

LEMMA 4. *L_p is an open subgroup of G and L_q is nowhere dense for all $1 \leqq q < p$.*

Proof. By definition, there is a compact neighbourhood N of 0 such that $o_H(x) \leqq p$, for all $x \in N$. Thus $N \subset \bigcup_{q=1}^{p} L_q$. It follows that at least one of the groups L_q $(1 \leqq q \leqq p)$ is not of zero Haar measure and hence is open. Using the definition again, we see that L_p must be such a group and moreover the only such group. The result follows.

Now we turn to the construction of the H-independent set K in G. It will be necessary for the present to assume that G is metrizable. Choose a compact neighbourhood N of 0 which is contained in the subgroup L_p. The subset I_1 of $N \times N$ consisting of those ordered pairs (x_1, x_2) such that either

$$n_1x_1 + n_2x_2 \in H_1$$

for some n_1, n_2 with $|n_1| \leqq 1$, $|n_2| \leqq 1$, $|n_1| + |n_2| > 0$ or $x_1 \in H_1$ or $x_2 \in H_1$, is nowhere dense. Thus we can find disjoint compact sets $U(0)$ and $U(1)$ with nonempty interior and with diameter less than half such that $U(0) \times U(1) \subset (N \times N) \backslash I_1$.

Now consider the subset I_2 of $U(0) \times U(0) \times U(1) \times U(1)$ consisting of those 4-tuples (x_1, x_2, x_3, x_4) such that either

$$n_1x_1 + n_2x_2 + n_3x_3 + n_4x_4 \in H_2$$

for some n_1, n_2, n_3, n_4 with $|n_i| \leqq 2$ $(i = 1, 2, 3, 4)$, $0 < |n_1| + |n_2| + |n_3| + |n_3|$ or $qx_i \in H_2$ for some i $(1 \leqq i \leqq 4)$ and some q $(1 \leqq q < \min(p, 3))$. Using Lemma 4, we see that this is again nowhere dense and so we are able to find disjoint compact sets $U(00)$, $U(01)$ in $U(0)$ and $U(10)$, $U(11)$ in $U(1)$, all with nonempty interior and diameter less than $\frac{1}{4}$, such that

$$U(00) \times U(01) \times U(10) \times U(11) \subset U(0) \times U(0) \times U(1) \times U(1) \backslash I_2.$$

We repeat this procedure in an obvious way subject to the constraint that at stage n, the n_i's and the q's are strictly less than $\min(p, n + 1)$. Put

$$K = \bigcap_{n=1}^{\infty} \bigcup_{k_i = 0,1} U(k_1 k_2 \cdots k_n).$$

Then K is a compact perfect subset of G. If $x \in K$, then it is clear from the construction that for all $1 \leqq q < p$, $qx \notin H_n$ for all sufficiently large n. Thus $o_H(x) = p$ for all $x \in K$. If $x_1, x_2, \cdots, x_N$ are distinct elements of K then, for all large enough n, they are in different sets of the form $U(k_1 k_2 \cdots k_n)$ $(k_i = 0, 1;\ i = 1, 2, \cdots, n)$. Consequently,

$$n_1 x_1 + n_2 x_2 + \cdots + n_N x_N \notin H$$

provided $|n_i| < p$ $i = 1, 2, \cdots, N$ and $\Sigma_{i=1}^{N} |n_i| > 0$. It is easily seen from this that K is H-independent.

Now we have to remove the metrizability restriction on G. There are quite standard methods available for solving problems of this kind. Unfortunately they appear to be useless for our problem. The standard methods rely on the existence of a compact G_δ subgroup N of G such that the situation remains more or less the same when we pass from G to the quotient group G/N. Such a technique would certainly be available if H were in the σ-ring generated by all compact G_δ sets in G; for then we would choose N to be a subgroup of H. However, even in the case when $H = \{0\}$, H could only have this property if G itself were metrizable. Thus we resort to somewhat different methods. The key idea is to reduce the problem to the case where G is a product of compact metrizable groups. Although this product may be large, it is possible to use our knowledge of the metrizable case to eliminate potential counterexamples by a "rolling hump" argument.

Our first step is a standard one. G contains an open subgroup isomorphic to $\mathcal{R}^n \times D$ where D is a compact group. A straightforward argument shows that it is enough to concentrate on groups of this form. If $H \cap \mathcal{R}^n$ is of first category in $\mathcal{R}^n$ then the construction above works in $\mathcal{R}^n$ to yield the desired conclusion. Obviously if K is

$(H \cap \mathcal{R}^n)$-independent in $\mathcal{R}^n$, then it is H-independent in $\mathcal{R}^n \times D$.

If $H \cap \mathcal{R}^n$ is not of first category then $\mathcal{R}^n \subset H$ so that $H \cap D$ must be of first category. Thus we have reduced the problem to the case of compact groups. Using standard structure theory and duality arguments we may represent D as a quotient by a compact subgroup N of the direct product of a family $\{C_\alpha: \alpha \in I\}$ of metrizable groups, (see, for example, **[8]** 444–5). The next step is to replace D by $C = \Pi_{\alpha \in I} C_\alpha$.

Let $\pi: C \to D$ be the canonical projection with kernel N. This is an open mapping, so that if H is a first category $\mathcal{F}_\sigma$-subgroup of D then $H' = \pi^{-1}(H)$ is a first category $\mathcal{F}_\sigma$-subgroup of C. Assuming, for the moment, that we can handle products of metrizable groups, let K' be a perfect H'-independent subset of C, and let $K = \pi(K')$. The H'-independence of K' implies its N-independence, so that the restriction of π to K' is a homeomorphism onto K. Thus K is perfect. A simple computation shows that K is H-independent and that if $o_{H'}(x) = p$ for all $x \in K'$, then $o_H(y) = p$ for all $y \in K$.

It only remains, therefore, to consider groups of the form $C = \Pi_{\alpha \in I} C_\alpha$ where each C_α is a compact metrizable group and I is uncountable. Let H be a first category $\mathcal{F}_\sigma$-subgroup of C. If $H \cap C_\alpha$ is of first category, for any α, we may construct K inside C_α in accordance with the procedure set out earlier. Thus, for each x, we may assume that $H \cap C_\alpha$ is an open subgroup of C_α.

Suppose that $H \cap C_\alpha \neq C_\alpha$ for distinct indices $\alpha = \alpha_1, \alpha_2, \alpha_3, \cdots$. Then it is easily seen that $H \cap \Pi_{i=1}^\infty C_{\alpha_i}$ is not open and hence is of first category in $\Pi_{i=1}^\infty C_{\alpha_i}$. As this last group is also metrizable we may construct a perfect H-independent set in this situation also. Thus all that remains is the case when $H \cap C_\alpha \neq C_\alpha$ for only finitely many α's. Evidently, nothing is lost if we discard these α's. As we have already argued, we may assume that for any sequence (α_i), $H \cap \Pi_{i=1}^\infty C_{\alpha_i}$ is an open subgroup of $\Pi_{i=1}^\infty C_{\alpha_i}$. With the extra hypothesis that $H \cap C_\alpha = C_\alpha$ for all α, this may be strengthened to allow us to assume that H contains all countably infinite products $\Pi_{i=1}^\infty C_{\alpha_i}$.

Thus, if $C^{(\sigma)}$ is defined to consist of all members $\mathbf{x} = (x_\alpha)_{\alpha \in I}$ of C with the property that $x_\alpha \neq 0$ for only countably many α's, then $C^{(\sigma)} \subset H$.

The final twist is to note that $C^{(\sigma)}$ is pseudocompact — i.e. every continuous real-valued function on $C^{(\sigma)}$ is bounded — and that the Stone–Čech compactification of $C^{(\sigma)}$ is C (see **[6]**). It follows immediately that H is pseudocompact. As it is also σ-compact it must be compact (**[5]** Ex. 5H) and hence equal to C. This gives the required contradiction.

5. Remarks and problems.

First we give the promised concrete description of the point derivation d where

$$d(\mu) = (\mu^\wedge \circ \varphi)'(0).$$

This is just the coefficient of the first term in the Taylor expansion of $\mu^\wedge \circ \varphi$ and so consists of the integral over G of the part of the measure concentrated on translates of $K + H$ but not in translates of H. Thus

$$d(\mu) = \sum_{x \in G} ((1-h) \cdot \mu)(x + K + H)$$

where, of course, h is the idempotent associated with the Raikov system generated by the $\mathcal{F}_\sigma$-subgroup H and K is a compact H-independent perfect set.

In fact, we can also describe the higher order derivations at h, corresponding to the higher order terms in the Taylor series. Thus, putting

$$\rho_n(\mu) = \sum_{x \in G} ((1-h) \cdot \mu)(x + S_n) \qquad (n \geqq 0)$$

(recall the definition of S_n from §3) we have

$$d_0(\mu) = \rho_0(\mu)$$
$$d_n(\mu) = (n!)(\rho_n(\mu) - \rho_{n-1}(\mu)) \qquad (n \geqq 1)$$

and

$$\mu \circ \varphi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} d_n(\mu) z^n.$$

Leibniz formula applies for the sequence of d_n's and contains much of the measure theoretic and combinatorial properties of the S_n's that we have exhibited and used in Lemmas 2 and 3. Thus, $d_n(\mu) = 1$ for a probability measure μ if and only if μ is concentrated on translates of S_n and $\mu(x + S_r) = 0$ for all $r < n$. If μ and ν are probability measures,

$$d_n(\mu * \nu) = \sum_{k=0} \binom{n}{k} d_k(\mu) d_{n-k}(\nu)$$

exhibits the fact that a product measure is concentrated on translates of S_n and assigns zero mass to translates of S_r for $r < n$ if and only if it is built up from products of measures on translates of S_k and S_{n-k} which annihilate translates of lower order sets S_r, for some k. The existence of $d_n(\mu)$ and the fact that $|d_n(\mu)| \leqq \|\mu\|$ implies that intersections of translates of S_n are of zero μ-measure when μ is in the L-ideal generated by the Raikov system.

Clearly, it ought to be possible to extend our main result to cover more general types of idempotents in Δ. An obvious next step is to remove the symmetry restriction on the Raikov system. However, here it must be borne in mind that there is on the real line an asymmetric

Raikov system $\mathcal{R}$ generated by a semigroup S for which $S-S$ is the whole of $\mathcal{R}$ ([7]). The uncountably generated Raikov systems appear much less tractable.

Another class of idempotents have been produced by Šreider ([4] 195) and shown to be different, in general, from Raikov idempotents. Šreider takes a subgroup Q of the group of discontinuous characters of G. Then

$$A = \{\mu : \gamma \text{ is } \mu\text{-measurable for all } \gamma \in Q\}$$

is an L-subalgebra and its orthogonal complement is an L-ideal. In fact, if Q is singly generated by γ, say, then the direct sum decomposition is given by a symmetric Raikov system $\mathcal{R}$ where an $\mathcal{F}_\sigma$ subset $F = \bigcup_n F_n$ (F_n compact) belongs to $\mathcal{R}$ if and only if $\gamma|_{F_n}$ is continuous. In this case $\mathcal{R}$ need not be singly generated. It would be interesting to know if there exist analytic discs around these idempotents.

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