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SETS IN R^d HAVING $(d-2)$-DIMENSIONAL KERNELS

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Let S be a d-dimensional set, $d \geqq 2$, and assume that for every $(d+1)$-member subset T of S, there corresponds a $(d-2)$-dimensional convex set $K_T \subseteq S$ such that every point of T sees K_T via S and $(\operatorname{aff} K_T) \cap S = K_T$. Furthermore, assume that when T is affinely independent, then K_T is the kernel of T relative to S. Then S is starshaped and the kernel of S is $(d-2)$-dimensional.

1. Introduction. Let S be a subset of R^d, $d \geqq 2$. For points x, y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Similarly, for $T \subseteq S$, we say *x sees T* (and *T sees x*) *via S* if and only if x sees each point of T via S. The set of points in S seen by T is called the *kernel of T relative to S* and is denoted $\ker_S T$. Finally, if $\ker_S S \equiv \ker S$ is not empty, then S is said to be *starshaped*.

An interesting problem is that of determining necessary and sufficient conditions for S to be a starshaped set whose kernel is k-dimensional, $0 \leqq k \leqq d$. Several papers have considered this question (Hare and Kenelly **[2]**, Kenelly, Hare, et al. **[3]**, Toranzos **[4]**), and Foland and Marr **[1]** have proved that a set S will have a zero-dimensional kernel provided S contains a noncollinear triple and every three noncollinear members of S see via S a unique common point. Hence the purpose of this paper is to obtain an analogue of these results for subsets of R^d whose kernel is $(d-2)$-dimensional.

The following familiar terminology will be used. Throughout the paper, $\operatorname{conv} S$, $\operatorname{aff} S$, $\operatorname{cl} S$, $\operatorname{bdry} S$, $\operatorname{rel\,int} S$, and $\ker S$ will denote the convex hull, affine hull, closure, boundary, relative interior, and kernel, respectively, of the set S. The cone of x over S, defined to be the union of all rays emanating from x through points of S, will be denoted $\operatorname{cone}(x, S)$. Also, if S is convex, $\dim S$ will represent the dimension of S.

2. Proof of the theorem.

THEOREM. *Let S be a d-dimensional set, $d \geqq 2$, and assume that for every $(d+1)$-member subset T of S, there corresponds a $(d-2)$-dimensional convex set $K_T \subseteq S$ such that every point of T sees K_T via S and $(\operatorname{aff} K_T) \cap S = K_T$. Furthermore, assume that when T is affinely independent, then K_T is the kernel of T relative to S. Then S is starshaped and the kernel of S is $(d-2)$-dimensional.*

Proof. The proof of the theorem is lengthy and will be accomplished by a sequence of lemmas. The first lemma and its corollary are immediate consequences of our hypothesis, while the second, third and fourth lemmas present the main arguments in the proof.

LEMMA 1. *If* $T = \{t_1, \cdots, t_{d+1}\}$ *is an affinely independent subset of* S, *then* $\cup\{[t_i, t_j]: 1 \leqq i < j \leqq d+1\} \not\subseteq S$.

Proof of Lemma 1. Otherwise, $T \subseteq \ker_S T = K_T$, contradicting the fact that K_T is a convex set of dimension $d-2$.

COROLLARY 1. *Let* $T = \{t_1, \cdots, t_{d+1}\}$ *be a subset of* S, *with* $t_1, \cdots, t_d$ *affinely independent and conv* $\{t_1, \cdots, t_d\} \subseteq S$. *Then* $K_T \subseteq$ *aff* $\{t_1, \cdots, t_d\}$.

LEMMA 2. *Assume that conv* $(K \cup \{p\}) \cup$ *conv* $(K \cup \{q\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2, p \notin$ *aff* K, *and* $q \notin$ *aff* $(K \cup \{p\})$. *Then for* $x \in S$ *and* $x \in \pi \equiv$ *aff* $(K \cup \{p\})$, x *sees each point of* K *via* S.

Proof of Lemma 2. To begin, note that for $k_1, \cdots, k_{d-1}$ any $d-1$ affinely independent points in K, the set $\{k_1, \cdots, k_{d-1}, p, q\} \equiv T$ is affinely independent. Hence the set K_T described in the theorem is exactly $\ker_S T$, and so $K \subseteq K_T$. Thus without loss of generality we may assume $K = K_T$ and therefore $(\operatorname{aff} K) \cap S = K$. Also, we assume that $x \notin \operatorname{aff} K$, for otherwise $x \in K$, finishing the proof.

Now by the hypothesis of the theorem, the points $k_1, \cdots, k_{d-1}, p, x$ see via S a convex set D of dimension $d-2$ such that $(\operatorname{aff} D) \cap S = D$, and since $\operatorname{conv}\{k_1, \cdots, k_{d-1}, p\} \subseteq S \cap \pi, D \subseteq \pi$ also (by the corollary to Lemma 1). Similarly, $k_1, \cdots, k_{d-1}, q, x$ see a $(d-2)$-dimensional convex set D' with $(\operatorname{aff} D') \cap S = D'$, and $D' \subseteq \pi' \equiv \operatorname{aff}(K \cup \{q\})$.

If either $D = K$ or $D' = K$, the argument is complete. Hence we assume $D \neq K$ and $D' \neq K$ to reach a contradiction. The set $D' \cup D$ cannot contain a set P of $d+1$ affinely independent points, for these points would see $k_1, \cdots, k_{d-1}, x$ via S, contradicting the fact that $\ker_S P$ is a convex set of dimension $d-2$. A similar argument implies that all points seen by $k_1, \cdots, k_{d-1}, x$ necessarily lie in the $(d-1)$-dimensional flat $\operatorname{aff}(D \cup D')$. Then since $[\operatorname{aff}(D \cup D')] \cap \pi \cap S = D$, the subset of π seen by $k_1, \cdots, k_{d-1}, x$ is exactly D.

We assert that $x \in (\operatorname{aff} D) \cap S = D$: Consider the $(d-1)$-dimensional flat π, and let D_1, D_2 denote distinct open halfspaces of π determined by D. Since $K \not\subseteq \operatorname{aff} D$, without loss of generality assume $k_1 \in D_1$. There are two cases to consider, depending on the location of the remaining k_i points.

Case 1. If for every $k_i \notin D$, we have $k_i \in D_1$, then the sets $\operatorname{conv}(D \cup \{k_i\})$, $k_i \notin D$, intersect in a $(d-1)$-dimensional convex set C in $\operatorname{cl}(D_1)$, $D \subseteq \operatorname{bdry} C$, and each point $k_1, \cdots, k_{d-1}$ sees C via S. In case $x \notin \operatorname{aff} D$, $\operatorname{cone}(x, D)$ would intersect $C \sim D$ at some point a, and clearly $[x, a] \subseteq S$. Therefore each of $k_1, \cdots, k_{d-1}, x$ would see a via S, contradicting the fact that the subset of π seen by $k_1, \cdots, k_{d-1}, x$ is exactly D. We conclude that if Case 1 occurs then $x \in \operatorname{aff} D$.

Case 2. If for some $k_i \notin D$, we have $k_i \in D_2$, then the sets $\operatorname{cone}(k_i, D)$, $k_i \notin D$, intersect in a $(d-1)$-dimensional convex set $C, D \subseteq C$, and $k_1, \cdots, k_{d-1}$ see C via S. Again, if $x \notin \operatorname{aff} D$, $\operatorname{cone}(x, D)$ would intersect $C \sim D$ at some point, impossible by the argument in Case 1. We conclude that $x \in \operatorname{aff} D$ if Case 2 occurs, and our assertion is proved.

Thus we have $x \in (\operatorname{aff} D) \cap S = D$, so x sees $k_1, \cdots, k_{d-1}$ via S. However, this is impossible since the subset of π seen by $x, k_1, \cdots, k_{d-1}$ is exactly D and $k_1 \notin D$. Our original assumption is false and either $D = K$ or $D' = K$. In either case, x sees K via S, completing the proof of Lemma 2.

COROLLARY 2. *Assume that* $\operatorname{conv}(K \cup \{p\}) \cup \operatorname{conv}(K \cup \{q\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2, p \notin \operatorname{aff} K$, *and* $q \notin \operatorname{aff}(K \cup \{p\})$. *If* $x \in (S \cap \operatorname{aff}(K \cup \{p\})) \sim \operatorname{aff} K$ *and* $y \in (S \cap \operatorname{aff}(K \cup \{q\})) \sim \operatorname{aff} K$, *then* $[x, y] \not\subseteq S$.

Proof. Otherwise the set $K \cup \{x, y\}$ would contain $d+1$ affinely independent points with each corresponding segment in S, violating Lemma 1.

LEMMA 3. *Assume that* $\operatorname{conv}(K \cup \{p\}) \cup \operatorname{conv}(K \cup \{q\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2, p \notin \operatorname{aff} K$, *and* $q \notin \operatorname{aff}(K \cup \{p\})$. *Let* $\pi = \operatorname{aff}(K \cup \{p\})$, $\pi' = \operatorname{aff}(K \cup \{q\})$. *Select* $r \notin \pi \cup \pi'$, *and let* π_1 *and* π_1' *denote the open halfspaces determined by* π *and* π', *respectively, and containing* r. *If* $u \in \pi \cup \pi'$ *and if* $[r, u] \subseteq S$, *then* $[r, u] \subseteq (\operatorname{cl} \pi_1) \cap (\operatorname{cl} \pi_1')$.

Proof of Lemma 3. If $u \in \operatorname{aff} K = \pi \cap \pi'$, the result is trivial. Hence without loss of generality we assume that $u \in \pi' \sim \operatorname{aff} K$. Then clearly $[r, u] \subseteq \operatorname{cl} \pi_1'$, and we need only show that $[r, u] \subseteq \operatorname{cl} \pi_1$.

It suffices to prove that $(r, u) \cap \pi = \emptyset$: Suppose on the contrary that $v \in (r, u) \cap \pi$. Now $v \notin \pi'$, for otherwise the line determined by u and v would lie in π' and $r \in \pi'$, contradicting our hypothesis. Hence

$v \notin \pi'$, and so $v \notin \operatorname{aff} K$. But then we have $v \in (S \cap \pi) \sim \operatorname{aff} K$, $u \in (S \cap \pi') \sim \operatorname{aff} K$ and $[v, u] \subseteq S$, violating the corollary to Lemma 2. Our assumption is false, $(r, u) \cap \pi = \emptyset$, and $[r, u] \subseteq \operatorname{cl} \pi_1$, finishing the proof of Lemma 3.

LEMMA 4. *Assume that* $\operatorname{conv}(K \cup \{p\}) \cup \operatorname{conv}(K \cup \{q\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2, p \notin \operatorname{aff} K$, *and* $q \notin \operatorname{aff}(K \cup \{p\})$. *If* $z \in S$, *then* z *sees* K *via* S.

Proof of Lemma 4. As in the proof of Lemma 2, let $\pi = \operatorname{aff}(K \cup \{p\})$, $\pi' = \operatorname{aff}(K \cup \{q\})$, and assume that $K = (\operatorname{aff} K) \cap S$. Furthermore, we may suppose that $z \notin \pi \cup \pi'$, for otherwise the result is an immediate consequence of that lemma. Then for $k_1, \cdots, k_{d-1}$ affinely independent in K, the points $k_1, \cdots, k_{d-1}, p, z$ are affinely independent and see via S a unique $(d-2)$-dimensional convex subset A. By the corollary to Lemma 1, since $\operatorname{conv}\{k_1, \cdots, k_{d-1}, p\} \subseteq S \cap \pi$, we have $A \subseteq \pi$, and by Lemma 2, A sees K via S. Similarly, $k_1, \cdots, k_{d-1}, q, z$ see a $(d-2)$-dimensional convex set $A', A' \subseteq \pi'$, and A' sees K via S.

As in Lemma 3, let π_1 and π_1' denote the open halfspaces determined by π and π', respectively, and containing z. Since $A \cup A' \subseteq \pi \cup \pi'$, it follows directly from the lemma that $\operatorname{conv}(A \cup \{z\}) \cup \operatorname{conv}(A' \cup \{z\}) \subseteq \operatorname{cl} \pi_1 \cap \operatorname{cl} \pi_1'$.

If $A = K$ or $A' = K$, the argument is complete. Hence we assume $A \neq K, A' \neq K$, to reach a contradiction. The argument is given in two steps.

Step 1. We show that for an appropriate choice of point t in π' and convex set D in $\alpha \equiv \operatorname{aff}(A \cup \{z\}), [t, d] \cup K \cup A$ lies in S and in the boundary of its convex hull for every d in $D \sim A$. To begin, select $t \in (\operatorname{rel\,int} \operatorname{conv}(K \cup A')) \sim \alpha$ and let α_1 denote the open halfspace determined by α and containing t. Then $t \in \pi_1 \cap \alpha_1$ and, by Lemma 3, $\operatorname{conv}(K \cup \{t\}) \subseteq \operatorname{cl} \pi_1 \cap \operatorname{cl} \alpha_1$. By the corollary to Lemma 1, for $a_1, \cdots, a_{d-1}$ affinely independent in A, the points $a_1, \cdots, a_{d-1}, z, t$ see some $(d-2)$-dimensional convex set D in α, and $(\operatorname{aff} D) \cap S = D$. Furthermore, by Lemma 3 applied to π and α, $\operatorname{conv}(D \cup \{t\}) \subseteq \operatorname{cl} \pi_1 \cap \operatorname{cl} \alpha_1$. Similarly, by the results in Lemmas 2 and 3, D sees A via S and D and A are in $\operatorname{cl} \pi_1'$. We conclude that $\operatorname{conv}(D \cup \{t\})$ lies in $\operatorname{cl} \pi_1 \cap \operatorname{cl} \pi_1' \cap \operatorname{cl} \alpha_1$.

Note that K and D lie in some common hyperplane: Otherwise, for T a subset of $K \cup D$ consisting of $d+1$ affinely independent points, the corresponding set K_T would contain $A \cup \{t\}$, contradicting the fact that $K_T = \ker_S T$ is a convex set of dimension $d-2$. Since A and D also lie in a common hyperplane, we have $K \cap A \subseteq (\operatorname{aff} K) \cap (\operatorname{aff} A) \subseteq \operatorname{aff}(K \cup D) \cap \operatorname{aff}(A \cup D) = \operatorname{aff} D$, $K \cap A \subseteq (\operatorname{aff} D) \cap S = D$, and D contains $K \cap A$. Similarly, $D \cap A \subseteq K \cap A$, and $D \cap \pi = D \cap A = K \cap A$. (Of course, $K \cap A$ may be empty.) By our choice of t, t and $K \sim A$ lie in the same open halfspace determined by α. Moreover, for any point d in $D \sim A$, $K \sim A$ and $[t, d)$ lie in α_1, and it is easy to see that $[t, d] \cup K \cup A$ lies in the boundary of its convex hull, the desired result for Step 1.

Step 2. Next we show that $\operatorname{conv}([t, d] \cup K \cup A) \subseteq S$. This will violate Lemma 1, since $\operatorname{conv}([t, d] \cup K \cup A)$ is a convex set of dimension d. Recall that $d \notin \pi$ so $[t, d] \cap \pi = \emptyset$. By previous comments, for z_0 on $[t, d]$, there corresponds a $(d-2)$-dimensional convex set $E_{z_0} = E_0$ in π such that z_0 sees E_0 via S and $E_0 = (\operatorname{aff} E_0) \cap S$. Lemma 3 implies that $\operatorname{conv}(E_0 \cup \{z_0\}) \subseteq \operatorname{cl} \pi_1 \cap \operatorname{cl} \pi_1' \cap \operatorname{cl} \alpha_1$. Also, using Lemma 2, z_0 and A see all points of $E_0 \cup D$ via S, so $E_0 \cup D$ cannot contain $d+1$ affinely independent points. Thus E_0 and D lie in a common hyperplane. Hence for z_1 and z_2 on $[t, d]$, the corresponding sets E_1 and E_2 are in hyperplanes containing D, $E_t = K$ and $E_d = A$. If K and A are parallel, then since each E_0 set is in a hyperplane containing D, the sets E_1 and E_2 must be parallel. In case $\operatorname{aff} K$ intersects $\operatorname{aff} A$, then $\operatorname{aff} K \cap \operatorname{aff} A \subseteq \operatorname{aff}(K \cup D) \cap \operatorname{aff}(A \cup D) = \operatorname{aff} D$, and $\operatorname{aff} K \cap \operatorname{aff} A \cap \operatorname{aff} D = \operatorname{aff} K \cap \operatorname{aff} A \neq \emptyset$. Also, for every E_0 set, $\operatorname{aff} D \cap \operatorname{aff} A \subseteq \operatorname{aff}(D \cup E_0) \cap \operatorname{aff}(A \cup E_0) = \operatorname{aff} E_0$, and $\operatorname{aff} E_0$ contains the $(d-3)$-dimensional set $\operatorname{aff} D \cap \operatorname{aff} A \cap \operatorname{aff} K = \operatorname{aff} K \cap \operatorname{aff} A$. Therefore each pair of distinct $\operatorname{aff} E_0$ sets will intersect in exactly $\operatorname{aff} K \cap \operatorname{aff} A$. Furthermore, it is not hard to show that for $z_1 \neq z_2$, the sets $\operatorname{conv}(E_1 \cup \{z_1\}) \sim \pi$ and $\operatorname{conv}(E_2 \cup \{z_2\}) \sim \pi$ are disjoint: If $\operatorname{conv}(E_1 \cup \{z_1\})$ intersected $\operatorname{conv}(E_2 \cup \{z_2\})$ at point $b \notin \pi$, then $E_1 \cup E_2 \cup \{b\}$ would contain $d+1$ affinely independent points with corresponding segments in S, violating Lemma 1. Hence the sets must be disjoint.

Now we select $c \in \operatorname{conv}([t, d] \cup K \cup A)$ to show that $c \in S$, and without loss of generality, we assume that $c \in \operatorname{conv}((t, d) \cup K \cup A)$. Our argument is motivated by a planar construction employed in [**1**, Lemma 2]. For $z_0 \in [t, d]$ and E_0 the corresponding subset of π seen by z_0, we have $\operatorname{conv}(K \cup A) \cap (\operatorname{aff} E_0) \subseteq E_0$, so either $c \in \operatorname{conv}([t, z_0] \cup K \cup E_0)$ or $c \in \operatorname{conv}([z_0, d] \cup A \cup E_0)$. Thus we may define sets F, G in the following manner: Let $F = \{z_0 \colon z_0 \in [t, d]$ and $c \in \operatorname{conv}([t, z_0] \cup K \cup E_0)\}$, $G = \{z_0 \colon z_0 \in [t, d]$ and $c \in \operatorname{conv}([z_0, d] \cup A \cup E_0)\}$. By previous comments, if $t < z_1 < z_2 < d$, then $\operatorname{conv}(E_1 \cup \{z_1\}) \cap \operatorname{conv}(E_2 \cup \{z_2\}) \subseteq K \cap A$. Hence, if $z_1 \in F$, we have $z_2 \in F$, and similarly if $z_2 \in G$, then $z_1 \in G$. Therefore, F and G are connected intervals whose union is $[t, d]$, and each of F and G is nonempty since $t \in G$ and $d \in F$. Clearly we may select a point

$u \in (t, d)$ such that $[t, u) \subseteq G$ and $(u, d] \subseteq F$, and without loss of generality, we assume that $u \in F$. We will show that for this u and for E_u the corresponding subset of π seen by u, $c \in \operatorname{conv}(E_u \cup \{u\})$.

Select a sequence $\{u_n\}$ in (t, u) converging to u, and let $\{E_n\}$ represent the corresponding sequence of $(d-2)$-dimensional convex sets in π. Since the sets $\operatorname{aff} E_n$ either are parallel to both A and K or intersect in the $(d-3)$-flat $\operatorname{aff} A \cap \operatorname{aff} K$, and since each E_n lies in the subset of π bounded by $\operatorname{aff} A$ and $\operatorname{aff} K$, clearly the sequence $\{E_n \cap \operatorname{conv}(A \cup K)\}$ converges to a $(d-2)$-dimensional convex set E' in $\operatorname{conv}(K \cup A) \subseteq S$.

We assert that $E' \subseteq E_u$. First we remark that for w in $(\pi \cap S) \sim A$, w cannot lie in $\operatorname{aff}(D \cup \{t\})$. Otherwise, by Lemma 2, w would see both A and D via S, and the set $A \cup D \cup \{w\}$ would contain $d+1$ affinely independent points with corresponding segments in S, violating Lemma 1. Therefore, by previous arguments, each point w of $(\pi \cap S) \sim A$ sees via S a unique $(d-2)$-dimensional subset J_w of $\operatorname{aff}(D \cup \{t\})$, $J_w = (\operatorname{aff} J_w) \cap S$, and either $\operatorname{aff} J_w$ is parallel to sets A, K, and D or $\operatorname{aff} J_w$ contains the $(d-3)$-dimensional flat $\operatorname{aff} A \cap \operatorname{aff} K = \operatorname{aff} A \cap \operatorname{aff} K \cap \operatorname{aff} D$. Furthermore, for w in E_n, the set J_w necessarily contains u_n, so J_w is uniquely determined by n, and therefore each w in E_n is associated with the same $(d-2)$-dimensional subset of $\operatorname{aff}(D \cup \{t\})$, call it J_n. Similarly, let J_u and J' denote the $(d-2)$-dimensional subsets of $\operatorname{aff}(D \cup \{t\})$ seen by E_u and E', respectively, and note that $u \in J_u$. By an earlier argument involving Lemma 1, distinct sets $\operatorname{conv}(E_n \cup J_n) \sim \pi$ are disjoint, and each of these is disjoint from $\operatorname{conv}(E_u \cup J_u) \sim \pi$ and from $\operatorname{conv}(E' \cup J') \sim \pi$. Also, $\operatorname{conv}(E_u \cup J_u) \sim \pi$ and $\operatorname{conv}(E' \cup J') \sim \pi$ are either disjoint or one is a subset of the other. In any event, J' is necessarily bounded in $\operatorname{aff}(D \cup \{t\})$ by $\operatorname{aff} J_u$ and $\operatorname{aff} J_n$ for every n, and since $\{u_n\}$ converges to u, this implies that $u \in J'$. Therefore, J' and J_u both contain u, and $J' = J_u$. Then by Lemma 1, for w' in E', w' must belong to E_u, $E' \subseteq E_u$, and the assertion is proved.

Now since $u_n \in G$, we have $c \in \operatorname{conv}([u_n, d] \cup A \cup E_n)$ for each n, and so $c \in \operatorname{conv}([u, d] \cup A \cup E_u)$. Since $c \in \operatorname{conv}([t, u] \cup K \cup E_u)$, we have $c \in \operatorname{conv}(E_u \cup \{u\}) \subseteq S$, the desired result. We conclude that $\operatorname{conv}([t, d] \cup K \cup A) \subseteq S$, finishing Step 2.

To complete the proof of Lemma 3, notice that $\operatorname{conv}([t, d] \cup K \cup A)$ is a d-dimensional subset of S. Clearly we have a violation of Lemma 1, our preliminary assumption must be false, and one of the sets A, A' must be K. (In fact, by Lemma 1, it is easy to see that $A = A' = K$.) Therefore z sees K via S, and Lemma 4 is proved.

At last, using the lemmas above, the proof of the theorem is immediate. Select a set T consisting of $d+1$ affinely independent points of S, and let $K = \ker_S T$. Since $\dim K = d-2$, clearly we may select points p, q in T such that $p \notin \operatorname{aff} K$ and $q \notin \operatorname{aff}(K \cup \{p\})$. Hence K, p, and q satisfy the hypotheses of Lemmas 2, 3 and 4, and so for $z \in S, z$ sees K via S. Thus $K \subseteq \ker S$, and since $\ker S \subseteq \ker_S T$, we have $K = \ker S$. Therefore S is indeed starshaped, and $\ker S$ has dimension $d-2$, the desired result.

The author would like to thank the referee for his conjecture of the following corollary.

COROLLARY 3. *The hypothesis of the theorem above provides a characterization of d-dimensional sets S, $d \geqq 2$, for which $K \equiv \ker S$ has dimension $d-2$, $(\operatorname{aff} K) \cap S = K$, and the maximal convex subsets of S have dimension $d-1$.*

Proof. If set S satisfies the properties above, then clearly to every $(d+1)$-member subset T of S, the set K serves as an appropriate K_T set. Furthermore, if T is affinely independent, we assert that $K = \ker_S T$: Clearly we may select points t_1 and t_2 in T with $t_1 \notin K$ and $t_2 \notin \operatorname{aff}(K \cup \{t_1\})$. For y any point which sees T via S, if $y \notin \operatorname{aff}(K \cup \{t_1\})$, then $\operatorname{conv}(K \cup [y, t_1])$ would be a full d-dimensional subset of S, contradicting the fact that maximal convex subsets of S have dimension $d-1$. Hence $y \in \operatorname{aff}(K \cup \{t_1\})$. Similarly $y \in \operatorname{aff}(K \cup \{t_2\})$, and $y \in (\operatorname{aff} K) \cap S = K$. We conclude that $K = \ker_S T$, and S indeed satisfies the hypothesis of the theorem.

Conversely, if S satisfies the hypothesis of the theorem, then the dimension of $K \equiv \ker S$ is $d-2$ and $(\operatorname{aff} K) \cap S = K$. We need only show that for M a maximal convex subset of S, $\dim M = d-1$. Clearly $\dim M \leqq d-1$, and since $K \subseteq M$, $\dim M \geqq d-2$. If $\dim M = d-2$, then $M \subseteq (\operatorname{aff} K) \cap S = K$, and $M = K$. However, since M is maximal, this implies that there are no points of S not in K, impossible since S is a full d-dimensional. Thus $\dim M = d-1$, finishing the proof of the corollary.

In conclusion, note that for $d \geqq 3$, the result fails without the requirement that $(\operatorname{aff} K_T) \cap S = K_T$, as the following example illustrates.

EXAMPLE 1. Let $\{S_n\}$ be a sequence of $(d-1)$-dimensional simplices in $R^d, \{E_n\}$ a corresponding sequence of $(d-2)$-dimensional simplices, so that E_n is a facet of S_n, $E_{n+1} \subseteq E_n$, $\cap \{S_i : 1 \leqq i \leqq n+1\} = E_{n+1}$, and $K = \cap \{E_n : 1 \leqq n\}$ is a singleton set. Define $S = \operatorname{cl}(\cup \{S_n : 1 \leqq n\})$. Then for T any finite subset of S and k the largest integer such that $S_k \cap T \neq \emptyset$, $E_k \subseteq \ker_S T$. Moreover, if T contains $d+1$ affinely independent points, then $E_k = \ker_S T$. However, $\dim(\ker S) = \dim K = 0$. Hence for $d \geqq 3$, the theorem fails without the requirement that $(\operatorname{aff} K_T) \cap S = K_T$. Of course, if $d = 2$, each set K_T is a singleton set so that $(\operatorname{aff} K_T) \cap S = K_T$ will be satisfied automatically.

An easy adaptation of Example 1 shows that S need not even be starshaped unless $(\operatorname{aff} K_T) \cap S = K_T$.

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