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ALGEBRAS WHICH SATISFY A SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

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Let A be an algebra of complex valued functions satisfying a second order linear partial differential equation in a plane domain. If the equation is hyperbolic or parabolic, the functions of A are locally functions of only one variable. If the equation is elliptic, there exists a unique complex function λ such that $f_x = \lambda f_y$ for each f in A, and after a change of variables each function in A is analytic. If an algebra of functions satisfies the maximum principle, and one nonconstant function and its square satisfy an elliptic equation, then every function in the algebra satisfies this equation.

1. Introduction. In this paper we study algebras of complex valued functions defined on a plane domain, which satisfy some linear second order partial differenital equation

$$Lw = aw_{xx} + 2bw_{xy} + cw_{yy} + dw_x + ew_y = 0, \tag{1}$$

with real coefficients. We start with an example which turns out to be typical of the significant cases.

Let L be a self-adjoint elliptic operator:

$$Lw = \frac{\partial}{\partial x}(aw_x + bw_y) + \frac{\partial}{\partial y}(bw_x + cw_y), \tag{2}$$

where a, b, c are C^2 real functions on a simply connected domain, satisfying the normalizing condition $ac - b^2 = 1$. For each C^2 function u satisfying $Lu = 0$, we define (up to an additive constant) a conjugate function v by

$$v(x, y) = \int^{(x,y)} -(bu_x + cu_y)dx + (au_x + bu_y)dy. \tag{3}$$

It is easy to check the following facts: $Lv = 0$; the conjugate of v is $-u$; the set of functions $u + iv$ is an algebra; $(u + iv)^{-1}$ is in the algebra if $u + iv \neq 0$.

The functions $u + iv$ turn out to be analytic after the appropriate change of variables. Moreover, the example illustrates the only way that the functions of an algebra can satisfy a linear second order elliptic partial differential equation.

Suppose A is a function algebra (a Banach algebra of complex continuous functions, with the sup norm) on the unit circle $\Gamma = \{z : |z| = 1\}$. If $\operatorname{Re} A = \operatorname{Re} A_0$, where A_0 is the disc algebra restricted to the circle, then **[4, 5]** $A = A_0 \circ \Phi$ for some homeomorphism Φ of Γ onto Γ. We obtain a similar result for algebras defined on a domain rather than on its boundary. Specifically, if A is an algebra of functions on a domain and $\operatorname{Re} A$ consists of harmonic functions, then A or $\bar{A}$ consists of analytic functions.

We also obtain a simple geometric characterization of functions which are analytic functions of a homeomorphism (i.e., interior mappings in the sense of Stoilow). Let u, v be sufficiently smooth real functions on a domain G. Then $u + iv$ or $u - iv$ is analytic on G if and only if $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$ on G. This result generalizes as follows. We define a family of inner products "$*$", each with its norm "$\| \ \|$". For each such inner product $*$, the equations $\nabla u * \nabla v = 0$ and $\|\nabla u\| = \|\nabla v\|$ characterize those functions $u + iv$ which are analytic after a change of variables determined by $*$. The equations imply that ∇u and ∇v are nonparallel wherever they are nonzero. The converse is essentially true. In particular, if ∇u and ∇v never vanish and are never parallel on a domain, then $u + iv$ is analytic after an appropriate change of variables.

In the final section we apply our results to algebras which satisfy a maximum principle on G, and obtain two extensions of results of Rudin [7] for such function algebras.

2. The parabolic and hyperbolic cases. In this section we consider algebras of complex C^2 functions which satisfy (1), where L is parabolic or hyperbolic. We show that no such algebra can separate points, and in fact must consist essentially of functions of one variable.

We assume that the coefficients of L are real C^2 functions on a domain G in the (x, y)-plane, and that a, b, c do not vanish simultaneously. A solution of (1) is a real or complex C^2 function which satisfies (1) identically on G.

An "algebra of functions" on G will always be assumed to contain at least one non-constant function.

A "change of variables" means a one-to-one transformation $(x, y) \to (\xi, \eta)$ where $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ are C^2 functions and the Jacobian $\xi_x \eta_y - \xi_y \eta_x$ does not vanish. It follows that the inverse functions $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ are also C^2. The equation (1) transforms into the following equivalent equation in the (ξ, η) variables:

$$L'w = a'w_{\xi\xi} + 2b'w_{\xi\eta} + c'w_{\eta\eta} + d'w_{\xi} + e'w_{\eta} = 0, \tag{4}$$

where

$$\begin{aligned} a' &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\ b' &= a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + c\xi_y\eta_y, \\ c' &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2, \\ d' &= L\xi, \\ e' &= L\eta. \end{aligned} \tag{5}$$

Clearly a', b', c' are C^1 functions, and d', e' are continuous.

If (1) is a parabolic equation (i.e., $ac - b^2 = 0$ in G), then for each point of G there is a neighborhood U, and a change of variables on U onto U' so that the equation takes the form

$$L'w = w_{\xi\xi} + d'w_{\xi} + e'w_{\eta} = 0 \tag{6}$$

on U' [**6**, p. 63].

If (1) is hyperbolic ($ac - b^2 < 0$ in G), then each point of G has a neighborhood U and a change of variables on U onto U' so that the equation takes the form

$$L''w = w_{\xi\eta} + d'w_{\xi} + e'w_{\eta} = 0 \tag{7}$$

on U' [**6**, p. 58].

We first look at algebras which satisfy a parabolic or hyperbolic equation in canonical form. To this end define the operators M and N as follows (d and e are continuous functions):

$$Mw = w_{xx} + dw_x + ew_y, \tag{8}$$

$$Nw = w_{xy} + dw_x + ew_y. \tag{9}$$

THEOREM 1. *Let A be an algebra of complex C^2 functions which satisfy the parabolic equation $Mf = 0$. Then each f in A is a function of y only.*

Proof. Let $f = u + iv \in A$. Since $f^2 \in A$, $u^2 - v^2$ and uv also satisfy $Mw = 0$. Setting $M(uv) = 0$ and using $Mu = Mv = 0$ we conclude that $u_x v_x = 0$. Similarly, $M(u^2 - v^2) = 0$ leads to $u_x^2 = v_x^2$. Hence $u_x = v_x = 0$, and f is a function of y.

COROLLARY. *If A is an algebra of complex C^2 functions which satisfy the parabolic equation* (1) *on G, then each function of A is locally a function of the same single variable after a change of coordinates.*

Now we turn to the hyperbolic case.

LEMMA 2. *Let A be an algebra of complex C^2 functions on G such that $Nf = 0$ for all $f \in A$. If A contains some function f such that f_x does not vanish on G, then every function in A is a function of x only. Similarly, if A contains a function g such that g_y does not vanish on G, then every function in A is a function of y only.*

Proof. Let $f = u + iv \in A$. Using the equations $Nu = Nv = N(uv) = N(u^2 - v^2) = 0$ we obtain

$$u_x v_y + u_y v_x = 0 \tag{10}$$
$$u_x u_y - v_y v_x = 0.$$

Considered as equations in u_x, v_x, the determinant is $-(u_y^2 + v_y^2)$. Hence $(u_x, v_x) \neq (0,0)$ implies $(u_y, v_y) = (0,0)$. Similarly, $(u_y, v_y) \neq (0,0)$ implies $(u_x, v_x) = (0,0)$, and f_x, f_y cannot both be nonzero at the same point. If f_x does not vanish in G, then $f_y \equiv 0$ in G, and f is a function of x. Similarly, if g_y does not vanish on G, then g is a function of y only. If f_x does not vanish on G, then $g_y \equiv 0$ for all $g \in A$. Otherwise, if $g_y(x_0, y_0) \neq 0$, we let $h = f + g$, and get the contradiction $h_x(x_0, y_0) = f_x(x_0, y_0) \neq 0$, and $h_y(x_0, y_0) = g_y(x_0, y_0) \neq 0$.

THEOREM 3. *Let A be an algebra of complex C^2 functions on a domain G, such that $Lf = 0$ for all $f \in A$, where L is hyperbolic. If $f_x(x_0, y_0) \neq 0$ or $f_y(x_0, y_0) \neq 0$ for some $f \in A$ and some $(x_0, y_0) \in G$, then there is a neighborhood U of (x_0, y_0) and a change of variables $(x, y) \rightarrow (\xi, \eta)$ on U onto U' such that every function in A is a function of ξ on U', or every function in A is a function of η on U'.*

Proof. We make a local change of variables so that $Lf = 0$ becomes (7) on U'. Since $f_x = f_\xi \xi_x + f_\eta \eta_x$, $f_y = f_\xi \xi_y + f_\eta \eta_y$, either f_ξ or f_η is nonzero in a sufficiently small neighborhood of $(\xi(x_0, y_0), \eta(x_0, y_0))$. The result then follows from Lemma 2.

The following example shows that when the functions of an algebra satisfy a hyperbolic equation, these functions need not be globally functions of the same variable. Let $S = \{(x, y): |x| < 1, |y| < 1\}$ and let G be S with the closed first quadrant removed. Let $f(x, y) = y^3$ in the second quadrant, $f(x, y) = 0$ in the third quadrant, and $f(x, y) = x^3$ in the fourth quadrant. All polynomials in f satisfy the hyperbolic equation $w_{xy} = 0$.

3. Elliptic case with Laplacian principal part. We consider now the following elliptic equation:

$$L_0 w = w_{xx} + w_{yy} + dw_x + ew_y = 0, \tag{11}$$

where d and e are continuous real functions on a domain G. We show that if A is an algebra of complex C^2 functions on G which satisfy (11), then A or $\bar{A}$ consists of analytic functions, and $d = e = 0$.

The following theorem gives a very simple and appealing geometric description of how the gradients of real functions u and v must behave in order for $u + iv$ to be a conformal mapping. We say that u and v are *conjugate harmonic functions* in a domain if either $u + iv$ or $u - iv$ is analytic in that domain.

THEOREM 4. *If u, v are C^2 functions on a domain G, then u and v are conjugate harmonic functions in G if and only if*

$$\nabla u \cdot \nabla v = 0, \qquad |\nabla u| = |\nabla v| \tag{12}$$

hold identically in G.

Note. The result above was stated by Dzyadyk **[3]** for C^1 functions but there appears to be a gap in the proof at the following point. Let ϕ be continuous on a domain G, zero on a closed set Z contained in G, and analytic on each component of $G - Z$. Then (?) $G - Z$ has only one component and ϕ is analytic in G. We do not know a proof of this statement.[1] However for our purposes we only require the result for C^2 functions, and for this case we furnish the elementary proof below.

Proof. If either f or $\bar{f}$ is analytic, then (12) follows from the Cauchy–Riemann equations. We assume therefore that (12) holds. If $f = u + iv$, then (12) is equivalent to $f_x^2 + f_y^2 = 0$. Hence $f_x = 0$ if and only if $f_y = 0$, and $f_x = \pm i f_y$. Let $Z = \{(x, y): u_x = u_y = v_x = v_y = 0\}$. Then $G - Z = Z^c$ is open, and $Z^0 \cup Z^c$ is dense in G. Clearly u and v are harmonic on Z^0. Since $f_x = \pm i f_y$ with one sign holding on each component of Z^c, f or $\bar{f}$ is analytic on each component of Z^c. Hence u

[1] We are indebted to Walter Rudin for pointing out that this statement is a Theorem of Radó (see, e.g. [**8**, Theorem 12.13]).

and v are harmonic on $Z^0 \cup Z^c$, and by the continuity of $u_{xx} + u_{yy}$, $v_{xx} + v_{yy}$, u and v are harmonic on G. The functions $g = u_x - iu_y$ and $h = v_x - iv_y$ are analytic in G, and hence have isolated zeros (unless u and v are constant). Therefore Z consists of isolated points, Z^0 is empty, and Z^c is connected and dense. Hence f or $\bar{f}$ is analytic on all of Z^c. Since f is continuous on G, f or $\bar{f}$ is analytic on G.

DEFINITION. We say u and v are *square-conjugates* for L if and only if u or v is nonconstant, and $Lu = Lv = L(uv) = L(u^2 - v^2) = 0$. The last condition is of course equivalent to $L(f) = L(f^2) = 0$, where $f = u + iv$.

THEOREM 5. *If u, v are C^2 functions in G which are square-conjugate for L_0, then u and v are conjugate harmonic functions in G; moreover, $d = e = 0$, and L_0 is the Laplacian.*

Proof. We calculate as follows:

$$L_0(uv) = uL_0(v) + vL_0(u) + 2\nabla u \cdot \nabla v,$$
$$L_0(u^2 - v^2) = 2uL_0(u) - 2vL_0(v) + 2(|\nabla u|^2 - |\nabla v|^2).$$

If u and v are square-conjugates, then $\nabla u \cdot \nabla v = 0$, and $|\nabla u| = |\nabla v|$, and u, v are harmonic conjugates by Theorem 4.

Since u and v are harmonic and $L_0 u = L_0 v = 0$, we also have

$$du_x + eu_y = 0, \qquad dv_x + ev_y = 0. \tag{13}$$

Let $f = u \pm iv$ (whichever is analytic in G). The determinant of the system (13) is $\pm|f'|^2$. Since u or v is nonconstant, the determinant vanishes at most at isolated points of G, and off this set $d = e = 0$. By continuity, $d = e = 0$ on G.

The following theorem says in particular that function algebras whose real parts are harmonic functions consist of analytic functions, or consist of conjugates of analytic functions.

THEOREM 6. *If A is an algebra of complex C^2 functions on G, and $L_0 f = 0$ for all $f \in A$, then A or $\bar{A}$ consists of analytic functions, and L_0 is the Laplacian.*

Proof. Let $f = u + iv$ be a nonconstant function in A. Then u and v are square-conjugates for L_0, f or $\bar{f}$ is analytic, and $d = e = 0$. We need only show that A cannot contain both a nonconstant analytic function and the conjugate of a nonconstant analytic function. Suppose on the contrary that $g, \bar{h} \in A$, with g and h analytic and nonconstant. Then $g\bar{h}$ or $\bar{g}h$ is analytic. If, for example, $g\bar{h}$ is analytic, then $\bar{h}$ is analytic except on the set of isolated points where g is zero. Since h is continuous, h and $\bar{h}$ are both analytic, which is a contradiction.

4. The general elliptic case.

We now consider an elliptic equation of the form

$$Lw = aw_{xx} + 2bw_{xy} + cw_{yy} + dw_x + ew_y = 0. \tag{14}$$

We will make standard assumptions on the coefficients in terms of the following definitions.

DEFINITION. A function f is Hölder continuous in G if for every compact subset K of G there are positive constants c, α, with $0 < \alpha \leqq 1$, such that $|f(z_1) - f(z_2)| \leqq c\,|z_1 - z_2|^\alpha$ for all $z_1, z_2 \in K$. A function f is in the class $H_m(G)$ if f and its partial derivatives up to order m are Hölder continuous in G.

DEFINITION. We will say that the operator L of (14) is a *regular elliptic operator* in G if $a, b, c \in H_1(G)$, d, e are continuous on G, and the two normalizing conditions hold: $ac - b^2 = 1$, $a > 0$.

LEMMA 7. *If u, v satisfy* (14), *then* $L(uv) = 0$ *if and only if*

$$\nabla u * \nabla v = 0, \tag{15}$$

and $L(u^2 - v^2) = 0$ *if and only if*

$$\|\nabla u\| = \|\nabla v\| \tag{16}$$

where

$$\nabla u * \nabla u \equiv au_xv_x + bu_xv_y + bu_yv_x + cu_yv_y\,.$$
$$\|\nabla u\|^2 \equiv \nabla u * \nabla u = au_x^2 + 2bu_xu_y + cu_y^2.$$

If $f = u + iv$, then (15) *and* (16) *together are equivalent to the complex form*

$$af_x^2 + 2bf_xf_y + cf_y^2 = 0. \tag{17}$$

Proof. The results are easily verified by computation, and do not depend on the ellipticity of L.

We now let L be the regular elliptic operator of (14), and consider the following Beltrami system associated with L:

$$\eta_x = b\xi_x + c\xi_y \tag{18}$$
$$\eta_y = -a\xi_x - b\xi_y.$$

Solving for ξ_x and ξ_y gives the equivalent system

$$\xi_x = -b\eta_x - c\eta_y \tag{19}$$
$$\xi_y = a\eta_x + b\eta_y.$$

Because of the hypotheses on a, b, c, we can invoke a known result which says that there is a global homeomorphism $(x, y) \to (\xi, \eta)$ from G onto G' such that ξ and η satisfy (18) and (19), and the Jacobian $\xi_x\eta_y - \xi_y\eta_x$ does not vanish on G. (See [**2**, p. 160], and for a more general result see [**1**].) There is no restriction on the domain G. Functions satisfying (18), (19) are necessarily in the class $H_2(G)$ [**9**, Theorem 2.4, p. 87], and in particular are C^2 functions on G. It follows from (4), (5), (18) that (14) becomes

$$a'(w_{\xi\xi} + w_{\eta\eta}) + (L\xi)w_\xi + (L\eta)w_\eta = 0 \tag{20}$$

in the (ξ, η) variables (cf. [**2**], p. 159).

THEOREM 8. *There is a square-conjugate pair of functions u, v for the regular elliptic operator L if and only if*

$$d = a_x + b_y; \qquad e = b_x + c_y. \tag{21}$$

If (21) *holds, and ξ, η are a change of variables satisfying* (18), (19), *then* (20) *is Laplace's equation, and ξ, η are square-conjugates for L. Functions u, v are square-conjugates for L if and only if $u + iv$ or $u - iv$ is an analytic function of $\xi + i\eta$ in G'.*

Note. The equations (21) are just the conditions that L be self-adjoint:

$$Lw = (aw_x + bw_y)_x + (bw_x + cw_y)_y. \tag{22}$$

Without the assumption that (14) is normalized $(ac - b^2 = 1)$, (21) becomes

$$Kd = (Ka)_x + (Kb)_y, \qquad Ke = (Kb)_x + (Kc)_y, \tag{23}$$

where $K = (ac - b^2)^{-1/2}$.

Proof. Assume that u, v are square-conjugates for L, so that u, v, uv, and $u^2 - v^2$ all satisfy (20) when considered as functions of ξ and η in G'. Because $ac - b^2 > 0$, the coefficient a' of (20) (cf. (5)) only vanishes when $\xi_x = \xi_y = 0$. Hence $a' \neq 0$ on G since $\xi_x\eta_y - \xi_y\eta_x \neq 0$. By Theorem 5, $L\xi = L\eta = 0$ and (20) is Laplace's equation. Since ξ and η are obviously square-conjugates for Laplace's equation $w_{\xi\xi} + w_{\eta\eta} = 0$, ξ and η are square-conjugates for L. We set $\eta_{xy} = \eta_{yx}$ in (18) and get

$$(a\xi_x + b\xi_y)_x + (b\xi_x + c\xi_y)_y = 0. \tag{24}$$

Subtracting $L\xi = 0$ from (24) gives

$$(a_x + b_y - d)\xi_x + (b_x + c_y - e)\xi_y = 0. \tag{25}$$

Similarly, from (19) and $L\eta = 0$ we get

$$(a_x + b_y - d)\eta_x + (b_x + c_y - e)\eta_y = 0. \tag{26}$$

Since the Jacobian $\xi_x\eta_y - \xi_y\eta_x \neq 0$, we conclude that $d = a_x + b_y$ and $e = b_x + c_y$.

Now assume that (21) holds; i.e., that L is the self-adjoint operator (22). It follows immediately from (18) and (19) that $L\xi = L\eta = 0$. Hence (20) is Laplace's equation and ξ, η are square-conjugates for L.

The square-conjugate pairs u, v for L in G correspond to the square-conjugate pairs for Laplace's equation in G'. Hence by Theorem 5 the square-conjugate pairs u, v for L coincide with the analytic functions $u + iv$ of $\xi + i\eta$ in G'.

COROLLARY 1. *If A is an algebra of complex C^2 functions on G such that $Lf = 0$ for some regular elliptic operator and all $f \in A$, then there is a change of variables $\zeta = \xi + i\eta$ on G onto G' such that $f \circ \zeta^{-1}$ is analytic on G' for all $f \in A$.*

Proof. As in Theorem 6, A or $\bar{A}$ consists of analytic functions of $\xi + i\eta$. If $\bar{f}$ is an analytic function of $\xi + i\eta$, f is an analytic function of $\xi - i\eta$.

We know that if (ξ, η) is a change of variables on G and ξ, η satisfy the Beltrami system (18), then ξ and η are square-conjugates for L. We show next that the Beltrami equations characterize square-conjugacy in general; i.e., without assuming the mapping $(x, y) \to (\xi, \eta)$ is one-to-one. In fact, the Beltrami systems are simply the Cauchy–Riemann equations after a change of variable.

We consider the following two Beltrami systems, which are the same as (18) and its negative. We continue to assume that $a, b, c \in H_1(G)$.

$$\begin{aligned} v_x &= bu_x + cu_y \\ v_y &= -au_x - bu_y; \end{aligned} \tag{27}$$

$$\begin{aligned} v_x &= -bu_x - cu_y \\ v_y &= au_x + bu_y. \end{aligned} \tag{28}$$

LEMMA 9. *If $(x, y) \to (\xi, \eta)$ is a change of variables on G onto G' such that ξ and η satisfy* (18), (19), *then* (27) *is equivalent to $u_\xi = v_\eta$, $u_\eta = -v_\xi$, and* (28) *is equivalent to $u_\xi = -v_\eta$, $u_\eta = v_\xi$.*

Proof. We write the first equation of (27) in terms of ξ and η, using (18) and (19):

$$\begin{aligned} v_\xi \xi_x + v_\eta \eta_x &= b(u_\xi \xi_x + u_\eta \eta_x) + c(u_\xi \xi_y + u_\eta \eta_y) \\ &= u_\xi (b\xi_x + c\xi_y) + u_\eta (b\eta_x + c\eta_y) \\ &= u_\xi \eta_x - u_\eta \xi_x. \end{aligned}$$

In the same way, the second equation in (27) yields

$$v_\xi \xi_y + v_\eta \eta_y = u_\xi \eta_y - u_\eta \xi_y.$$

Hence we have the following system representing (27) in the (ξ, η) variables:

$$\begin{aligned} v_\xi \xi_x + v_\eta \eta_x &= u_\xi \eta_x - u_\eta \xi_x \\ v_\xi \xi_y + v_\eta \eta_y &= u_\xi \eta_y - u_\eta \xi_y. \end{aligned} \tag{29}$$

Since $\xi_x \eta_y - \xi_y \eta_x \neq 0$, we can solve for v_ξ and v_η, and we get

$$v_\xi = -u_\eta, \qquad v_\eta = u_\xi. \tag{30}$$

Of course (30) is equivalent to (29) and (27), and similarly (28) is equivalent to

$$v_\xi = u_\eta, \qquad v_\eta = - u_\xi. \tag{31}$$

THEOREM 10. *If L is a self-adjoint regular elliptic operator, and u, v are C^2 functions on G, then the following are equivalent:*

(a) *u, v are square-conjugates for L in G*

(b) *u, v satisfy one of the Beltrami systems* (27), (28) *throughout G*

(c) *u, v satisfy* (15), (16) *in G.*

Proof. Let ξ and η be a change of variables on G onto G' such that (18) and (19) hold. Then by Theorem 8, $Lw = 0$ becomes Laplace's equation in the (ξ, η) variables.

Assume (a) holds. Then u and v are square-conjugates for Laplace's equation in G'. Hence u and v are conjugate harmonic functions of ξ and η; i.e., (30) or (31) holds, and therefore (27) or (28) holds.

To show that (b) implies (a), we assume that u and v satisfy (27) or (28), and hence that u and v are conjugate harmonic functions of ξ and η in G'. Hence u and v are square-conjugates for Laplace's equation in G', and therefore square-conjugates for L in G.

We have already shown (Lemma 7) that (a) implies (c), so assume (c) holds. Let $f = u + iv$, so that (17) holds:

$$af_x^2 + 2bf_xf_y + cf_y^2 = 0.$$

Substituting $f_x = f_\xi \xi_x + f_\eta \eta_x$, $f_y = f_\xi \xi_y + f_\eta \eta_y$ we get

$$f_\xi^2(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2) + f_\eta^2(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2) = 0. \tag{32}$$

Here we used the fact that

$$a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + c\xi_y\eta_y = 0,$$

which follows from (18). The coefficients of f_ξ^2 and f_η^2 are equal and nonzero. Hence $f_\xi^2 + f_\eta^2 = 0$, which is equivalent to

$$\begin{aligned} u_\xi v_\xi + u_\eta v_\eta &= 0 \\ u_\xi^2 + u_\eta^2 &= v_\xi^2 + v_\eta^2. \end{aligned} \tag{33}$$

By Theorem 4, u and v are conjugate harmonic functions on G', and hence square-conjugates for L.

COROLLARY 1. *If u, v is any square-conjugate pair for L, then $J = u_x v_y - u_y v_x$ is nonzero on any open subset of G on which $f = u + iv$ is one-to-one. The zeros of J and f_x and f_y are isolated.*

Note. This result is proved for solutions of Beltrami systems in [**9**, p. 91]. We include a brief proof here for the reader's convenience.

Proof. To be specific, assume u, v satisfy the Beltrami system (27). Let ξ, η be new variables such that $u + iv$ is an analytic function of $\xi + i\eta$, with $u_\xi = v_\eta$ and $u_\eta = -v_\xi$. Then

$$\begin{aligned} J &= \frac{\partial(u, v)}{\partial(\xi, \eta)} \frac{\partial(\xi, \eta)}{\partial(x, y)} \\ &= (u_\xi v_\eta - u_\eta v_\xi)(\xi_x \eta_y - \xi_y \eta_x) \\ &= (u_\xi^2 + v_\xi^2)(\xi_x \eta_y - \xi_y \eta_x) \\ &= |f'|^2(\xi_x \eta_y - \xi_y \eta_x). \end{aligned}$$

The zeros of f' are isolated, and f' is not zero on any open set on which f is one-to-one.

COROLLARY 2. *If A is an algebra of complex C^2 functions on G which satisfy* (14), *where L is a regular elliptic operator, then L is self-adjoint and* (27) *or* (28) *holds for every $u + iv$ in A.*

Proof. Either every f is an analytic function of $\xi + i\eta$ or every $\bar{f}$ is an analytic function of $\xi + i\eta$ by Theorem 6.

Next we characterize those pairs u, v of $H_2(G)$ functions which are square-conjugates for some regular elliptic operator L. We show there is at most one such L for any pair u, v. We also give a simple geometric condition on u and v which characterizes the fact that $u + iv$ is an analytic function composed with a homeomorphism.

THEOREM 11. *Let f be a nonconstant function in $H_2(G)$. If f satisfies*

$$L(f) = L(f^2) = 0 \tag{34}$$

for some regular elliptic operator L on G, then f satisfies

$$f_x = \lambda f_y \tag{35}$$

for some complex function $\lambda \in H_1(G)$ *with* $\operatorname{Im}\lambda \neq 0$, *and* λ *is determined up to complex conjugation by the coefficients of* L.

Conversely, if f *satisfies* (35) *in* G *for a complex function* λ *in* $H_1(G)$, *with* $\operatorname{Im}\lambda \neq 0$, *then there is a unique regular elliptic operator* L *such that* $L(f) = L(f^2) = 0$.

Proof. If $L(f) = L(f^2) = 0$, then by Lemma 7 we have

$$af_x^2 + 2bf_xf_y + cf_y^2 = 0. \tag{36}$$

The zeros of f_x and f_y are isolated by Corollary 1 of Theorem 10, so the quadratic equation (36) gives

$$f_x = (-b/a \pm i/a)f_y, \tag{37}$$

with $\lambda = -b/a \pm i/a$ uniquely determined except for the sign of $\operatorname{Im}\lambda$. Since a never vanishes and f_y vanishes at most at isolated points, the sign of $\operatorname{Im}\lambda$ is constant in G.

Now assume that (35) holds, with $\operatorname{Im}\lambda > 0$ to be specific. We let

$$a = 1/\operatorname{Im}\lambda, \qquad b = -\operatorname{Re}\lambda/\operatorname{Im}\lambda, \qquad c = |\lambda|^2/\operatorname{Im}\lambda. \tag{38}$$

Then $ac - b^2 = 1$, $a > 0$, and $a, b, c \in H_1(G)$. It is easy to check that (36) holds, so $L(f) = L(f^2) = 0$ by (c) of Theorem 10. Equation (37) shows that L is uniquely determined by λ, given that $ac - b^2 = 1$ and $a > 0$.

COROLLARY 1. *If* L *is a regular elliptic operator and* $L(f) = L(f^2) = 0$, *then* $L(\phi \circ f) = 0$ *for every* ϕ *analytic on* $f(G)$.

Proof. If $g = \phi \circ f$, then $g_x = (\phi' \circ f)f_x$ and $g_y = (\phi' \circ f)f_y$, so $g_x = \lambda g_y$ if $f_x = \lambda f_y$.

COROLLARY 2. *If* $f \in H_2(G)$, *and* f *is nonconstant, there is at most one regular elliptic operator* L *on* G *such that* $L(f) = L(f^2) = 0$, *and there is at most one* $\lambda \in H_1(G)$ *with* $\operatorname{Im}\lambda \neq 0$ *such that* $f_x = \lambda f_y$.

COROLLARY 3. *If* $f = u + iv \in H_2(G)$ *and* $J = u_xv_y - u_yv_x$ *does not vanish on* G, *then there is a unique regular elliptic operator* L *on* G *such that* $L(f) = L(f^2) = 0$.

Proof. If $J \neq 0$, then $f_y \neq 0$, and if $\lambda = f_x/f_y$, then $\operatorname{Im}\lambda = -J/|f_y|^2 \neq 0$.

The Cauchy–Riemann equations can be written $f_x = -if_y$, where $f = u + iv$. The following can therefore be considered a generalization of the Cauchy–Riemann characterization of analyticity:

THEOREM 12. *If* $f \in H_2(G)$ *and* $f_x = \lambda f_y$ *for some* $\lambda \in H_1(G)$ *with* $\operatorname{Im}\lambda \neq 0$, *then* $f = \phi \circ \zeta$ *where* $\zeta \in H_2(G)$ *is a homeomorphism of* G, *and* ϕ *is analytic on* $\zeta(G)$.

Proof. We know that f or $\bar{f}$ is an analytic function of $\zeta = \xi + i\eta$, where $\xi + i\eta$ is a homeomorphism and ξ, η satisfy (18), and $\zeta \in H_2(G)$ by [**9**, Theorem 2.4, p. 87]. If $\bar{f}$ is an analytic function of ζ, then f is an analytic function of $\bar{\zeta}$.

A geometric interpretation of the condition $f_x = \lambda f_y$, $\operatorname{Im}\lambda \neq 0$, can be given as follows. If the complex quantities f_x and f_y are considered as vectors in two-space, the condition implies that these vectors are nonparallel whenever they are nonzero. But f_x and f_y are nonzero and nonparallel at the same time that ∇u and ∇v are nonzero and nonparallel, as can be seen by considering the 2×2 matrix whose rows are ∇u and ∇v and whose columns are f_x and f_y. Thus for the case when ∇u and ∇v do not vanish, the hypothesis of Theorem 12 is simply that ∇u and ∇v are nonparallel and $u, v \in H_2(G)$.

5. Algebras satisfying a maximum principle.

In this section we use the results of §4 to describe certain algebras which satisfy a maximum principle. These results extend those of Rudin [**7**].

DEFINITION. We will say that an algebra of continuous complex functions on G satisfies the maximum principle on G if for every compact subset K of G and every $f \in A$, $\max\{|f(z)|: z \in K\} = \max\{|f(z)|: z \in \partial K\}$.

THEOREM 13. *Let* A *be an algebra of complex functions in* $H_2(G)$ *which satisfies the maximum principle. If*

$$L(f) = L(f^2) = 0 \tag{39}$$

for some nonconstant $f \in A$ *and some regular elliptic operator* L, *then* (39) *holds for all* $f \in A$. *If*

$$f_x = \lambda f_y \tag{40}$$

for some nonconstant $f \in A$ *and some* $\lambda \in H_1(G)$ *with* $\operatorname{Im}\lambda \neq 0$, *then every function in* A *satisfies* (40).

Proof. Let f be a nonconstant function in A which satisfies (39). By Theorem 8, there is a change of variables $(x, y) \to (\xi, \eta)$ such that (39) becomes Laplace's equation, and f is an analytic function of $\zeta = \xi + i\eta$. (Replace ζ with $\bar{\zeta}$ if $\bar{f}$ is analytic.) Rudin has shown [7, Theorem 2] that, in an algebra satisfying the maximum principle, if one nonconstant function is analytic then every function is analytic. Thus every $g \in A$ is an analytic function of ζ. Again using Theorem 8, we conclude that every $g \in A$ satisfies (39). If f is nonconstant and satisfies (40), then by Theorem 12, $f = \phi \circ \zeta$ where $\zeta \in H_2(G)$ is a homeomorphism of G, and ϕ is analytic on $\zeta(G)$. Since $f_x - \lambda f_y = 0 = (\phi' \circ \zeta)(\zeta_x - \lambda \zeta_y)$ and the zeros of ϕ' are isolated, $\zeta_x = \lambda \zeta_y$. Again by Rudin's result, every $g \in A$ is an analytic function of ζ. It follows that $g_x = \lambda g_y$.

We next give a local criterion that an algebra satisfying the maximum principle consists of analytic functions after a change of variables.

THEOREM 14. *Let A be an algebra of functions in $H_2(G)$ which satisfies the maximum principle. Suppose that at each point z in G there exists an open sphere $S_z \subset G$ centered at z and a function $\lambda_z \in H_1(S_z)$ with* $\operatorname{Im} \lambda_z \neq 0$, *and a function f_z in A, nonconstant in S_z, such that*

$$\frac{\partial f_z}{\partial x} = \lambda_z \frac{\partial f_z}{\partial y} \tag{41}$$

in S_z. Then there is a change of variables $\zeta = \xi + i\eta$ from G onto G' such that $\zeta \in H_2(G)$ and $f \circ \zeta^{-1}$ is analytic on G' for all $f \in A$.

(Note that if at every point $z \in G$ the algebra contains a function with non-vanishing Jacobian at z then the conditions of the theorem are satisfied, by Corollary 3 to Theorem 11.)

Proof. It is sufficient to show that there exists $\lambda \in H_1(G)$, $\operatorname{Im} \lambda \neq 0$, such that $f_x = \lambda f_y$ for all $f \in A$. The result will then follow from Theorem 11 and Corollary 1 to Theorem 8. By applying Theorem 13 to each domain S_z, we conclude that $f_x = \lambda_z f_y$ in S_z for every $f \in A$, $z \in G$. We will show that if two spheres S_{z_1} and S_{z_2} overlap, then $\lambda_{z_1} = \lambda_{z_2}$ in the intersection, and hence λ is defined globally. But in $S_{z_1} \cap S_{z_2}$, the function f_{z_1} satisfies (41) and the corresponding equation with λ_{z_1} replaced by λ_{z_2}. Since the zeros of $\partial f_z / \partial y$ are isolated in S_z (Corollary 1 to Theorem 10), we must have $\lambda_{z_1} = \lambda_{z_2}$ in the intersection.

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