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DECOMPOSING MODULES INTO PROJECTIVES AND INJECTIVES

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A ring R is called a right PCI-ring if and only if for any cyclic right R-module C either $C \cong R$ or C is injective. Faith has shown that right PCI-rings are either semiprime Artinian or simple right semihereditary right Ore domains. Thus if R_1 and R_2 are right PCI-rings then $R = R_1 \oplus R_2$ is not a right PCI-ring unless R_1 and R_2 are both semiprime Artinian but R has the property that every cyclic right R-module is the direct sum of a projective right R-module and an injective right R-module, and rings with this property on cyclic right R-modules will be called right CDPI-rings. On the other hand, if S is a semiprime Artinian ring then the ring of 2×2 upper triangular matrices with entries in S is also a right CDPI-ring. The structure of right Noetherian right CDPI-rings is discussed. These rings are finite direct sums of right Artinian rings and simple rings. A classification of right Artinian right CDPI-rings is given. However the structure of simple right Noetherian right CDPI-rings is more difficult to determine precisely and the problem of finding it reduces to a conjecture of Faith.

1. Introduction. Recall that if X is a nonempty subset of a ring R (and by a ring we shall always mean a ring with identity element) then the *left annihilator of* X is the set of all elements r of R such that $rx = 0$ for every element x of X, and is denoted by $l(X)$. Similarly the *right annihilator of* X is $r(X) = \{r \in R: xr = 0$ for all x in $X\}$. A subset A of R is called a *left* (respectively *right*) *annihilator* in case $A = l(X)$ ($A = r(X)$) for some nonempty subset X of R. A ring R is a *Baer ring* if and only if for every right annihilator A in R there exists an idempotent element e such that $A = eR$, equivalently for every left annihilator B in R there exists an idempotent element f such that $B = Rf$. Examples of Baer rings can be found in [6]. Baer rings are examples of *right PP-rings*, that is rings such that every principal right ideal is projective. On the other hand, Small [9], Theorem 1, showed that if R is a right PP-ring and R does not contain an infinite collection of orthogonal idempotents then R is a Baer ring.

A right CDPI-ring R is a right PP-ring (in fact it is right semihereditary, see [10], Lemma 2.4) and has the property that R/E is an injective right R-module for every essential right ideal E of R (see Corollary 2.2). Rings with this latter property we shall call *right RIC-rings* ("RIC" for restricted injective condition). If a ring

R is a Baer ring, then R is a right CDPI-ring if and only if R/E is an injective right R-module for every right ideal E of R with zero left annihilator (Theorem 2.4). Recall that Osofsky [8] proved that a ring R is semiprime Artinian if and only if every cyclic right R-module is injective.

A ring R is a *right CEPI-ring* provided every cyclic right R-module is the extension of a projective right R-module by an injective right R-module. The class of right CEPI-rings coincides with the class of right PP- right RIC-rings (Theorem 2.9) but strictly contains the class of right CDPI-rings since there is an example in [10] of a right and left Artinian right and left CEPI-ring which is not a right CDPI-ring.

Let us call a ring R a *right PCI-domain* provided R is a right PCI-ring and a domain. Goodearl [5] called a ring R a *right SI-ring* in case every singular right R-module is injective. By [10], Corollary 4.8, if R is a right Noetherian right CDPI-ring then R is a right SI-ring and hence by [5], Theorem 3.11,and [3], Theorems 14 and 17, R is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where A is a right Artinian right CDPI-ring and for each integer $1 \leqq i \leqq n$, the ring B_i is a right CDPI-ring Morita equivalent to a right Noetherian simple right PCI-domain, and conversely. The ring A can be characterized as a certain ring $(S, M, 0, T)$ of 2×2 "matrices"

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$$

with s in a semiprime Artinian ring S, t in a semiprime Artinian ring T and m in a certain left S-, right T-bimodule M, under the usual matrix addition and multiplication (Corollary 3.8).

When it comes to the rings B_i $(1 \leqq i \leqq n)$ the natural question which arises is the following one.

Question 1.1. Given a right Noetherian simple right PCI-domain D, is any ring S Morita equivalent to D a right CDPI-ring?

This question is related to a conjecture of Faith [3], p. 111, and to show the connection between them we make the following definitions. Let m be a positive integer. A ring R is a *right FGDPI-ring* (*right* $FGDPI_m$*-ring*) if and only if every finitely generated (m-generator) right R-module is the direct sum of a projective right R-module and an injective right R-module. Right Noetherian semiprime right FGDPI_2-rings are right FGDPI-rings and are left Goldie (Theorem 5.7). It follows that (see Corollary 4.12) the answer to 1.1 is "yes" if and only if D is a left Ore domain and this is precisely Faith's conjecture, and in this case the rings B_i $(1 \leqq i \leqq n)$ are just the rings Morita equivalent to right Noetherian simple right PCI-domains. Recall that if the ring D is a left Ore domain then Faith [3], Theorem 22 and subsequent remarks, proved that D is a left Noetherian left PCI-domain and we call such rings *Noetherian simple PCI-domains*. Examples of these rings can be found in [2]. Faith's conjecture can be expressed in yet another way (see Theorems 4.11 and 5.7):

Conjecture 1.2. If D is a right Noetherian simple right PCI-domain then the ring D_2 is a right CDPI-ring where D_2 is the complete ring of 2×2 matrices with entries in D.

We shall call a ring R a *Noetherian simple PCI-domain* if and only if R is a right and left Noetherian simple right and left PCI-domain. Examples of Noetherian simple PCI-domains have been produced by Cozzens [2]. For any positive integer m a ring R is a right Noetherian right FGDPI_m-ring if and only if R is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where A is a right Artinian right FGDPI_m-ring and for each integer $1 \leqq i \leqq n$ the ring B_i is a simple right and left Noetherian ring Morita equivalent to a Noetherian simple PCI-domain (see Corollary 5.8). There is a corresponding structure theorem for right Noetherian right FGDPI-rings. We have not been able to find explicitly the structure of right Artinian right FGDPI_m-rings (m an integer greater than 1) or right Artinian right FGDPI-rings.

We mention one further interesting fact about semiprime rings. If R is a semiprime ring then the following statements are equivalent:

(i) R is a right Noetherian right FGDPI_2-ring,

(ii) R is a left Noetherian left FGDPI_2-ring,

(iii) R is a right Noetherian right FGDPI-ring, and

(iv) R is a left Noetherian left FGDPI-ring (see Corollary 5.9).

Note also that if R is a right Noetherian right FGDPI_2-ring then R is a left SI-ring and in particular R is left hereditary (see Corollary 5.10).

2. Right CDPI-rings.

In this section we first look at characterizations of right CDPI-rings, we then examine the relationship between right CEPI-rings and right RIC-rings and finally we generalize the theorem of Osofsky mentioned in the Introduction.

LEMMA 2.1 (See [10], Lemma 5.1). *A ring R is a right CDPI-ring if and only if for every right ideal E of R there exists an idempotent element e such that E is contained in the right ideal eR and the right R-module eR/E is injective.*

COROLLARY 2.2. *Let R be a right CDPI-ring and E be a right* *ideal of R with zero left annihilator. Then the right R-module R/E is injective.*

If X is a nonempty subset of a ring R then by $rl(X)$ we shall mean $r(l(X))$, the right annihilator of the left annihilator of X. The proof of the next result is an easy adaptation of the proof of [10], Lemma 5.7.

LEMMA 2.3. *Let R be a Baer ring. Then R is a right CDPI-ring if and only if $rl(E)/E$ is an injective right R-module for each right ideal E of R.*

THEOREM 2.4. *Let R be a Baer ring. Then R is a right CDPI-ring if and only if R/E is an injective right R-module for each right ideal E of R with zero left annihilator.*

Proof. In view of Corollary 2.2 we need prove only the sufficiency. Suppose that R is a ring such that R/E is injective for every right ideal E with $l(E) = 0$. Let A be a right ideal of R. Since R is a Baer ring there exists an idempotent element a of R such that $rl(A) = aR$. Let $B = \{r \in R: ar \in A\}$. Since a is idempotent it follows that $A = aA$ and hence $A \subseteq B$. Then $aR = rl(A) \subseteq rl(B)$. But $(1 - a) R \subseteq B \subseteq rl(B)$ and hence $Rrl(B)$. Thus $l(B) = 0$ and by hypothesis R/B is injective. Since the mapping $\varphi: R/B \to a\ R/A$ defined by $\varphi(r + B) = ar + A$ $(r \in R)$ is an R-isomorphism it follows that aR/A is injective. By Lemma 2.1 R is a right CDPI-ring.

COROLLARY 2.5. *Let R be a ring which does not contain an infinite collection of orthogonal idempotent elements. Then R is a right CDPI-ring if and only if R is a right PP-ring and R/E is an injective right R-module for every right ideal E of R with zero left annihilator.*

Proof. The necessity is a consequence of Corollary 2.2 and [10], Lemma 2.4. The sufficiency follows by the theorem and [9], Theorem 1.

An immediate consequence of Corollary 2.5 is the next result.

COROLLARY 2.6. *Let R be a semiprimary ring. Then R is a right CDPI-ring if and only if R is a right PP-ring such that R/E is an injective right R-module for every right ideal E of R with zero left annihilator.*

COROLLARY 2.7. *Let R be a ring which does not contain an* *infinite direct sum of nonzero right ideals. Then R is a right CDPI-ring if and only if R is a right nonsingular ring such that R/E is an injective right R-module for every right ideal E of R with zero left annihilator.*

Proof. The necessity follows by Corollary 2.2 and [10], Lemma 2.4. Conversely, suppose that R is a right nonsingular ring such that R/E is an injective right R-module for each right ideal E with $l(E) = 0$. Since R is right nonsingular it follows that R is a right RIC-ring. Also by [4], Lemma 1.4 and Theorem 2.3 (iii), R is a right Goldie ring. By [10], Corollary 4.3 and Lemma 4.4, R is a right PP-ring. Finally by Corollary 2.5 R is a right CDPI-ring.

Next we consider briefly right CEPI-rings. Let E be a right ideal of a right CEPI-ring R. There exists a right ideal F of R containing E such that F/E is projective and R/F is injective. Since F/E is projective there exists a right ideal G of R such that $E \cap G = 0$ and $F = E \oplus G$. Moreover, $G \cong F/E$ is projective. We have proved:

LEMMA 2.8. *A ring R is a right CEPI-ring if and only if for every right ideal E of R there exists a projective right ideal G of R such that $E \cap G = 0$ and $R/(E \oplus G)$ is an injective right R-module.*

In [10], Lemma 2.4, we proved that a right CEPI-ring is a right semihereditary right RIC-ring. Now we have the following result.

THEOREM 2.9. *A ring R is a right CEPI-ring if and only if R is a right PP- right RIC-ring.*

Proof. As we have just remarked the necessity is proved in [10], Lemma 2.4. Conversely, suppose that R is a right PP-right RIC-ring. Let E be a right ideal of R. By Zorn's lemma there exists a maximal collection S of nonzero elements x_λ $(\lambda \in \Lambda)$ of R such that if $H = \sum x_\lambda R$ then $H = \bigoplus_\Lambda x_\lambda R$ and $E \cap H = 0$. Since R is a right PP-ring, H is projective. Let a be a nonzero element of R. If $a \notin S$ then either $aR \cap H \neq 0$ or $E \cap (aR \oplus H) \neq 0$. It follows that $E \oplus H$ is an essential right ideal of R. Since R is a right RIC-ring, the right R-module $R/(E \oplus H)$ is injective. By Lemma 2.8 R is a right CEPI-ring.

Finally in this section we give the following generalization of Osofsky's theorem [8].

THEOREM 2.10. *A ring R is semiprime Artinian if and only if R is a right self-injective right RIC-ring.*

Proof. The necessity is a consequence of Osofsky's theorem. Conversely, let R be a right self-injective right RIC-ring. Since R is right self-injective, given any right ideal A of R there exists an idempotent element e of R such that A is an essential submodule of the right ideal eR. Since R is a right RIC-ring it follows that eR/A is injective. By Lemma 2.1 R is a right CDPI-ring. Let C be a cyclic right R-module. There exists a projective module P and an injective module Q such that $C = P \oplus Q$. Since P is therefore cyclic it follows that P is isomorphic to a direct summand of R and hence P is injective. Thus C is injective. Thus every cyclic right R-module is injective and R is semiprime Artinian by Osofsky's theorem [8].

3. Semiprimary right CDPI-rings. Right CDPI-rings are right RIC-rings (see [10], Lemma 2.4). In addition, by [10], Lemma 2.5 and Theorem 4.1, semiprimary right RIC-rings are right SI-rings. Also, by [5], Proposition 3.5, semiprimary right SI-rings are left SI-rings. Thus we have the following result.

LEMMA 3.1. *Semiprimary right CDPI-rings are right and left SI-rings.*

Let R be a right SI-ring. By [5], Proposition 3.3, R is right hereditary. If in addition R is semiprimary then R is a Baer ring by [9], Theorem 1. Noting this fact, the next result of this section is proved by adapting the proof of [10], Theorem 5.13.

LEMMA 3.2. *A ring R is a semiprimary (right) SI-ring if and only if R is semiprime Artinian or there exist semiprime Artinian rings S and T and a left S-, right T-bimodule M such that M is a faithful left S-module and R is isomorphic to the ring $(S, M, 0, T)$.*

For the remainder of this section we shall fix the following notation: S and T are semiprime Artinian rings, M is a left S-, right T-bimodule (not necessarily faithful as a left S-module) and R is the ring $(S, M, 0, T)$. That is, R consists of all "matrices"

$$(s, m, 0, t) = \begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$$

with s in S, m in M and t in T, addition and multiplication in R being the usual matrix addition and multiplication. For each non-empty subset X of M let $\mathrm{Ann}_S(X)$ denote the annihilator of X in

S; that is, $\mathrm{Ann}_S(X) = \{s \in S: sX = 0\}$. Let $I = \mathrm{Ann}_S(M)$ and let q be the central idempotent element of S such that $I = Sq$. The right socle of R will be denoted by A. It can easily be checked that $A = (I, M, 0, T)$ and A is an essential right ideal of R. By [5], Proposition 3.1, R is a right SI-ring and in view of Lemma 3.2 we can take R as a typical semiprimary right SI-ring. The Jacobson radical of R will be denoted by J. Clearly $J = (0, M, 0, 0)$. Moreover $A = J \oplus eR$ where e is the idempotent $(q, 0, 0, 1)$ of R (here 1 is the identity element of the ring T). Note that $eJ = 0$ and recall that $A = \cap \{E: E$ is an essential right ideal of $R\}$.

LEMMA 3.3. *Let R be a semiprimary right SI-ring with Jacobson radical J and let X be a right R-module. Then X is injective if and only if given any homomorphism $\varphi: J \to X$ there exists an element x of X such that $\varphi(j) = xj$ for every element j of J.*

Proof. The necessity is an immediate consequence of Baer's criterion for injectivity (see for example [1], Lemma 18.3). Conversely, suppose that X has the stated property. By Lemma 3.2 we can suppose without loss of generality that in the above notation $R = (S, M, 0, T)$. Let $Z = Z(X)$ be the singular submodule of X. Since R is a right SI-ring it follows that Z is injective and hence there exists a submodule Y of X such that $X = Z \oplus Y$. Note that Y is nonsingular. Let E be an essential right ideal of R and $\varphi: E \to Y$ be an R-homomorphism. Let α be the restriction of φ to J. By hypothesis there exists an element x of X such that $\alpha(j) = xj$ $(j \in J)$. If $x = z + y_1$ where $z \in Z$, $y_1 \in Y$, then clearly $\alpha(j) = y_1 j$ $(j \in J)$. Let y_2 be the element $\varphi(e)$ of Y, where $e = (q, 0, 0, 1)$ as above. Let y be the element $y_1(1 - e) + y_2 e$ of Y. If $a \in A$ then $a = j + er$ for some elements j of J and r of R and

$$\varphi(a) = \varphi(j) + \varphi(e)er = y_1 j + y_2 er = ya .$$

Thus $\varphi(a) = ya (a \in A)$. Now let $b \in E$. Since A is an essential submodule of E there exists an essential right ideal K of R such that $bK \subseteq A$. For any element k of K, $\varphi(b)k = \varphi(bk) = ybk$ and hence $(\varphi(b) - yb)k = 0$. It follows that $(\varphi(b) - yb)K = 0$. Since Y is nonsingular it follows that $\varphi(b) = yb$. Hence $\varphi(b) = yb(b \in E)$, and by Baer's criterion Y, and hence X, is injective.

It is clear from the proof of Lemma 3.3 that in Lemma 3.3 we can replace J be the right socle A.

In view of Corollary 2.6 interest centres on right ideals of R with zero right annihilator. Let E be a right ideal of R. Let $F = \{a \in S: (a, 0, 0, 0) \in E\}$. Then F is a right ideal of S and there exists an idempotent element f of S such that $F = fS$. If $\bar{f}$ is the element $(f, 0, 0, 0)$ of R then $\bar{f}R = (fS, fM, 0, 0)$. If N is the T-submodule $(1 - f)M$ then $M = fM \oplus N$ and $E = \bar{f}R \oplus E_1$ where E_1 is the right ideal $E \cap (0, N, 0, T)$. For, if $r = (a, b, 0, c) \in E$ with a in S, b in M and c in T then $(a, 0, 0, 0) = (a, b, 0, c)(1, 0, 0, 0) \in E$ and hence $a = fa$ and $r - \bar{f}r \in E_1$. Now $E_1 = (E_1 \cap J) \oplus C$ for some right ideal C contained in E_1. Let $D = \{t \in T\colon (0, y, 0, t) \in C$ for some element y of $M\}$. Then D is a right ideal of T and there exists an idempotent element g of T such that $D = gT$. Let m be an element of M such that $c = (0, m, 0, g) \in C$. For any element c_1 of C it can easily be checked that $c_1 - cc_1 \in C \cap J = 0$. It follows that c is an idempotent element of R and $C = cR$. In particular c idempotent implies that $m = mg$. Thus there exists a T-submodule X of N such that E consists of all "matrices" $(fa, fb + x + mt, 0, gt)$ with a in S, b in M, x in X and t in T. Now suppose that $l(E) = 0$. It can easily be checked that if e is an idempotent element of S such that $\mathrm{Ann}_S(x) = Se$ then $X = (1 - f)X$ implies that $e(1 - f) \in Se$ and

$$(e(1 - f), -e(1 - f)m, 0, 1 - g)$$

belongs to $l(E)$. Thus $e(1 - f) = 0$ and $g = 1$. But $e(1 - f) = 0$ implies that $e = ef$ and $Se \subseteq Sf$. This gives the following result after a little checking.

LEMMA 3.4. *A right ideal E of the above ring R has zero left annihilator if and only if there exists a T-submodule X of M, an idempotent element e of S such that $Se = \mathrm{Ann}_S(X)$, an idempotent element f of S such that $Se \subseteq Sf$, and an element m of M such that E consists of all "matrices" $(fa, fb + x + mt, 0, t)$ with a in S, b in M, x in X and t in T.*

LEMMA 3.5. *If $X = \mathrm{Ann}_M(\mathrm{Ann}_S(X))$ for every T-submodule X of M then R is a right CDPI-ring.*

Proof. By $\mathrm{Ann}_M(\mathrm{Ann}_S(X))$ we mean the set of elements m of M such that $\mathrm{Ann}_S(X)m = 0$. In the notation of the previous lemma let E be the right ideal of all "matrices" $(fa, fb + x + mt, 0, t)$ with a in S, b in M, x in X and t in T. Let $s \in \mathrm{Ann}_S(fM + X)$; then $sfM = sX = 0$. But $sX = 0$ implies that $s = se$ and hence $sf = sef = se = s$. It follows that $sM = 0$ and hence $\mathrm{Ann}_S(fM + X) = \mathrm{Ann}_S(M)$. By hypothesis $fM + X = \mathrm{Ann}_M(\mathrm{Ann}_S(fM + X)) = M$. It follows that the ideal $(0, M, 0, T)$ is contained in E. Let $\varphi\colon J \to R/E$ be an R-homomorphism. If $b = (0, 0, 0, 1)$ then $j = jb$ for every element j of J and it follows that $\varphi = 0$. By Corollary 2.6 and Lemmas 3.2–3.4 R is a right CDPI-ring.

In particular if $S = M = T$ then R is a right CDPI-ring. This special case was proved in [10], Theorem 5.15. Another special case is when M is a simple right T-module and again R is a right CDPI-ring. This corresponds to the Jacobson radical J of R being a minimal right ideal (see [10], Theorem 5.9). We can express Lemma 3.5 in terms of J as follows.

COROLLARY 3.6. *Let R be a semiprimary right SI-ring such that $F = J \cap rl(F)$ for every right ideal F contained in the Jacobson radical J of R. Then R is a right CDPI-ring.*

THEOREM 3.7. *In the above notation let R be the semiprimary right SI-ring $(S, M, 0, T)$. Then R is a right CDPI-ring if and only if for every T-submodule X of M such that $\mathrm{Ann}_S(X) = \mathrm{Ann}_S(M)$ and every T-homomorphism $\varphi: M \rightarrow M/X$ there exists an element a of S such that $\varphi(m) = am + X$ for all m in M.*

Proof. Suppose first that R is a right CDPI-ring. Let X be a T-submodule of M such that $\mathrm{Ann}_S(X) = \mathrm{Ann}_S(M) = Sq$ and $\varphi: M \rightarrow M/X$ a T-homomorphism. Let V be a set of coset representatives of X in M and define a mapping $\tau: M \rightarrow V$ by $\varphi(m) = \tau(m) + X$ $(m \in M)$. Let E be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, R/E is an injective right R-module. Define $\bar{\varphi}: J \rightarrow R/E$ by $\bar{\varphi}(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ $(m \in M)$. Since $\bar{\varphi}$ is an R-homomorphism there exists an element $r = (a, b, 0, c)$ of R such that $\bar{\varphi}(j) = rj + E$ $(j \in J)$. It can easily be checked that this gives $\varphi(m) = am + X$ $(m \in M)$.

Conversely, in the notation of Lemma 3.4 let E be the right ideal of R consisting of all "matrices" $(fa, fb + x + mt, 0, t)$ with a in S, b in M, x in X and t in T. Let Y be the T-submodule $fM + X$ of M and let H be the right ideal consisting of all "matrices" $(0, y + mt, 0, t)$ with y in Y and t in T. By [5], Proposition 3.3, R is right hereditary. Thus to prove that R/E is an injective right R-module it is sufficient to prove that R/H is an injective right R-module because $H \subseteq E$ (see [1], Exercise 18.10).

Let $\alpha: J \rightarrow R/H$ be an R-homomorphism, where again J is the Jacobson radical of R. If $p = (0, 0, 0, 1)$ then p is an idempotent element of R and $J = Jp$. It follows that if K is the right ideal containing H such that $\mathrm{Im}\,\alpha = K/H$ then K is contained in the ideal $(0, M, 0, T)$. For each element x of M choose an element x^M of M and an element x^T of T such that $\alpha(0, x, 0, 0) = (0, x^M, 0, x^T) + H$. Since α is a homomorphism we note the following three facts.

(i) There exist elements y_0 in Y and t_0 in T such that $0^M = y_0 + mt_0$, $0^T = t_0$.

(ii) For all elements x_1, x_2 in M there exist elements y_1 in Y and t_1 in T such that $(x_1 + x_2)^M - x_1^M - x_2^M = y_1 + mt_1$, $(x_1 + x_2)^T - x_1^T - x_2^T = t_1$.

(iii) For all elements x in M and c in T there exist elements y_2 in Y and t_2 in T such that $(xc)^M - x^M c = y_2 + mt_2$, $(xc)^T - x^T c = t_2$. Define $\beta: M \to M/Y$ by $\beta(x) = (x^M - mx^T) + Y$ for every element x of M. By (i), (ii) and (iii) β is a T-homomorphism. But $Y = fM + X$ implies that $\mathrm{Ann}_S(Y) = \mathrm{Ann}_S(M)$. Therefore by hypothesis there exists an element s_1 of S such that $\beta(x) = s_1 x + Y$ $(x \in M)$. Let s be the element $(s_1, 0, 0, 0)$ of R. Then for each element j of J there exists an element x of M such that $j = (0, x, 0, 0)$ and hence $\alpha(j) = (0, x^M, 0, x^T) + H = sj + H$. Thus $\alpha(j) = sj + H$ $(j \in J)$. By Corollary 2.6 and Lemmas 3.2–3.4 R is a right CDPI-ring. This proves the theorem.

Combining Lemmas 3.1, 3.2 and Theorem 3.7 we have:

COROLLARY 3.8. *A ring R is a semiprimary right CDPI-ring if and only if R is semiprime Artinian or there exist semiprime Artinian rings S and T and a left S-, right T-bimodule M such that M is a faithful left S-module and for every T-submodule X of M such that $\mathrm{Ann}_S(X) = 0$ and T-homomorphism $\varphi: M \to M/X$ there exists an element a of S with $\varphi(m) = am + X$ for every m in M, and R is isomorphic to the ring $(S, M, 0, T)$.*

COROLLARY 3.9. *In the above notation let R be the semiprimary right SI-ring $(S, M, 0, T)$. Suppose that R is a right CDPI-ring. Then there does not exist a left S-, right T-sub-bimodule X of M and a nonzero T-submodule Y of M such that $\mathrm{Ann}_S(X) = \mathrm{Ann}_S(M)$, $X \cap Y = 0$ and Y can be embedded in X.*

Proof. Suppose that M contains a sub-bimodule X and a submodule Y with the given properties. Let X_1 be a T-submodule of X such that there is a T-isomorphism $\varphi: X_1 \to Y$. Since T is semiprime Artinian there exists a T-submodule N of M such that $M = X_1 \oplus Y \oplus N$. Define $\alpha: M \to M/X$ by $\alpha(x_1 + y + n) = \varphi(x_1) + X$ for all x_1 in X_1, y in Y and n in N. If R is a right CDPI-ring then by the theorem there exists an element s of S such that for each element x_1 of X_1, $\varphi(x_1) + X = \alpha(x_1) = sx_1 + X$. It follows that $\varphi(x_1) \in X \cap Y = 0$ for each element x_1 of X_1, a contradiction. Thus R is not a right CDPI-ring.

COROLLARY 3.10. *Suppose that S and T are simple rings and the above ring $R = (S, M, 0, T)$ is a right CDPI-ring. Then M is a simple left S-, right T-bimodule.*

Proof. Let X be a nonzero left S-, right T-sub-bimodule of M. Since S is simple it follows that $\mathrm{Ann}_S(X) = \mathrm{Ann}_S(M) = 0$. If Y is a simple T-submodule of M then Y can be embedded in X, because T is simple and simple right T-modules are isomorphic. By Corollary 3.9 $X \cap Y \neq 0$ and hence $Y \subseteqq X$. It follows that $X = M$.

We can express Corollary 3.10 in the following form.

COROLLARY 3.11. *Let R be a semiprimary right CDPI-ring with Jacobson radical J. If R contains precisely two maximal ideals then J is a minimal ideal of R.*

4. **Category equivalence.** Let R be a ring and A, B be right R-modules. A monomorphism $\varphi: A \to B$ is called *essential* if and only if $\mathrm{Im}\,\varphi$ is an essential submodule of B; that is, $\mathrm{Im}\,\varphi \cap C \neq 0$ for every nonzero submodule C of B. The first lemma in this section is elementary and well known but we shall include its proof for completeness.

LEMMA 4.1. *A right R-module C is singular if and only if there exists an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules such that $\alpha: A \to B$ is an essential monomorphism.*

Proof. Suppose that C is singular. For each element c of C let $R_c = R$ and let $F = \bigoplus_C R_c$. Let $\pi: F \to C$ be the canonical projection. For each element c of C there exists an essential right ideal E_c of $R = R_c$ such that $cE_c = 0$. Let $E = \bigoplus_C E_c$. Then E is an essential submodule of F and $E \subseteqq \mathrm{Ker}\,\pi$. If $K = \mathrm{Ker}\,\pi$ and $i: K \to F$ is inclusion then $0 \to K \xrightarrow{i} F \xrightarrow{\pi} C \to 0$ is an exact sequence such that i is an essential monomorphism. Conversely, suppose that there exists an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules such that α is an essential monomorphism. Let $c \in C$ and let b be an element of B such that $\beta(b) = c$. It can easily be checked that $\mathrm{Ker}\,\beta = \mathrm{Im}\,\alpha$ is an essential submodule of B implies that $G = \{r \in R: br \in \mathrm{Ker}\,\beta\}$ is an essential right ideal of R. Also, $cG = \beta(b)G = \beta(bG) = 0$. It follows that C is singular.

COROLLARY 4.2. *A right R-module C is a finitely generated singular module if and only if there exists an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules such that B is finitely generated and $\alpha: A \to B$ is an essential monomorphism.*

LEMMA 4.3. *A ring R is a right RIC-ring if and only if every* *finitely generated singular right R-module is injective.*

Proof. The sufficiency follows from the fact that if E is an essential right ideal of R then R/E is a cyclic singular right R-module. Conversely, suppose that R is a right RIC-ring. Let n be a positive integer and X a right R-module generated by elements $x_1, x_2, \cdots, x_n$. If $n = 1$ there is nothing to prove. Suppose that $n > 1$ and let $Y = x_1R + x_2R + \cdots + x_{n-1}R$. Then Y is a singular module. If Y is injective then there exists a submodule Z of X such that $X = Y \oplus Z$. It follows that Z is a cyclic singular module and hence Z is injective. Thus X is injective. The result follows by induction on n.

COROLLARY 4.4. *Any ring Morita equivalent to a right RIC-ring is itself a right RIC-ring.*

Proof. By Corollary 4.2 since category equivalence preserves exact sequences, finitely generated modules and essential monomorphisms (see [1], Propositions 21.4, 21.6(5) and 21.8(2)).

THEOREM 4.5. *A ring R is a rihgt CEPI-ring if and only if every finitely generated right R-module is the extension of a projective right R-module by an injective right R-module.*

Proof. The given condition is clearly sufficient for R to be a right CEPI-ring. Conversely, suppose that R is a right CEPI-ring. Let n be a positive integer and X be a right R-module generated by elements $x_1, x_2, \cdots, x_n$. If $n = 1$ there is nothing to prove and so we suppose that $n > 1$. Let $Y = x_1R + x_2R + \cdots + x_{n-1}R$. Suppose there is a submodule A of Y such that A is projective and Y/A is injective. Since X/Y is cyclic and R is a right CEPI-ring it follows that there exists a submodule B of X such that $Y \subseteq B$, B/Y is projective and X/B is injective. Now consider B/A. Since Y/A is injective there exists a submodule C of B such that $A \subseteq C$ and $B/A = (Y/A) \oplus (C/A)$. Since $C/A \cong B/Y$ is projective and A is projective it follows that $C \cong A \oplus (C/A)$ is projective. Moreover, $B/C \cong Y/A$ is injective and hence $X/C \cong (B/C) \oplus (X/B)$ is injective. The result follows by induction on n.

COROLLARY 4.6. *Any ring Morita equivalent to a right CEPI-ring is itself a right CEPI-ring.*

Proof. By the theorem since category equivalence preserves exact sequences, finitely generated modules, projective modules and injective modules (see [1], Propositions 21.4, 21.6(2) and 21.8(2)).

It is interesting to compare Theorem 2.5 with the next result.

THEOREM 4.7. *A ring R is a right SI-ring if and only if every right R-module is the extension of a projective right R-module by an injective right R-module.*

Proof. Suppose that every right R-module is the extension of a projective module by an injective module. In particular, this means that R is a right CEPI-ring. By [10], Lemma 2.4, R is right nonsingular. Let X be a singular right R-module. There exists a submodule Y of X such that Y is projective and X/Y is injective. Suppose that $Y \neq 0$ and let y be a nonzero element of Y. Since Y is projective there exists a homomorphism $\varphi: Y \to R$ such that $\varphi(y) \neq 0$. But there exists an essential right ideal E of R such that $yE = 0$ and hence $\varphi(y)E = 0$. This contradicts the fact that R is right nonsingular. Thus $Y = 0$ and X is injective. It follows that R is a right SI-ring.

Conversely, suppose that R is a right SI-ring. Let A be a right R-module and $\mathfrak{A}$ the collection of cyclic submodules of A. By Zorn's lemma there is a maximal collection $\mathfrak{B}$ of members of $\mathfrak{A}$ whose sum is direct. Let Λ be an index set and x_λ elements of A such that $\mathfrak{B}$ is the collection of submodules $x_\lambda R (\lambda \in \Lambda)$. Let $B = \bigoplus_\Lambda x_\lambda R$. The choice of B ensures that B is an essential submodule of A. Since R is a right SI-ring it follows that R is right hereditary (see [5], Proposition 3.3) and hence B is projective. Moreover A/B is a singular right R-module and is injective because R is a right SI-ring. It follows that every right R-module is the extension of a projective module by an injective module.

COROLLARY 4.8. *If R is a right Noetherian right RIC-ring then every right R-module is the extension of a projective right R-module by an injective right R-module.*

Proof. By the theorem and [10], Theorem 4.1.

In particular Corollary 4.8 tells us that any right Noetherian right CDPI-ring R has the property that every right R-module is the extension of a projective module by an injective module.

Next we consider right FGDPI-rings. The proof of Corollary 4.6 gives immediately:

LEMMA 4.9. *Any ring Morita equivalent to a right FGDPI-ring is itself a right FGDPI-ring.*

Before examining the relationship between right FGDPI-rings and right CDPI-rings we first introduce some notation. Let R be a ring, n a positive integer and R_n the complete ring of $n\times n$ matrices with entries in R. Let (r_{ij}) denote the $n\times n$ matrix whose (i, j)th entry is the element r_{ij} or R. For any right R-module X let $X^{(n)}$ denote the right R-module $X\oplus X\oplus\cdots\oplus X$ (n copies). Then $X^{(n)}$ can be made into an R_n-module by defining:

$$(x_1, x_2, \cdots, x_n)(r_{ij}) = \left(\sum_{k=1}^{n} x_k r_{k1}, \sum_{k=1}^{n} x_k r_{k2}, \cdots, \sum_{k=1}^{n} x_k r_{kn}\right),$$

where $x_i\in X$ and $r_{ij}\in R$ $(1\leqq i, j\leqq n)$. Let e_{ij} denote the matrix unit in R_n with 1 in the (i, j)th position and zeros elsewhere. For any right R_n-module Y, Ye_{11} is a right R-module. It is easy to check that for any right R-module X the right R-modules X and $X^{(n)}e_{11}$ are isomorphic. Recall the following result.

LEMMA 4.10 (See [7], Corollary 2.3). *With the above notation, a right R_n-module X is projective (respectively injective) if and only if the right R-module Xe_{11} is projective (respectively injective).*

THEOREM 4.11. *Let n be a positive integer. A ring R is a right FGDPI$_n$-ring if and only if R_n is a right CDPI-ring.*

Proof. Suppose that R_n is a right CDPI-ring. Let X be a right R-module generated by elements $x_1, x_2, \cdots, x_n$. If $Y = X^{(n)}$ then Y is the cyclic right R_n-module $(x_1, x_2, \cdots, x_n)R_n$. There exists a projective right R_n-module P and an injective right R_n-module Q such that $Y = P\oplus Q$. Then $Ye_{11} = (Pe_{11})\oplus(Qe_{11})$, as R-modules. Since the right R-modules X and Ye_{11} are isomorphic it follows that X is the direct sum of a projective module and an injective module by Lemma 4.10. Thus R is a right FGDPI$_n$-ring.

Conversely, suppose that R is a right FGDPI$_n$-ring. Let $A = aR_n$ be a cyclic right R_n-module. Then $Ae_{11} = aR_ne_{11} = \sum_{k=1}^{n} ae_{k1}R$ is an n-generator right R-module. By hypothesis there exists a projective right R-module B and an injective right R-module C such that $Ae_{11} = B\oplus C$. Now $R_n = R_ne_{11}R_n$ implies that $Ae_{11}R_n = AR_ne_{11}R_n = A$ and hence $A = (BR_n) + (CR_n)$. Since $B = Be_{11}$ and $C = Ce_{11}$ it follows that

$$BR_n = \sum_{k=1}^{n} Be_{1k} \qquad \text{and} \qquad CR_n = \sum_{k=1}^{n} Ce_{1k}\,.$$

It can easily be checked that $B\cap C = 0$ implies that $(BR_n)\cap(CR_n) = 0$. That is $A = (BR_n)\oplus(CR_n)$. Moreover, $(BR_n)e_{11} = B$ and $(CR_n)e_{11} = C$. By Lemma 4.10 BR_n is a projective right R_n-module and CR_n is an injective right R_n-mudule. It follows that R_n is a right CDPI-ring.

COROLLARY 4.12. *A ring R is a right FGDPI-ring if and only if R_n is a right CDPI-ring for every positive integer n.*

It is interesting to contrast Theorem 4.11 with the next result.

THEOREM 4.13. *Let R be a right CDPI-ring and e be an idempotent element of R such that $R = ReR$. Then the subring eRe of R is a right CDPI-ring.*

Proof. Let S denote the ring eRe and let I be a right ideal of S. If J is the right ideal IR of R then $J \subseteq eR$ since $I = eI$. By hypothesis there exist right ideals F and G of R such that $J \subseteq F \subseteq eR$, $J \subseteq G \subseteq eR$, F/J is a projective right R-module, G/J is an injective right R-module and $eR/J = (F/J) \oplus (G/J)$. Since $eR/G \cong F/J$ is projective there exists a right ideal H of R such that $eR = G \oplus H$. Then Ge and He are right ideals of S, $S = (Ge) \oplus (He)$ and hence $S/(Ge)$ is a projective right S-module. Moreover, $eR = F + G$, $F \cap G = J$ together imply $S = (Fe) + (Ge)$ and $(Fe) \cap (Ge) = Je = IRe = IeRe = I$. Thus S/I is the direct sum $((Fe)/I) \oplus ((Ge)/I)$ of the right S-modules $(Fe)/I$ and $(Ge)/I$. Also, $(Fe)/I \cong S/(Ge)$ is a projective right S-module. It remains to prove that $(Ge)/I$ is an injective right S-module. Note that $G = GR = GReR = GeR$. Thus it is sufficient to prove the following result.

LEMMA 4.14. *Let R be a ring and e be an idempotent element of R such that $R = ReR$. Let $A \subseteq B$ be right ideals of the ring $S = eRe$ and $\bar{A} = AR$, $\bar{B} = BR$. If $\bar{B}/\bar{A}$ is an injective right R-module then B/A is an injective right S-module.*

Proof. Let C be a right ideal of S and $\varphi: C \to B/A$ an S-homomorphism. Let V be a set of coset representatives of A in B and define a mapping $\alpha: C \to V$ by $\alpha(c) + A = \varphi(c)$ $(c \in C)$. Define $\bar{\varphi}$: $CR \to \bar{B}/\bar{A}$ by

$$\bar{\varphi}\Big(\sum_{i=1}^{n} c_i r_i\Big) = \sum_{i=1}^{n} \alpha(c_i) e r_i + \bar{A}$$

for all positive integers n and elements c_i of C and r_i of R $(1 \leqq i \leqq n)$. Clearly $\bar{\varphi}$ is independent of the choice of V. Suppose n is a positive integer, $r_i \in R$ and $c_i \in C$ $(1 \leqq i \leqq n)$ and

$$\sum_{i=1}^{n} c_i r_i = 0 .$$

For any element x of R,

$$\sum_{i=1}^{n} c_i e r_i x e = 0$$

and hence

$$\sum_{i=1}^{n} \varphi(c_i) e r_i x e = 0 .$$

That is, for all x in R,

$$\sum_{i=1}^{n} \alpha(c_i) e r_i x e \in A .$$

Since $R = ReR$ it follows that $1 \in ReR$ and hence

$$\sum_{i=1}^{n} \alpha(c_i) e r_i \in AR = \bar{A} .$$

Thus $\bar{\varphi}$ is well defined and clearly $\bar{\varphi}$ is an R-homomorphism. By hypothesis there exists an element b of $\bar{B}$ such that $\bar{\varphi}(r) = br + \bar{A}$ $(r \in C)$. It follows that $be \in \bar{B}e = BRe = BeRe = B$. Let $c \in C$. Then $c = ce = ec$ and $\varphi(c) = \alpha(c) + A = \alpha(c)e + A$ and $\bar{\varphi}(c) = \alpha(c)e + \bar{A} = bc + \bar{A} = bec + \bar{A}$. This implies that $\alpha(c)e - bec \in \bar{A} \cap S = A$ and hence $\varphi(c) = bec + A$. Thus $\varphi(c) = bec + A$ $(c \in C)$. It follows that B/A is an injective right S-module. This completes the proof of Lemma 4.14 and hence also of Theorem 4.13.

5. **Right FGDPI-rings.** Let R be a semiprime right Goldie ring. Goldie [4], Theorems 4.1 and 4.4, proved that R has a (classical) right quotient ring Q and Q is semiprime Artinian. Levy [7], Theorem 5.3, proved that if R has the additional property that every finitely generated torsion-free right R-module is a submodule of a free right R-module then Q is the left quotient ring of R and hence by [4], Theorem 4.4, R is a left Goldie ring. In actual fact to prove that Q was the left quotient ring of R all Levy needed was the fact that every 2-generator right R-submodule of Q is contained in a free right R-module. Thus we can state Levy's result in the following form.

LEMMA 5.1. *Let R be a semiprime ring Goldie ring with right quotient ring Q such that every 2-generator right R-submodule of Q is contained in a free right R-module. Then R is a left Goldie ring.*

Next we restate [7], Theorem 6.1, as follows.

LEMMA 5.2. *Let R be a semiprime right and left Goldie right* *(and left) semihereditary ring. Then every finitely generated right R-module X is the direct sum of its singular submodule $Z(X)$ and a projective R-submodule P.*

COROLLARY 5.3. *Let R be a semiprime right and left Goldie ring. Then R is a right FGDPI-ring if and only if R is a right RIC-ring.*

Proof. The necessity follows by [10], Lemma 2.4. Conversely, suppose that R is a right RIC-ring. Let X be a finitely generated right R-module with singular submodule Z. By [10] Corollary 4.3 and Lemma 4.4, R is right semihereditary. By Lemma 5.2 there exists a projective submodule P of X such that $X = Z \oplus P$. By Lemma 4.3 Z is injective. It follows that R is a right FGDPI-ring.

Let R be a semiprime right Noetherian ring with right quotient ring Q and suppose Q is a finitely generated right R-module. Let a be a regular element of R and consider the ascending chain $a^{-1}R \subseteqq a^{-2}R \subseteqq a^{-3}R \subseteqq \cdots$ of R-submodules of Q. Since Q is a Noetherian right R-module there exists a positive integer n such that $a^{-n}R = a^{-n-1}R$. Then $a^{-n-1} = a^{-n}b$ for some element b of R and hence $1 = ab = ba$. It follows that $R = Q$.

LEMMA 5.4. *Let R be a prime right Noetherian right $FGDPI_2$-ring. Then R is a left Goldie ring.*

Proof. Let Q be the right quotient ring of R. In view of Lemma 5.1 it is sufficient to prove that every 2-generator right R-submodule of Q is contained in a free right R-module. Let X be a 2-generator right R-submodule of Q. By hypothesis there exists a projective R-submodule P of X and an injective R-submodule I of X such that $X = P \oplus I$. Suppose that $I \neq 0$. For any regular element c of R we have $I = Ic$ (see [7], Theorem 3.1). Since I is torsion-free, for all elements x of I and regular elements c of R there exists a unique element x of I such that $\bar{x}c = x$. By defining $xc^{-1} = \bar{x}$ for all x in I and c regular in R we can make I into a right Q-module. Since $I \neq 0$ and Q is simple Artinian it follows that I contains a simple right Q-module. Since Q is simple Artinian all simple right Q-modules are isomorphic. Because I is a finitely generated right R-module it follows that Q is a finitely generated right R-module. As our remarks above show, in this case $R = Q$ and hence R is left Goldie. Now suppose that $Q \neq R$. Then $I = 0$, $X = P$ and hence X is contained in a free right R-module. Thus every 2-generator right R-submodule of Q is contained in a free right R-module. By Lemma 5.1 R is a left Goldie ring.

LEMMA 5.5. *Let S and T be subrings of a ring R such that $R = S \oplus T$. Let n be a positive integer. Then R is a right $FGDPI_n$-ring if and only if S and T are both right $FGDPI_n$-rings.*

Proof. Suppose that R is a right FGDPI_n-ring. Let X be an n-generator right S-module. We can make X into an n-generator right R-module by defining $x(s + t) = xs$ for all x in X, s in S and t in T. By hypothesis there exists a projective right R-module P and an injective right R-module I such that $X = P \oplus I$. It can easily be checked that P is a projective right S-module and I is an injective right S-module. It follows that S is a right FGDPI_n-ring. Similarly T is a right FGDPI_n-ring.

Conversely, suppose first that $n = 1$; that is, S and T are both right CDPI-rings. Let E be a right ideal of $R = S \oplus T$. Then there exists a right ideal E_1 of S and a right ideal E_2 of T such that $E = E_1 \oplus E_2$. Since S and T are right CDPI-rings there exist idempotent elements e_1 of S and e_2 of T such that $E_1 \subseteqq e_1S$, $E_2 \subseteqq e_2T$, $A = (e_1S)/E_1$ is an injective right S-module and $B = (e_2T)/E_2$ is an injective right T-module. The Abelian group $C = A \oplus B$ can be made into a right R-module by defining $(a, b)(s + t) = (as, bt)$ for all a in A, b in B, s in S and t in T. If $f = e_1 + e_2$ then f is an idempotent element of R and $E \subseteqq fR$. Moreover, $(fR)/E$ is isomorphic to the right R-module C. If F is a right ideal of R then $F = F_1 \oplus F_2$ for some right ideals F_1 of S and F_2 of T, and it can easily be checked that any R-homomorphism $\varphi: F \to C$ can be lifted to an R-homomorphism $\bar{\varphi}: R \to C$. Thus C is injective. It follows that R is a right CDPI-ring. Now suppose that n is any positive integer and S and T are both right FGDPI_n-rings. By Theorem 4.11 the matrix rings S_n and T_n are right CDPI-rings. But clearly $R_n \cong S_n \oplus T_n$ and the above argument shows that R_n is a right CDPI-ring. By Theorem 4.11 R is a right FGDPI_n-ring.

It is clear that one consequence of Lemma 5.5 is the following result.

COROLLARY 5.6. *Let S and T be subrings of a ring R such that $R = S \oplus T$. Then R is a right FGDPI-ring if and only if both S and T are right FGDPI-rings.*

THEOREM 5.7. *Let R be a semiprime right Noetherian ring. Then the following statements are equivalent.*

(i) *R is a right $FGDPI_2$-ring.*

(ii) *R is a right FGDPI-ring.*

(iii) *R is a left Goldie right RIC-ring.*

(iv) *R is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where A is a semiprime Artinian ring and for each $1 \leqq i \leqq n$ the ring B_i is a simple right and left Noetherian ring Morita equivalent to a Noetherian simple PCI-domain.*

Proof. (ii) $\Rightarrow$ (i) is clear. (iii) $\Rightarrow$ (ii) is a consequence of Corollary 5.3. (iv) $\Rightarrow$ (iii) is a consequence of [5], Theorem 3.11. It remains to prove (i) $\Rightarrow$ (iv). Suppose that R is a right FGDPI$_2$-ring. By [5], Theorem 3.11, R is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where A is semiprime Artinian and B_i is a simple right Noetherian ring Morita equivalent to a right Noetherian simple right PCI-domain D_i for each $1 \leqq i \leqq n$. By Lemmas 5.4 and 5.5 the ring B_i is a left Goldie ring for each $1 \leqq i \leqq n$. Thus, for each $1 \leqq i \leqq n$, D_i is left Goldie and hence a Noetherian simple PCI-domain by [3], Theorem 22 and subsequent remarks. It follows that B_i is left Noetherian ($1 \leqq i \leqq n$). This proves (iv).

COROLLARY 5.8. *For any positive integer m a ring R is a right Noetherian right FGDPI$_m$-ring if and only if R is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where A is a right Artinian right FGDPI$_m$-ring and the ring B_i is a simple right and left Noetherian ring Morita equivalent to a Noetherian simple PCI-domain for each $1 \leqq i \leqq n$.*

Proof. By the theorem and Lemma 5.5.

COROLLARY 5.9. *Let R be a semiprime ring. Then the following statements are equivalent.*

(i) *R is a right Noetherian right FGDPI$_2$-ring.*
(ii) *R is a left Noetherian left FGDPI$_2$-ring.*
(iii) *R is a right Noetherian right FGDPI-ring.*
(iv) *R is a left Noetherian left FGDPI-ring.*

Proof. By the theorem, Lemma 5.5 and Corollary 5.6.

COROLLARY 5.10. *Let R be a right Noetherian right FGDPI$_2$-ring with Jacobson radical J. Then the ring R/J is a left Noetherian left FGDPI-ring. Moreover R is a left SI-ring and in particular R is left hereditary.*

Proof. By Corollary 5.8 R/J is a right Noetherian right FGDPI$_2$-ring and by Corollary 5.9 R/J is a left Noetherian left FGDPI$_2$-ring. In §1 we noted that right Noetherian right CDPI-rings are right SI-rings. Also by [5], Proposition 3.5, right Artinian right SI-rings are left SI-rings. The result follows by [5], Theorem 3.11 and Proposition 3.3.

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