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POSITIVE OPERATORS AND THE ERGODIC THEOREM

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Let T be a positive linear operator on $L_1(X, \mathscr{F}, \mu)$ satisfying $\sup_n \|(1/n) \sum_{i=0}^{n-1} T^i\|_1 < \infty$, where $(X, \mathscr{F}, \mu)$ is a finite measure space. It will be proved that the two following conditions are equivalent: (I) For every f in $L_\infty(X, \mathscr{F}, \mu)$ the Cesàro averages of $T^{*n}f$ converge almost everywhere on X. (II) For every f in $L_1(X, \mathscr{F}, \mu)$ the Cesàro averages of $T^n f$ converge in the norm topology of $L_1(X, \mathscr{F}, \mu)$. As an application of the result, a simple proof of a recent individual ergodic theorem of the author is given.

Let $(X, \mathscr{F}, \mu)$ be a finite measure space and T a positive linear operator on $L_1(X, \mathscr{F}, \mu)$. If T is a contraction, then we denote by C and D the conservative and dissipative parts of T, respectively (cf. Foguel [4]). In [5] Helmberg proved that if T is a contraction then the two following conditions are equivalent: (I) For every $f \in L_\infty(X, \mathscr{F}, \mu)$ the Cesàro averages

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{*i}f$$

converge a.e. on X. (II) $\lim_n T^{*n}1_D = 0$ a.e. on X and there exists a function $0 \leqq u \in L_1(X, \mathscr{F}, \mu)$ satisfying $Tu = u$ and $\{u > 0\} = C$. It is easily seen that condition (II) is equivalent to each of the following conditions. (III) For every $u \in L_1(X, \mathscr{F}, \mu)$ the Cesàro averages

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i}u$$

converge in the norm topology of $L_1(X, \mathscr{F}, \mu)$. (IV) For every $A \in \mathscr{F}$ the Cesàro averages

$$\frac{1}{n}\sum_{i=0}^{n-1} \int T^{*i}1_A d\mu$$

converge. (Cf. Lin and Sine [6].)

The main purpose of this paper is to prove that the equivalence of conditions (I), (III), and (IV) holds, even if T is not a contraction but satisfies $\sup_n \|(1/n) \sum_{i=0}^{n-1} T^i\|_1 < \infty$. That is, we shall prove the

THEOREM 1. *Let $(X, \mathscr{F}, \mu)$ be a finite measure space and T a positive linear operator on $L_1(X, \mathscr{F}, \mu)$ satisfying $\sup_n \|(1/n) \sum_{i=0}^{n-1} T^i\|_1 < \infty$. Then the three following conditions are equivalent:*

(I) *For every* $f \in L_\infty(X, \mathscr{F}, \mu)$, $(1/n)\sum_{i=0}^{n-1} T^{*i}f$ *converges* a.e. *on* X.

(III) *For every* $u \in L_1(X, \mathscr{F}, \mu)$, $(1/n)\sum_{i=0}^{n-1} T^i u$ *converges in the norm topology of* $L_1(X, \mathscr{F}, \mu)$.

(IV) *For every* $A \in \mathscr{F}$, $(1/n)\sum_{i=0}^{n-1}\int T^{*i}1_A d\mu$ *converges.*

Proof. (I) $\Rightarrow$ (IV): Immediate from Lebesgue's bounded convergence theorem.

(IV) $\Rightarrow$ (III): The Vitali-Hahn-Saks theorem shows that the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1} T^i 1$$

converges weakly in $L_1(X, \mathscr{F}, \mu)$. By this and the fact that $\lim_n \|(1/n)T^n 1\|_1 = 0$, due to Derriennic and Lin [2], we see that $(1/n)\sum_{i=0}^{n-1} T^i 1$ converges in the norm topology of $L_1(X, \mathscr{F}, \mu)$ (cf. Theorem VIII.5.1 in [3]). Thus (III) follows easily from a standard approximation argument.

(III) $\Rightarrow$ (I): Define a function $0 \leqq t \in L_\infty(X, \mathscr{F}, \mu)$ by the relation:

$$t(x) = \limsup_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}1(x) \qquad (x \in X)\,.$$

Since $T^*t \geqq t$, if we set

$$s(x) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}t(x) \qquad (x \in X)\,,$$

then we have

$$(1) \qquad s = T^*s\,.$$

Let us put

$$Y = \{s > 0\} \quad \text{and} \quad Z = \{s = 0\}\,.$$

Then, by [2] and [7], we have:

$$(2) \qquad \begin{aligned} &u \in L_1(X, \mathscr{F}, \mu) \quad \text{and} \quad \{u \neq 0\} \subset Z \quad \text{imply} \\ &\{Tu \neq 0\} \subset Z \quad \text{and} \quad \lim_n \left\|(1/n)\sum_{i=0}^{n-1} T^i u\right\|_1 = 0\,. \end{aligned}$$

Using condition (III), take a function $0 \leqq h \in L_1(X, \mathscr{F}, \mu)$ so that

$$\lim_n \left\|h - (1/n)\sum_{i=0}^{n-1} T^i 1\right\|_1 = 0\,.$$

Since $Th = h$, for all $0 \leqq f \in L_\infty(X, \mathscr{F}, \mu)$ we have

$$\int (T^*f)hd\mu = \int f(Th)d\mu = \int fhd\mu .$$

By this, T^* may be regarded as a positive linear contraction on $L_1(X, \mathscr{F}, hd\mu)$, and therefore for every $f \in L_\infty(X, \mathscr{F}, \mu)$ $(\subset L_1(X, \mathscr{F}, hd\mu))$ the limit

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}f(x) = \lim_n s(x)\frac{\sum_{i=0}^{n-1} T^{*i}f(x)}{\sum_{i=0}^{n-1} T^{*i}s(x)} \tag{3}$$

exists a.e. on $\{h > 0\} \cap Y$, by the Chacon-Ornstein theorem (cf. [4]).

To prove the almost everywhere existence of the limit (3), we now define

$$\bar{f}(x) = \limsup_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}f(x) \qquad (x \in X)$$

and

$$\underline{f}(x) = \liminf_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}f(x) \qquad (x \in X) .$$

Since $T^*\bar{f} \geqq \bar{f} \geqq \underline{f} \geqq T^*\underline{f}$, if we set

$$f^*(x) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}\bar{f}(x) \qquad (x \in X)$$

and

$$f_*(x) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^{*i}\underline{f}(x) \qquad (x \in X) ,$$

then: $0 \leqq f^* - f_* \in L_\infty(X, \mathscr{F}, \mu)$ and $T^*(f^* - f_*) = f^* - f_*$. This and (2) imply $f^* - f_* = 0$ a.e. on Z, and thus $\bar{f} = \underline{f}$ a.e. on $\{h > 0\}$. Hence $f^* = f_*$ a.e. on $\{h > 0\}$, because $T^*1_{\{h=0\}} = 0$ a.e. on $\{h > 0\}$. Consequently we have

$$\begin{aligned}\int (f^* - f_*)d\mu &= \int \Big(\frac{1}{n}\sum_{i=0}^{n-1} T^i 1\Big)(f^* - f_*)d\mu \\ &= \int h(f^* - f_*)d\mu = 0 ,\end{aligned}$$

and so $f^* - f_*$ a.e. on X. This completes the proof.

As an easy application of Theorem 1, we shall show the following individual ergodic theorem due to the author [8]. His arguments given in [8] are rather long and complicated.

THEOREM 2. *Let $(X, \mathscr{F}, \mu)$ be a finite measure space and T a bounded (not necessarily positive) linear operator on $L_1(X, \mathscr{F}, \mu)$.*

Let τ denote the linear modulus of T in the sense of Chacon and Krengel [1]. *Assume the conditions*:

$$\sup_n \left\|(1/n)\sum_{i=0}^{n-1}\tau^i\right\|_1 < \infty , \tag{4}$$

$$\sup_n \left\|(1/n)\sum_{i=0}^{n-1}\tau^i\right\|_\infty < \infty . \tag{5}$$

Then, for every $f \in L_\infty(X, \mathscr{F}, \mu)$, $(1/n)\sum_{i=0}^{n-1} T^i f$ converges a.e. *on X.*

Proof. Let $f \in L_\infty(X, \mathscr{F}, \mu)$. Since $|T^n f| \leqq \tau^n |f|$ for each $n \geqq 1$, we have $\lim_n \|(1/n)T^n f\|_1 \leqq \lim_n \|(1/n)\tau^n |f|\|_1 = 0$, and by (5), the set

$$\left\{(1/n)\sum_{i=0}^{n-1} T^i f\colon n \geqq 1\right\}$$

is weakly sequentially compact in $L_1(X, \mathscr{F}, \mu)$. Hence, a well-known mean ergodic theorem (cf. Theorem VIII.5.1 in [3]) implies that $\lim_n \|g - (1/n)\sum_{i=0}^{n-1} T^i f\|_1 = 0$ for some $g \in L_1(X, \mathscr{F}, \mu)$ with $Tg = g$. Condition (5) implies $g \in L_\infty(X, \mathscr{F}, \mu)$, and hence $f - g \in L_\infty(X, \mathscr{F}, \mu)$. It is easily seen that $f - g$ belongs to the L_1-norm closure of the set $\{h - Th\colon h \in L_\infty(X, \mathscr{F}, \mu)\}$, because $L_\infty(X, \mathscr{F}, \mu)$ is a dense subspace of $L_1(X, \mathscr{F}, \mu)$. So, given an $\varepsilon > 0$, we can choose an $h \in L_\infty(X, \mathscr{F}, \mu)$ so that

$$\|(f - g) - (h - Th)\|_1 < \varepsilon .$$

Write $k = (f - g) - (h - Th)$. Then

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} T^i(f - g)\right| \leqq \frac{1}{n}\sum_{i=0}^{n-1}\tau^i|k| + \frac{1}{n}(|h| + \tau^n|h|)$$

and

$$\lim_n \frac{1}{n}(|h| + \tau^n|h|) = 0 \quad \text{a.e. on } X,$$

because Theorem 1 implies that the Cesàro averages of $\tau^n|h|$ converge a.e. on X. Thus

$$\limsup_n \left|\frac{1}{n}\sum_{i=0}^{n-1} T^i(f - g)\right| \leqq \lim_n \frac{1}{n}\sum_{i=0}^{n-1}\tau^i|k| ,$$

and by Fatou's lemma,

$$\left\|\lim_n (1/n)\sum_{i=0}^{n-1}\tau^i|k|\right\|_1 \leqq \varepsilon\left(\sup_n \left\|(1/n)\sum_{i=0}^{n-1}\tau^i\right\|_1\right) .$$

Consequently we have

$$\lim_n (1/n) \sum_{i=0}^{n-1} T^i(f - g) = 0 \quad \text{a.e. on } X,$$

and this establishes Theorem 2.

REMARK. It is known (cf. [2]) that Theorem 2 need not hold in general if we replace $f \in L_\infty(X, \mathscr{F}, \mu)$ by $f \in L_1(X, \mathscr{F}, \mu)$. But the author does not know whether, in Theorem 2, $f \in L_\infty(X, \mathscr{F}, \mu)$ can be replaced by $f \in L_p(X, \mathscr{F}, \mu)$ with $1 < p < \infty$.

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