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TANGENT FRAME FIELDS ON SPIN MANIFOLDS

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In this note we prove the following theorems.

THEOREM A. **Let M^n be a spin manifold with $n \equiv 7 \bmod 8$ and $n > 7$. Then M admits at least 8 nonhomotopic tangent 4-frame fields.**

THEOREM B. **Let M^n be a spin manifold with $n \equiv 3 \bmod 8$ and $n > 3$. Suppose that $w_{n-4}M = 0$ and $w_4 M \cdot w_{n-5}M = 0$. Then M^n admits a tangent 4-frame field iff**

$$w_{n-3}M = 0 \quad \textbf{and} \quad \chi_2 M = 0 .$$

1. **Introduction.** Here M^n denotes a closed connected smooth manifold of dimension n. A tangent k-frame field on M^n is an ordered set of k linearly independent vector fields on M^n. The classical theorem of Hopf states that M^n possesses a tangent 1-frame field iff the Euler characteristic $\chi M = 0$. A table of necessary and sufficient conditions for tangent 2-frame fields on orientable manifolds appears in [10] while conditions for tangent 3-frame fields are tabulated in and [3]. In particular, Atiyah and Dupont prove in [1] that any orientable manifold M^n with $n \equiv 3 \bmod 4$ admits a tangent 3-frame field. This result is best possible since neither the sphere S^{8i+3} nor $S^3 \times CP^{4i+2}$ admits a tangent 4-frame field.

Recall that an orientable manifold M^n is called a spin manifold if the Stiefel-Whitney class $w_2 M$ is trivial. The mod 2 semicharacteristic $\chi_2 M^n$ is defined if $n = 2s + 1$ by

$$\chi_2 M = \left(\sum_{i=0}^{s} \dim H_i(M; Z/2) \right) \bmod 2 .$$

Let σM denote the signature of M^n whenever n is divisible by 4. Finally δ represents the Bockstein-coboundary operator associated to the exact coefficient sequence $Z \to Z \to Z/2$.

Theorem A is a best possible result for $n \equiv 7 \bmod 16$. In [8, p. 690] Szczarba constructed certain spin manifolds M^n with $n \equiv 3 \bmod 4$ as the quotient spaces of free and differentiable actions of generalized quarternion groups on S^n. The span of these spherical space forms M^n with $n \equiv 7 \bmod 16$ and $n > 7$ is precisely 4 by Theorem 1.1 of [2].

An immediate consequence of Theorem A and the result of Thurston given by [14, Corollary 1] is the following.

COROLLARY. *Let M^n be a spin manifold with $n \equiv 7 \bmod 8$ and*

$n > 7$. *Then M possesses a C^∞ codimension 4 foliation with trivial normal bundle.*

We shall derive the following consequence of Theorem A and a theorem of Atiyah-Dupont given in [1, p. 25].

PROPOSITION. Let M^n be a spin manifold with $n \equiv 0 \bmod 8$ and $n > 8$. Suppose that $H_1(M; Z)$ has no 2-torsion, $\delta w_{n-6}M = 0$, and $u^2 = 0$ for all u in $H^2(M; Z/2)$. Then M admits a tangent 5-frame field iff

$$w_{n-4}M = 0\ , \quad \chi M = 0\ , \quad \text{and} \quad \sigma M \equiv 0 \bmod 16\ .$$

The above proposition was proved by Atiyah-Dupont under the assumption that M^n is 3-connected. Both Theorem A and B were announced in [7] and generalize Theorem 1.2 of [9]. Indeed, their proofs are applications of the Postnikov methods developed by Emery Thomas and applied in [9], [11], [12], [13], [5], and [6]. We thank Samuel Gitler, James Heitsch, and João de Carvalho for helpful conversations.

2. **Proof of Theorem A.** The k-invariants in a modified Postnikov resolution for the fibration

$$V_{n,4} \longrightarrow B\,\mathrm{Spin}\,(n-4) \xrightarrow{\pi} B\,\mathrm{Spin}\,(n) \tag{2.1}$$

through dimension n where $n \equiv 3 \bmod 4$ and $n > 7$ are listed with their defining relations below.

$$\begin{aligned} &k^0 = w_{n-3} \\ &k^1\colon \mathrm{Sq}^2\mathrm{Sq}^1 w_{n-3} = 0 \\ &k^2\colon (\mathrm{Sq}^4 + \cdot w_4)w_{n-3} = 0 \\ &k^3\colon \mathrm{Sq}^2 k^1 = 0\ . \end{aligned} \tag{2.2}$$

(See resolution II of [6, p. 56].) Let

$$\tau\colon M^n \longrightarrow B\,\mathrm{Spin}\,(n)$$

classify the tangent bundle of M where $n \equiv 7 \bmod 8$ and $n > 7$. We must show that τ lifts to $B\,\mathrm{Spin}\,(n-4)$ in (2.1). Set $n = 8t + 7$. Since the *Wu* classes v_iM are trivial for $i > 4t$, the classes w_iM are trivial for $i > 8t$ by the formula

$$W = \mathrm{Sq}\,V\ .$$

The proof of Theorem 1.3 of [11] evaluates $k^1(\tau)$ and $k^3(\tau)$ by secondary and tertiary operations applied to $w_{8t+2}M = 0$ respectively. Thus

$$k^1(\tau) = 0 = k^3(\tau)$$

because of zero indeterminacy. Let U denote the Thom class of the Thom complex $T\tau$ associated to the tangent bundle τ. In [9] Thomas proves that

$$U \cdot k^2(\tau) = \psi(U)$$

with zero indeterminacy where ψ is a stable secondary operation associated to the relation in the Steenrod algebra

$$\mathrm{Sq}^4\mathrm{Sq}^{8t+4} + \mathrm{Sq}^2(\mathrm{Sq}^{8t+4}\mathrm{Sq}^2) + \mathrm{Sq}^1(\mathrm{Sq}^{8t+4}\mathrm{Sq}^3 + \mathrm{Sq}^{8t+6}\mathrm{Sq}^1) = 0 .$$

We recall the following facts from [9] and [13]. Let

$$s\colon M \times M \longrightarrow M \times M$$

denote the involution which interchanges factors and let

$$c\colon M \times M \longrightarrow T\tau$$

denote the collapsing map associated to an embedding of τ as a neighborhood of the diagonal in $M \times M$. Select a basis

$$\alpha_1, \cdots, \alpha_r \tag{2.3}$$

for the graded vector space $\sum_{i=0}^{[n/2]} H^i(M; Z/2)$. Let $\beta_1, \cdots, \beta_r$ be the dual basis by Poincaré duality such that

$$\alpha_i \cdot \beta_j = \delta_{ij}\mu$$

if $\deg \alpha_i + \deg \beta_j = n$. Here μ generates $H^n(M; Z/2)$ while clearly $r = \chi_2 M$. We set

$$A = \sum_{i=1}^{r} \alpha_i \otimes \beta_i . \tag{2.4}$$

Then $c^*U = A + s^*A$ and $A \cdot s^*A = \chi_2 M(\mu \otimes \mu)$.

Suppose that $\psi(A)$ is defined. The indeterminacy of $\psi(A)$ is trivial iff $w_4 M = 0$ since

$$\mathrm{Sq}^4(v \otimes \mu) = \mathrm{Sq}^4 v \otimes \mu = v \cdot w_4 M \otimes \mu$$

for any class v in $H^{n-4}(M; Z/2)$. We consider the universal example (E, m, v) for the operation ψ on classes of dimension $8t + 7$.

$$\Omega C \xrightarrow{\ i\ } E \xrightarrow{\ p\ } K(Z/2, 8t + 7) . \tag{2.5}$$

Here p is the principal fibration induced from the path-loop fibration on

$$C = K(Z/2, 16t + 11) \times K(Z/2, 16t + 13) \times K(Z/2, 16t + 14)$$

by the classifying map $K(Z/2, 8t + 7) \to C$ with component operations

$$(\mathrm{Sq}^{8t+4}, \mathrm{Sq}^{8t+4}\mathrm{Sq}^2, \mathrm{Sq}^{8t+4}\mathrm{Sq}^3 + \mathrm{Sq}^{8t+6}\mathrm{Sq}^1)$$

applied to the fundamental class ι of $K(Z/2, 8t+7)$. Now m denotes the homotopy-commutative multiplication on E while v in $H^{2n}(E; Z/2)$ represents ψ.

We now exploit a technique of [4] in order to evaluate $\psi(U)$.

Let

$$\bar{A}: M \times M \longrightarrow E$$

denote any lifting of the class A in (2.4) under the assumption that $\psi(A)$ is defined. Then the map (2.6) $g = m \circ (\bar{A}, \bar{A} \circ s): M \times M \to E$ defines a lifting of $A + s^*A$ such that

$$g^*v = \bar{A}^*v + s^*\bar{A}v = 0$$

since s^* is the identity on $H^{2n}(M \times M; Z/2)$.

Let

$$\bar{U}: T\tau \longrightarrow E$$

be any lifting of the Thom class U and set $f = \bar{U} \circ c$. Since c^* is a monomorphism, $\psi(U)$ vanishes if we can show that

$$c^*\psi(U) = f^*v = g^*v = 0 .$$

Since f and g are liftings of c^*U, there exists a map

$$h: M \times M \longrightarrow \Omega C$$

unique up to homotopy such that f and $m(i \circ h, g)$ are homotopic. We identify h with a triple (x, y, z) of classes in $H^*(M \times M; Z/2)$. Thus

$$f^*v = g^*v + \mathrm{Sq}^4x + \mathrm{Sq}^2y + \mathrm{Sq}^1z = \mathrm{Sq}^4x . \tag{2.7}$$

The map $i \circ h$ is invariant under s since both f and g are invariant. Thus the homotopy class $[h] + [h \circ s]$ lies in the image of

$$[M \times M, K(Z/2, 8t+6)] \longrightarrow [M \times M, \Omega C] .$$

Consequently,

$$x + s^*x \varepsilon \mathrm{Sq}^{8t+4}H^{8t+6}(M \times M; Z/2) . \tag{2.8}$$

Note that Sq^4 is trivial on any class in $H^i(M; Z/2) \otimes H^{2n-4-i}(M; Z/2)$ with bi-degree $(i, 2n-4-i)$ different from $(n-4, n)$ and $(n, n-4)$.

The following lemma implies by (2.8) that the symmetric class $x + s^*x$ contains no nontrivial classes of bi-degree $(n-4, n)$ or $(n, n-4)$. Thus x is symmetric in the classes with bi-degree $(n, n-4)$ and $(n-4, n)$. We conclude that

$$0 = \mathrm{Sq}^4 x = f^* v .$$

LEMMA. *Let M^n be any orientable manifold with $n = 4j + 3$ and $j > 0$.*

Let

$$P\colon H^{2n-4}(M \times M;\ Z/2) \longrightarrow H^{n-4}(M;\ Z/2) \otimes H^n(M;\ Z/2)$$

be the projection morphism corresponding to the Kunneth formula. Then the kernel of P contains

$$\mathrm{Sq}^{n-3} H^{n-1}(M \times M;\ Z/2) .$$

Proof. Let $\alpha \otimes \beta$ be a class with bi-degree $(i, 4j + 2 - i)$ in $H^{n-1}(M \times M;\ Z/2)$. By the Cartan formula and dimensionality

$$\mathrm{Sq}^{4j}(\alpha \otimes \beta) = \alpha^2 \otimes \mathrm{Sq}^{4j-2}\beta + \mathrm{Sq}^{i-1}\alpha \otimes \mathrm{Sq}^{4j-i+1}\beta + \mathrm{Sq}^{i-2}\alpha \otimes \mathrm{Sq}^{4j-i+2}\beta .$$

The image of $\mathrm{Sq}^{4j}(\alpha \otimes \beta)$ under P is clearly trivial unless $i = 2j$. Further,

$$\mathrm{Sq}^{2j+1}\beta = \mathrm{Sq}^1\mathrm{Sq}^{2j}\beta = 0 \quad \text{in}$$
$$H^n(M;\ Z/2) \quad \text{when} \quad i = 2j .$$

To complete the proof of Theorem A, we must justify the assumption that $\psi(A)$ is defined. We leave this verification to the reader, since we shall make similar calculations in the more complicated proof of Theorem B. Finally, by [1, Proposition 6.13], the existence of a tangent 4-frame field on M given by a lifting of τ to $B\,\mathrm{Spin}\,(n-4)$ implies the existence of 8 nonhomotopic tangent 4-frame fields.

REMARK. The proof of Theorem A shows that any lifting of τ to any stage in the Postnikov resolution itself lifts to $B\,\mathrm{Spin}\,(n-4)$ since all the k-invariants of τ are trivial with zero indeterminacy.

3. Proof of Theorem B.

Let M^n be a spin manifold with $n = 8t + 3$ for positive t such that

$$w_4 M \cdot w_{n-5} M = 0 \quad \text{and} \quad w_{n-4} M = 0 .$$

We adopt the notation of §2 freely. We must show that

$$\tau\colon M \longrightarrow B\,\mathrm{Spin}\,(n)$$

has a lifting in the fibration (2.1) iff

$$w_{n-3} M = 0 \quad \text{and} \quad \chi_2 M = 0 .$$

Suppose the primary obstruction $w_{8t} M$ vanishes. For $n = 11$, the obstructions $k^1(\tau)$ and $k^3(\tau)$ vanish since they lie in the image of Sq^2 and a spin-trivial secondary operation respectively. For $n > 11$, the proof of Theorem 1.3 of [11] establishes the triviality of $k^1(\tau)$ and $k^3(\tau)$, whenever defined. (Note corrigenda (ii) in [10].)

In [9] Thomas proves that

$$U\cdot k^2(\tau) = \Gamma(U)$$

with zero indeterminacy where Γ is a nonstable secondary operation associated to the relation

$$\mathrm{Sq}^4\mathrm{Sq}^{8t} + \mathrm{Sq}^1(\mathrm{Sq}^{8t+2}\mathrm{Sq}^1 + \mathrm{Sq}^{8t}\mathrm{Sq}^3) + \mathrm{Sq}^2(\mathrm{Sq}^{8t}\mathrm{Sq}^2) = 0$$

which holds on mod 2 classes of degree $< 8t + 4$. Let (E, m, v) denote the universal example for the operation Γ on classes of degree $8t + 3$. Since Γ is nonstable,

$$m^*v = v \otimes 1 + 1 \otimes v + p^*\iota \otimes p^*\iota$$

in $H^{2n}(E \times E; Z/2)$. Suppose $\Gamma(A)$ is defined. The map g in (2.6) associated to any lifting $\bar{A}$ defines a lifting of c^*U such that

$$g^*v = \bar{A}^*v + s^*\bar{A}^*v + A\cdot s^*A = \chi_2 M(\mu \otimes \mu) .$$

Let $\bar{U}\colon T\tau \to E$ be any lifting of the Thom class U and set $f = \bar{U}\circ c$. The argument in §2 shows that

$$f^*v = g^*v .$$

(Recall that the lemma in §2 was formulated for $n \equiv 3 \bmod 4$.) Thus

$$U\circ k^2(\tau) = \Gamma(U) = \bar{U}^*v = (\chi_2 M)U\cdot\mu$$

and so by the Thom isomorphism

$$k^2(\tau) = (\chi_2 M)\mu .$$

The following lemma concludes the proof of Theorem B.

LEMMA. *$\Gamma(A)$ is defined.*

Proof. Now $\mathrm{Sq}^{8t+2}\mathrm{Sq}^1 A = 0 = \mathrm{Sq}^{8t}\mathrm{Sq}^3 A$ in the spin manifold $M \times M$ since

$$\mathrm{Sq}^{8t+2}\mathrm{Sq}^1 = \mathrm{Sq}^2(\mathrm{Sq}^{8t}\mathrm{Sq}^1)$$
$$\mathrm{Sq}^{8t}\mathrm{Sq}^3 = \mathrm{Sq}^2(\mathrm{Sq}^{8t-1}\mathrm{Sq}^2) + \mathrm{Sq}^1(\mathrm{Sq}^{8t}\mathrm{Sq}^2) .$$

Note that $\mathrm{Sq}^2 A$ is symmetric since

$$\mathrm{Sq}^2 A + s^*\mathrm{Sq}^2 A = c^*\mathrm{Sq}^2 U = 0 .$$

Thus $\mathrm{Sq}^2 A$ contains nonzero summands only of bi-degree $(4t+2, 4t+3)$ and $(4t+3, 4t+2)$. Let $\beta \otimes \gamma$ be any class with bi-degree $(4t+1, 4t+2)$. Now

$$\mathrm{Sq}^{4t}\mathrm{Sq}^1\gamma = \mathrm{Sq}^2\mathrm{Sq}^{4t-1}\gamma + \mathrm{Sq}^1\mathrm{Sq}^{4t}\gamma = 0$$

so by the Cartan formula

$$\mathrm{Sq}^{8t}\mathrm{Sq}^2 A = \sum \mathrm{Sq}^{4t}\mathrm{Sq}^2\alpha_i \otimes \mathrm{Sq}^{4t}\beta_i \tag{3.1}$$

where only the summands with degree $\alpha_i = 4t$ or $4t+1$ are possibly nonzero.

Suppose that the Wu class $v_{4t} = 0$. Then

$$\mathrm{Sq}^{4t}\beta = \beta \cdot v_{4t} = 0$$

for any β in $H^{4t+3}(M; Z/2)$. If v_{4t} is nonzero, we are free to choose v_{4t} to be a class in (2.3). Set $\alpha_j = v_{4t}$. We consider any summand in (3.1) with

$$\text{degree } \alpha_i = 4t\,, \quad \text{degree } \beta_i = 4t+3\,.$$

Now $\mathrm{Sq}^{4t}\beta_i = \beta_i \cdot v_{4t} = \beta_i \cdot \alpha_j = 0$ for $i \neq j$. If $i = j$,

$$\mathrm{Sq}^{4t}\mathrm{Sq}^2\alpha_j = \mathrm{Sq}^4\mathrm{Sq}^{4t-2}v_{4t}\,.$$

By dimensionality $\mathrm{Sq}^{4t-2}v_{4t} = w_{8t-2}M$. We conclude that

$$\mathrm{Sq}^{4t}\mathrm{Sq}^2\alpha_j = \mathrm{Sq}^4 w_{n-5}M = w_4 M \cdot w_{n-5}M = 0\,.$$

But all summands in (3.1) with degree $\alpha_i = 4t+1$ must vanish by symmetry so

$$\mathrm{Sq}^{8t}\mathrm{Sq}^2 A = 0\,.$$

The class $\mathrm{Sq}^{8t}A$ is symmetric since

$$\mathrm{Sq}^{8t}A + s^*\mathrm{Sq}^{8t}A = c^*\mathrm{Sq}^{8t}U = 0\,.$$

Recall that degree $\alpha_i \leqq 4t+1$ for every α_i in (2.3). By symmetry the possibly nonzero summands in $\mathrm{Sq}^{8t}A$ are the classes

$$\mathrm{Sq}^{4t+1}\alpha_i \otimes \mathrm{Sq}^{4t-1}\beta_i + \mathrm{Sq}^{4t}\alpha_i \otimes \mathrm{Sq}^{4t}\beta_i$$

where $\alpha_i \otimes \beta_i$ has bi-degree $(4t+1, 4t+2)$.

We claim that either $\mathrm{Sq}^{4t}\alpha_i$ or $\mathrm{Sq}^{4t}\beta_i$ is trivial. Choose a basis

$$x_1 v_{4t},\ x_2 v_{4t},\ \cdots,\ x_j v_{4t}$$

for $v_{4t}H^1(M; Z/2)$. Extend this basis to a basis

$$\alpha_1,\ \cdots,\ \alpha_r$$

for $H^{4t+1}(M; Z/2)$ with $\alpha_i = x_i v_{4t}$ for $i \leqq j$. Let $\beta_1, \cdots, \beta_r$ denote the dual basis for $H^{4t+2}(M; Z/2)$. For $j < i \leqq r$ and any class z in $H^1(M; Z/2)$,

$$\mathrm{Sq}^{4t}\beta_i \cdot z = \mathrm{Sq}^{4t}(\beta_i z) = \beta_i(z v_{4t}) = 0 .$$

Thus $\mathrm{Sq}^{4t}\beta_i = 0$ for $j < i$. For $i \leqq j$

$$\mathrm{Sq}^{4t}(x_i v_{4t}) = x_i w_{n-3}M + x_i^2 w_{n-4}M = 0 .$$

We conclude by symmetry that $\mathrm{Sq}^{8t}A = 0$.

4. **Proof of Proposition.** Let M^n be a spin manifold with $n \equiv 0 \bmod 8$ and $n > 8$. We assume that $H_1(M; Z)$ has no 2-torsion, $\delta w_{n-6}M = 0$, and $u^2 = 0$ for all u in $H^2(M; Z/2)$. Let

$$\tau \colon M \longrightarrow B\,\mathrm{Spin}\,(n)$$

classify the tangent bundle of M. The following diagram is the Moore-Postnikov resolution for the fibration

$$\pi \colon B\,\mathrm{Spin}\,(n-5) \longrightarrow B\,\mathrm{Spin}\,(n)$$

through dimension n.

$$\begin{array}{lccc}
 & & B\,\mathrm{Spin}\,(n-5) & \\
 & & \downarrow & \\
 & & E_4 & \\
 & & \downarrow & \\
 & & E_3 & \xrightarrow{k^3} K(Z \oplus Z/8,\, n) \\
(4.1) & & \downarrow & \\
 & & E_2 & \xrightarrow{k^2} K(Z/2 \oplus Z/2,\, n-1) \\
 & & \downarrow & \\
 & & E_1 & \xrightarrow{k^1} K(Z/2,\, n-2) \\
 & & \downarrow & \\
 & M \xrightarrow{\tau} & B\,\mathrm{Spin}\,(n) & \xrightarrow{w_{n-4}} K(Z/2,\, n-4) .
\end{array}$$

Let $f \colon M \longrightarrow E_3$ be any lifting for τ. Then

$$f^*k^3 \in H^n(M; Z \oplus Z/8) \cong Z \oplus Z/8 .$$

Atiyah and Dupont in [1, p. 25] show that

$$f^*k^3 = (0, 0) \quad \text{iff} \quad \chi M = 0 \quad \text{and} \quad \sigma M \equiv 0 \bmod 16 .$$

We must show that τ lifts to E_3 iff $w_{n-4}M = 0$. Assume $w_{n-4}M = 0$ so τ lifts to E_1.

The following diagram contains the first stage of a modified Postnikov resolution for the fibration

$$B\,\mathrm{Spin}\,(n-6) \longrightarrow B\,\mathrm{Spin}\,(n)$$

through dimension $n-1$.

(4.2)
$$\begin{array}{l} \bar{E}_1 \\ \downarrow \\ B\,\mathrm{Spin}\,(n) \xrightarrow{\delta w_{n-6} \times w_{n-4}} K(Z, n-5) \times K(Z/2, n-4)\,. \end{array}$$

Let $h\colon \bar{E}_1 \to E_1$ denote the induced map.

Then

$$h^* k^1 = \mathrm{Sq}^2 y$$

where y has the defining relation

$$\mathrm{Sq}^2(\delta w_{n-6}) + \mathrm{Sq}^1 w_{n-4} = 0\,.$$

The map τ lifts to $\bar{E}_1$ since $\delta w_{n-6}M = 0 = w_{n-4}M$. The indeterminacy of $k^1(\tau)$ is given by

$$\mathrm{Sq}^2\mathrm{Sq}^1 H^{n-5}(M;\, Z/2) = 0\,.$$

Now Sq^2 vanishes on $H^{n-4}(M;\, Z/2)$ iff $u^2 = 0$ for all u in $H^2(M;\, Z/2)$ by Poincaré duality and the Cartan formula. We conclude that $\tau(k^1) = 0$ so τ lifts to E_2 in (4.1).

We write $g^* k^2 = (u, v)$ where $g\colon M \to E_2$ is any lifting of τ and the classes u and v belong to $H^{n-1}(M;\, Z/2)$. Suppose that $g^* k^2$ is nonzero. Then at least one class, say u, is nontrivial. Now

$$0 = \delta u \in H^n(M;\, Z) \approx Z\,.$$

Select any class x in $H^{n-1}(M;\, Z)$ such that $\rho_2 x = u$ where ρ_2 denotes reduction mod 2. Next choose a class a in $H_{n-1}(M;\, Z)$ such that the evaluation $x(a)$ is an odd multiple of a generator for $H_0(M;\, Z) \approx Z$. There exists such a class a because $H^{n-1}(M;\, Z)$ has no 2-torsion.

Let $i\colon N \to M$ be the inclusion of an oriented codimension one submanifold N (not necessarily connected) of M such that

$$i_*(\mu_N) = a\,.$$

Here μ_N denotes the fundamental homology class of N. Since

$$x(a) = x(i_*\mu_N) = (i^*x)(\mu_N)\,,$$

it follows that $i^*u = \rho_2(i^*x) \neq 0$. Note that the lifting

$$g \circ i \colon N \longrightarrow E_2$$

of the stable tangent bundle of N does not lift to E_3 since

$$(g \circ i)^* k^2 = (i^*u, i^*v) \neq (0, 0) .$$

The following lemma applied to the connected components of N yields a contradiction to the assumption that g^*k^2 is nonzero. Thus τ lifts to E_3 and the proposition is proved.

LEMMA. *Let N be any codimension* 1, *closed, connected, orientable submanifold of M with inclusion denoted by i. Then any lifting of*

$$\tau \circ i \colon N \longrightarrow B\,\mathrm{Spin}\,(n)$$

to any space E_j in the resolution (4.1) *further lifts to* $B\,\mathrm{Spin}\,(n-5)$.

Proof. The normal bundle to N in M is trivial by orientability. So N is a spin manifold whose stable tangent bundle is classified by the composite $\tau \circ i$. The Moore-Postnikov resolution in (4.1) is essentially a modified Postnikov resolution through dimension $n-1$. One component of the class k^2 is the image of a class z in $H^{n-1}(E_1; Z/2)$ with defining relation

$$(\mathrm{Sq}^4 + \cdot w_4) w_{n-4} = 0 .$$

The corresponding spaces in the modified Postnikov resolution (2.1) for the fibration

$$B\,\mathrm{Spin}\,(n-5) \longrightarrow B\,\mathrm{Spin}\,(n-1)$$

clearly map into E_1 and E_2 in (4.1). The map of resolutions begins with the inclusion

$$B\,\mathrm{Spin}\,(n-1) \longrightarrow B\,\mathrm{Spin}\,(n) .$$

With respect to the induced maps, the class z goes to k^2 in (2.1) while the other component of k^2 in (4.1) maps to k^3 in (2.1). The proof of Theorem A shows that any lifting of $\tau(N)$ to any stage in the modified Postnikov resolution (2.1) for the fibration

$$B\,\mathrm{Spin}\,(n-5) \longrightarrow B\,\mathrm{Spin}\,(n-1)$$

itself lifts to $B\,\mathrm{Spin}\,(n-5)$. (See the remark in §2.) Thus the same property holds for any lifting of the stable tangent bundle in the resolution (4.1).

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