PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No. 1, 1978

NEIGHBORNETS

HEIKKI J. K. JUNNILA

By a "neighbornet" of a topological space X we mean a binary relation V on X such that for each $x \in X$, $V\{x\}$ is a neighborhood of x; thus a neighbornet is, in effect, an assignment of neighborhoods to the points of X. Such neighborhood assignments and the corresponding relations have been in use since the beginning of the study of general topology, at first in the theory of metric spaces and later in the theory of uniform spaces; in the last twenty or thirty years they have been used in connection with spaces defined by covering axioms and with various generalizations of metric and uniform spaces. Even though the concept of a neighbornet is not new, neighbornets have mostly been considered as tools, not as objects of intrinsic interest. With this paper we hope to show that neighbornets deserve to be studied also on their own. We shall show, for example, that from simple properties of neighbornets of semi-stratifiable spaces the solution follows easily to J. Ceder's problem, whether all M_3-spaces are M_2-spaces.

1. Preliminaries.

NOTATION. The set $\{1, 2, \cdots\}$ of the natural numbers is denoted by N. $\langle x_n \rangle$ denotes the sequence whose nth term is x_n (for $n \in N$).

Let A be a set and $\mathscr{L}$ a family of subsets of A. We denote by $\bigcup \mathscr{L}$ and $\bigcap \mathscr{L}$ the sets $\bigcup \{L \mid L \in \mathscr{L}\}$ and $\bigcap \{L \mid L \in \mathscr{L}\}$, respectively (note that, in general, $\bigcup \mathscr{L}_i$ is not the same as $\bigcup_{i \in I} \mathscr{L}_i$); in case the family $\mathscr{L}$ is empty, we let $\bigcup \mathscr{L} = \varnothing$ and $\bigcap \mathscr{L} = A$. The family $\{A - L \mid L \in \mathscr{L}\}$ is denoted by $^{-}\mathscr{L}$ and when $a \in A$, the family $\{L \in \mathscr{L} \mid a \in L\}$ is denoted by $(\mathscr{L})_a$.

The word "iff" is an abbreviation of "if and only if". To avoid unnecessary repetition of the words "a topological space" in the following, we let the symbol X stand throughout for a topological space. For each $x \in X$, η_x denotes the neighborhood filter of x.

For the meaning of notation and terminology used without definition in this paper, see [18].

Relations. All relations considered below are binary. Let B be a set. Relations on B are usually identified with subsets of $B \times B$. However, as this can be done in two different ways and as there is disagreement among mathematicians as to which of these ways is the "right" one (see e.g., [20], p. 24-26), we do not adopt here any specific convention. For our purposes, it is enough to know that relations can be represented as sets in such a way that if R and S are relations on B, then $(R \cup S)\{b\} = R\{b\} \cup S\{b\}$ and $(R \cap S)\{b\} = R\{b\} \cap S\{b\}$ for each $b \in B$. We use relations mainly as abbreviations for certain indexed families of sets and we usually define a relation R on B by defining subsets $R\{b\}$, $b \in B$, of B (these sets $R\{b\}$ are often denoted by $R(b)$ but we use the notation $R(b)$, or Rb, only if $R\{b\}$ is a singleton set; Rb is then defined by the equation $\{Rb\} = R\{b\}$).

Let R be a relation on B. For each $C \subset B$, we denote the set $\bigcup \{R\{c\} \mid c \in C\}$ by RC (or by $R(C)$). When S is a relation on B, we define a relation $S \circ R$ on B by setting $(S \circ R)\{b\} = S(R\{b\})$ for each $b \in B$. We let $R^1 = R$ and $R^{n+1} = R \circ R^n$ for $n \in N$; further, we let $R^\infty = \bigcup_{n \in N} R^n$. It is easily seen that R^∞ is a transitive relation and that we have $T = T^\infty$ if T is a transitive relation. We define a relation R^{-1} on B by requiring that for all $a \in B$ and $b \in B$, $a \in R^{-1}\{b\}$ iff $b \in R\{a\}$. For each $k \in N \cup \{\infty\}$, we abbreviate $(R^k)^{-1}$ to R^{-k}; note that we have $R^{-n-1} = R^{-1} \circ R^{-n}$ for each $n \in N$.

When R is a relation on B and $\mathscr{N} = \{N_b \mid b \in B\}$ a family of subsets of B, indexed by the elements of B, we say that R and $\mathscr{N}$ are *associated* with each other if $R\{b\} = N_b$ for each $b \in B$. For an indexed family $\mathscr{N} = \{N_b \mid b \in B\}$ of subsets of B, we denote by $A\mathscr{N}$ the relation associated with $\mathscr{N}$ (using one of the standard representations of relations as sets, one has $A\mathscr{N} = \bigcup \{N_b \times \{b\} \mid b \in B\}$ or $A\mathscr{N} = \bigcup \{\{b\} \times N_b \mid b \in B\}$). For a relation R on B, we denote by $\mathscr{A}R$ the indexed family associated with R; clearly, $\mathscr{A}R = \{R\{b\} \mid b \in B\}$.

We have established a one-to-one correspondence between relations on a set and certain indexed families of subsets of the set. Now we show that some special relations can be characterized in terms of arbitrary (that is, possibly unindexed) families of sets. Let $\mathscr{L}$ be a family of subsets of B. We denote by $S\mathscr{L}$ and $D\mathscr{L}$ the relations associated with the indexed families $\{\bigcup (\mathscr{L})_b \mid b \in B\}$ and $\{\bigcap (\mathscr{L})_b \mid b \in B\}$, respectively.

LEMMA 1.1. *Let R be a relation on a set of B. Then*

(i) *R is symmetric iff $R = S\mathscr{L}$ for some family $\mathscr{L}$ of subsets of B.*

(ii) *R is transitive and reflexive iff $R = D\mathscr{L}$ for some family $\mathscr{L}$ of subsets of B.*

Proof. Sufficiency. Let $\mathscr{L}$ be a family of subsets of B. Then the relation $S\mathscr{L}$ is obviously symmetric. The relation $D\mathscr{L}$ is transitive since for all $a \in B$ and $b \in B$, we have $\bigcap (\mathscr{L})_b \subset \bigcap (\mathscr{L})_a$ if $b \in \bigcap (\mathscr{L})_a$. Since $\bigcap (\mathscr{L})_b = B$ is $(\mathscr{L})_b = \varnothing$, we see that $D\mathscr{L}$ is reflexive.

Necessity. If R is symmetric, let $\mathscr{L}$ consist of all sets $\{a, b\} \subset B$, where $a \in R\{b\}$. If R is transitive, let $\mathscr{L} = \mathscr{A}R$.

For a family $\mathscr{L}$ of subsets of B, we have $D\mathscr{L} = D\mathscr{L}'$, where $\mathscr{L}' = \mathscr{L} \cup \{B\}$; hence in (ii) above, we can choose $\mathscr{L}$ to be a cover of B.

Let $\mathscr{L}$ be a family of subsets of B and let $k \in \boldsymbol{N} \cup \{\infty\}$. We abbreviate $(S\mathscr{L})^k$ to $S^k\mathscr{L}$. In the usual notation for star-sets, we have that $S^k\mathscr{L}C = St^k(C, \mathscr{L})$ for each $C \subset B$; in particular $S\mathscr{L}\{b\} = \bigcup (\mathscr{L})_b = St(b, \mathscr{L})$ for each $b \in B$. Since $D\mathscr{L}$ is reflexive and transitive, we have $(D\mathscr{L})^k = D\mathscr{L}$. For all $a \in B$ and $b \in B$, we have $a \in \bigcap (\mathscr{L})_b$ iff $b \in \bigcap (^-\mathscr{L})_a$; if follows that $(D\mathscr{L})^{-1} = D^-\mathscr{L}$.

REMARK. To help to abbreviate expressions involving relations, we adopt the convention that in such an expression, unless indicated otherwise by parenthesis, operations involving relations are to be performed before those involving subsets of the domain of the relations (for example, when R and S are relations on B and C and D subsets of B, $S \cap RC - D$ is to be interpreted as $((S \cap R)C) - D)$.

2. Definitions and basic properties.

DEFINITION 2.1. A relation V on X is a *neighbornet* of X if $V\{x\}$ is a neighborhood of x for each $x \in X$.

When R is a relation on X, we denote by $\dot{R}$ the relation associated with the indexed family $\{\operatorname{Int} R\{x\} \mid x \in X\}$ (note that, in general, $\dot{R}$ is different from R^0, the interior of R in the product space $X \times X$). With this notation, neighbornets can be characterized as follows: a relation V on X is a neighbornet iff the relation $\dot{V}$ is reflexive.

A relation R on X is said to be *open* if all the sets $R\{x\}$, $x \in X$, are open. For a neighbornet V of X, $\dot{V}$ is an open neighbornet contained in V.

Neighbornets can be regarded as generalizations of relations of the form $S\mathscr{U}$, where $\mathscr{U}$ is an open cover. One advantage of this generalization is that it allows us to study the dual properties of relations and their inverses; the relations $S\mathscr{U}$ are always symmetric so that this duality does not appear in connection with them (the need to overcome the restriction to symmetric relations has been observed e.g., in [8]). The duality between neighbornets and their inverses is a special case of the duality between openness and closedness and it will be very clearly illustrated in connection with transitive neighbornets (Theorem 3.14); at this point we just give a general characterization of the inverses of neighbornets.

DEFINITION 2.2. A relation R on X is *cushioned* if we have $\bar{A} \subset RA$ for every $A \subset X$.

LEMMA 2.3. *A relation V on X is a neighbornet iff the relation V^{-1} is cushioned.*

Proof. Sufficiency. Suppose that V^{-1} is cushioned and let $x \in X$. We have $x \notin V^{-1}(X \sim V\{x\})$ and it follows by the cushionedness of V^{-1} that $x \notin \overline{X \sim V\{x\}}$, in other words, that $x \in \operatorname{Int} V\{x\}$.

Necessity. Assume that V is a neighbornet and let $A \subset X$. For every $x \in \bar{A}$, the neighborhood $V\{x\}$ of x intersects the set A; it follows that $\bar{A} \subset V^{-1}A$.

We see thus that neighbornets coincide with what might be called "co-cushioned" relations.

It is obvious that if $\mathscr{U}$ is an open cover of X and, for each $x \in X$, U_x an element of the family $(\mathscr{U})_x$, then the relation $A\mathscr{O}$ associated with the indexed family $\mathscr{O} = \{U_x | x \in X\}$ is a neighbornet. Relations associated with indexed families of sets chosen in a similar manner from closed covers generally fail to be cushioned, but it is easily seen that the following holds: when $\mathscr{F}$ is a closure-preserving and closed cover of X and for each $x \in X$, F_x an element of the family $(\mathscr{F})_x$, then the relation $A\{F_x | x \in X\}$ is cushioned.

The following lemma gives a useful property of open neighbornets:

LEMMA 2.4. *Let V be an open neighbornet of X and let $A \subset X$. Then the set $\{x \in X | V^{-1}\{x\} \subset A\}$ is closed.*

Proof. The set in question is the complement of the open set $V(X - A)$.

3. **Special neighbornets.** In this section we study symmetric, unsymmetric and transitive neighbornets and indicate their relationship to some families of sets frequently encountered in topological investigations.

(a) Symmetric and cushioned neighbornets.

DEFINITION 3.1. A cover $\mathscr{L}$ of X is *semi-open* if the set $St(x, \mathscr{L})$ is a neighborhood of x for each $x \in X$.

In an implicit form, semi-open covers have often appeared in topological studies (see e.g., [9] and [1]).

A cover $\mathscr{L}$ is semi-open iff the relation $S\mathscr{L}$ is a neighbornet; as $S\mathscr{L}$ is symmetric, it follows by Lemma 2.3 that a cover $\mathscr{L}$ is semi-open iff we have $\bar{A} \subset St(A, \mathscr{L})$ for each $A \subset X$. It follows that, in addition to all open covers, also all closure-preserving closed covers are semi-open.

By using Lemma 1.1, we get the following characterization of symmetric neighbornets.

LEMMA 3.2. *A relation V on X is a symmetric neighbornet iff we have $V = S\mathscr{L}$ for some semi-open cover $\mathscr{L}$ of X.*

From Lemma 2.3 it follows that each symmetric neighbornet is cushioned. Conversely, when V is a cushioned neighbornet, it follows by using the same lemma that the symmetric relation $V \cap V^{-1}$ is a neighbornet. Thus we have the following result.

LEMMA 3.3. *A neighbornet is cushioned iff it contains a symmetric neighbornet.*

As the name suggests, there is a close connection between cushioned relations and cushioned refinements (see [17] or [18] for the definition of the latter term). This connection is made clear by the following result.

LEMMA 3.4. *A cover $\mathscr{K}$ of X has a cushioned refinement iff there exists a cushioned relation R on X such that $\mathscr{S}R \subset \mathscr{K}$.*

Proof. Sufficiency. Assume that there is a cushioned relation R on X such that we have $\mathscr{S}R \subset \mathscr{K}$. For every $K \in \mathscr{K}$, let $C_K = \{x \in X \mid R\{x\} = K\}$. Then the family $\{C_K \mid K \in K\}$ covers X and we have $RC_K \subset K$ for each $K \in \mathscr{K}$. By using the cushionedness of R, it follows that we have

$$\begin{aligned}\overline{\bigcup \{C_K \mid K \in \mathscr{K}'\}} &\subset R(\bigcup \{C_K \mid K \in \mathscr{K}'\}) \\ &= \bigcup \{RC_K \mid K \in \mathscr{K}'\} \subset \bigcup \{K \mid K \in \mathscr{K}'\}\end{aligned}$$

for each $\mathscr{K}' \subset \mathscr{K}$. The family $\{C_K \mid K \in \mathscr{K}\}$ is thus a cushioned refinement of $\mathscr{K}$.

Necessity. Let $\{F_K \mid K \in \mathscr{K}\}$ be a cushioned refinement of $\mathscr{K}$. For each $x \in X$, let $K(x) \in \mathscr{K}$ be such that $x \in F_{K(x)}$. When we denote by R the relation associated with the indexed family $\{K(x) \mid x \in X\}$, we have $\mathscr{S}R \subset \mathscr{K}$. The relation R is cushioned, because for each $B \subset X$ we have $B \subset \bigcup \{F_{K(b)} | b \in B\}$ and thus further, $\bar{B} \subset \bigcup \{K(b) | b \in B\} = RB$.

If the cover $\mathscr{K}$ in the preceding lemma is open, the relation R appearing in the lemma is a neighbornet. Thus by using Lemmas 3.3 and 3.2, we get the following result:

COROLLARY 3.5. *An open cover of X has a cushioned refinement iff the cover has a semi-open point-star refinement.*

The result of Lemma 3.4 is essentially due to E. Michael (Proposition 2.1 of [17]; note that cushioned neighbornets are the same as the relations that are called semi-neighborhoods of the diagonal in [17]).

(b) Unsymmetric and antisymmetric neighbornets.

DEFINITION 3.6. A relation R on a set A is *unsymmetric* provided that R is reflexive and for all $a \in A$ and $b \in A$, if $a \in R\{b\}$ and $b \in R\{a\}$, then $R\{a\} = R\{b\}$.

We note that a relation R is unsymmetric iff we have $R \cap R^{-1}\{a\} = \{b \in A | R\{b\} = R\{a\}\}$ for each $a \in A$, whereas R is antisymmetric iff we have $R \cap R^{-1}\{a\} = \{a\}$ for each $a \in A$. Thus every antisymmetric relation is unsymmetric.

LEMMA 3.7. *Let $\mathscr{C}$ be a cover of A. Then there is an unsymmetric relation R on A such that $\mathscr{A}R \subset \mathscr{C}$.*

Proof. Well-order $\mathscr{C}$ and for each $a \in A$, let C_a be the least member of $(\mathscr{C})_a$. Then it is easily seen that the relation associated with the indexed family $\{C_a | a \in A\}$ is unsymmetric.

COROLLARY 3.8. (i) *When $\mathscr{U}$ is an open cover of X, there is an unsymmetric neighbornet V of X such that $\mathscr{A}V \subset \mathscr{U}$.*

(ii) *When $\mathscr{K}$ is a closure-preserving and closed cover of X, there is an unsymmetric cushioned relation S on X such that $\mathscr{A}S \subset \mathscr{K}$.*

The first part of the corollary shows that, in a sense, unsymmetric neighbornets are no more special objects than open covers. In contrast to this, we will see in §4 that the existence of an antisymmetric neighbornet is a rather stringent condition for a topological space. Let us note, however, that many nontrivial spaces (as for example all spaces with a half-interval topology) do have antisymmetric neighbornets.

Let R be an unsymmetric relation. Then it follows by the remark made after Definition 3.6 that the relation $R \cap R^{-1}$ is an equivalence relation. For neighbornets with this last-mentioned property, we have the following result.

LEMMA 3.9. *Let U be a neighbornet of X such that $U \cap U^{-1}$ is an equivalence relation and let V be an open neighbornet contained in U. Denote by H the set $\{x \in X \mid V^{-1}\{x\} \subset U\{x\}\}$ and denote by R the relation $U \cap U^{-1}$. Then the family $\{H \cap R\{x\} \mid x \in X\}$ is closed and discrete.*

Proof. We will show first that if x and y are points of X such that $V\{y\} \cap H \cap R\{x\} \neq \varnothing$, then $R\{y\} = R\{x\}$. Let $x \in X$ and $y \in X$ be such that set $V\{y\} \cap H \cap R\{x\}$ is nonempty and let z be a point of this set. Then $y \in V^{-1}\{z\}$ and since $z \in H$, $y \in U\{x\}$. On the other hand, we have that $z \in V\{y\} \subset U\{y\}$. It follows that $z \in R\{y\}$; thus we have $R\{y\} \cap R\{x\} \neq \varnothing$. As R is an equivalence relation, it follows that $R\{y\} = R\{x\}$.

From the foregoing it follows that for each $y \in X$, the neighborhood $V\{y\}$ of y can intersect at most one set of the family $\{H \cap R\{x\} \mid x \in X\}$; this family is thus discrete. It remains to show that the sets $H \cap R\{x\}$, $x \in X$, are closed. Let x be a point of X and let u be a point of the set $\overline{H \cap R\{x\}}$. Then $V\{u\} \cap H \cap R\{x\} \neq \varnothing$ so that $R\{u\} = R\{x\}$; thus we have $u \in R\{x\}$. To show that $u \in H$, let v be a point of the set $V^{-1}\{u\}$. Then the set $V\{v\}$ is a neighborhood of u. By the foregoing we have that $u \in \overline{H \cap R\{u\}}$ and it follows that the neighborhood $V\{v\}$ of u intersects the set $H \cap R\{u\}$; from this it follows by the first part of the proof that we have $R\{v\} = R\{u\}$. As the set $R\{u\}$ is contained in the set $U\{u\}$, we see that $v \in U\{u\}$. We have shown that $V^{-1}\{u\} \subset U\{u\}$, in other words, that $u \in H$. Thus $u \in H \cap R\{x\}$ and we have shown that the set $H \cap R\{x\}$ is closed.

The foregoing lemma can be used with neighbornets U such that either U or U^{-1} is unsymmetric. When applied to antisymmetric neighbornets, it yields the following result: when U is an antisymmetric neighbornet and $V \subset U$ an open neighbornet, then the family $\{\{x\} \mid x \in X$ and $V^{-1}\{x\} \subset U\{x\}\}$ is closed and discrete.

(c) Transitive neighbornets.

DEFINITION 3.10. A sequence $\langle U_n \rangle$ of neighbornets of X is a *normal sequence* if $U_{n+1}^2 \subset U_n$ for every $n \in N$. A neighbornet U of X is *normal* if U is a member of some normal sequence of neighbornets of X.

By a result in [21], all normal sequences of neighbornets have quasi-metrics "associated" with them.

DEFINITION 3.11. A filter (-base) $\mathscr{U}$ of reflexive relations on X is said to be a *(base for a) quasi-uniformity* on X provided that for each $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V^2 \subset U$. A quasi-uniformity U on X is said to be *compatible* with X provided that for each $x \in X$, $\{U\{x\} | U \in \mathscr{U}\}$ is a base for η_x.

It is easily seen that the collection of all normal neighbornets of a topological space X forms a quasi-uniformity that contains every other quasi-uniformity of X; this quasi-uniformity is called the *fine* quasi-uniformity of X in [8].

We proceed to study transitive neighbornets. The following result is a direct consequence of the fact that for a transitive relation R we have $R^2 \subset R$.

LEMMA 3.12. *A transitive neighbornet is normal, unsymmetric and open.*

DEFINITION 3.13. Let $\mathscr{N}$ be a family of subsets of X. The family $\mathscr{N}$ is *interior-preserving* if we have $\operatorname{Int}(\bigcap \{N | N \in \mathscr{N}'\}) = \bigcap \{\operatorname{Int} N | N \in \mathscr{N}'\}$ for every subfamily $\mathscr{N}'$ of $\mathscr{N}$.

The terminology "interior-preserving" is justified by the observation that a family $\mathscr{N}$ is interior-preserving iff the family $^{-}\mathscr{N}$ is closure-preserving. Interior-preserving open families are called Q-collections in [22] and fundamental open families in [6].

THEOREM 3.14. *The following conditions are mutually equivalent for a relation U on X:*

(i) *U is a transitive neighbornet.*

(ii) *We have $U = D\mathscr{O}$ for some interior-preserving and open family $\mathscr{O}$.*

(iii) *We have $U^{-1} = D\mathscr{F}$ for some closure-preserving and closed family $\mathscr{F}$.*

Proof. It is easily seen that a family $\mathscr{O}$ of subsets of X is interior-preserving and open iff the relation $D\mathscr{O}$ is a neighbornet (i.e., iff $x \in \operatorname{Int} \bigcap (\mathscr{O})_x$ for each $x \in X$); the equivalence of (i) and (ii) follows directly from this fact by using (ii) of Lemma 1.1.

As we have $(D\mathscr{L})^{-1} = D^{-}\mathscr{L}$ for every family $\mathscr{L}$ of subsets of X and as a family $\mathscr{O}$ is interior-preserving iff the family $^{-}\mathscr{O}$ is closure-preserving, we see that conditions (ii) and (iii) are mutually equivalent.

Theorem 3.14 has a number of interesting corollaries:

COROLLARY 3.15. *If U is a transitive neighbornet of X, then the family $\{UA \mid A \subset X\}$ is interior-preserving and open and the family $\{U^{-1}A \mid A \subset X\}$ is closure-preserving and closed.*

Proof. Let $\mathscr{O}$ be an interior-preserving family of open subsets of X. Then we see, using that $x \in \operatorname{Int} \bigcap (\mathscr{O})_x$ for each $x \in X$, that both of the families $\{\bigcup \mathscr{O}' \mid \mathscr{O}' \subset \mathscr{O}\}$ and $\{\bigcap \mathscr{O}' \mid \mathscr{O}' \subset \mathscr{O}\}$ are interior-preserving and open. By using the relationship existing between interior-preserving families and closure-preserving families, we see that the analogous result holds for closure-preserving and closed families. Corollary 3.15 now follows from Theorem 3.14 and the preceding observations, since for any relation R on X, we have that $RA = \bigcup \{R\{a\} \mid a \in A\}$ for each $A \subset X$.

It follows from Corollary 3.15 that for every neighbornet V of X, the family $\{V^{-\infty}A \mid A \subset X\}$ is closure-preserving and closed.

COROLLARY 3.16. *When $\mathscr{O}$ is an interior-preserving and open cover of X, the family $\{St(x, \mathscr{O}) \mid x \in X\}$ has a closure-preserving and closed refinement.*

Proof. As $\mathscr{O}$ is a cover, we have $D\mathscr{O} \subset S\mathscr{O}$ and it follows that $D^{-}\mathscr{O} = (D\mathscr{O})^{-1} \subset (S\mathscr{O})^{-1} = S\mathscr{O}$; thus we have $\bigcap (^{-}\mathscr{O})_x \subset St(x, \mathscr{O})$ for each $x \in X$. As we have $x \in \bigcap (^{-}\mathscr{O})_x$ for each $x \in X$, it follows that the family $\mathscr{K} = \{\bigcap (^{-}\mathscr{O})_x \mid x \in X\}$ is a refinement of the family $\{St(x, \mathscr{O}) \mid x \in X\}$. As the family $^{-}\mathscr{O}$ is closure-preserving and closed, the family $\mathscr{K}$ also has these properties.

When $\mathscr{L}$ is a family of sets, we denote by $\mathscr{L}^F$ the family that consists of all finite unions of sets from $\mathscr{L}$. When $\mathscr{N}$ is another family of sets, we say that $\mathscr{N}$ is an *F-refinement* of $\mathscr{L}$ if $\mathscr{N}$ is a refinement of the family $\mathscr{L}^F$. Since every point-finite family of sets is interior-preserving, the following result is an immediate consequence of the preceding corollary.

COROLLARY 3.17. *Every point-finite open cover of a topological space has a closure-preserving closed F-refinement.*

This result is of great importance in the theory of metacompact spaces. However, as we do not intend to study covering axioms in this paper, we mention just one result that follows from Corollary 3.17.

COROLLARY 3.18. *Every locally compact metacompact space has a closure-preserving cover by compact closed subsets.*

Proof. If X is locally compact and metacompact, then X has a point-finite open cover such that every set of the cover is contained in some compact subset of X; any closed F-refinement of the cover consists of compact subsets of X.

We close this section with some remarks on the role of transitive neighbornets in the theory of quasi-uniformities. It is easy to show that the collection of all transitive neighbornets of a topological space is a base for a compatible quasi-uniformity for the space; this quasi-uniformity is called the *fine transitive quasi-uniformity* of the space ([8]). If the fine transitive quasi-uniformity of a space coincides with the fine quasi-uniformity of the space, then the space is said to be transitive ([8]). It follows directly from the definitions that a topological space is transitive iff each normal neighbornet of the space contains a transitive neighbornet. The equivalence of (i) and (ii) of Theorem 3.14 is used as a tool for studying quasi-uniformities with a transitive base in [7] and [8]. In [14] there is an example of a space that has a compatible quasi-uniformity with a countable base but that does not have a compatible quasi-uniformity with a countable transitive base; this space is nontransitive. A space that has a compatible quasi-uniformity with a countable transitive base is nonarchimedeanly quasi-metrizable ([7]).

4. Some applications. In this section we apply the concepts defined in the previous sections to the theory of semi-stratifiable spaces. The reason for choosing this theory as an example is that in a semi-stratifiable space, many covering properties can be translated into properties of neighbornets.

Before studying neighbornets of semi-stratifiable spaces, we observe that several generalizations of metric spaces have been given characterizations that can be expressed in a unified way by using sequences of neighbornets.

(a) Generalizations of metric spaces.

Let us recall that a *network at a point* $x \in X$ is a family (or a sequence) $\mathscr{L}$ of subsets of X such that for each $O \in \eta_x$, we have $x \in L \subset O$ for some $L \in \mathscr{L}$. A family of subsets of X is a *network* for X if the family is a network at each point of X.

DEFINITION 4.1. A sequence $\langle R_n \rangle$ of relations on X is *basic* (*strongly basic*) if for each $x \in X$, the sequence $\langle R_n\{x\} \rangle$ (the family $\{R_n O \mid n \in N$ and $O \in \eta_x\}$) is a network at x. The sequence $\langle R_n \rangle$ is *doubly* (*infinitely*) *basic* if the sequence $\langle R_n^2 \rangle$ (the sequence $\langle R_n^\infty \rangle$) is basic. If the sequence $\langle R_n^{-1} \rangle$ is (strongly, doubly, infinitely) basic, we say that the sequence $\langle R_n \rangle$ is (*strongly, doubly, infinitely*) *co-basic*.

LEMMA 4.2. *A sequence $\langle R_n \rangle$ of reflexive relations on X is basic* (*strongly basic*) *iff we have $F = \bigcap_{n \in N} R_n^{-1} F$ ($F = \bigcap_{n \in N} \overline{R_n^{-1} F}$) for each closed set $F \subset X$.*

We will now give translations of various definitions and characterizations of generalized metric spaces into the terminology defined above; Lemma 4.2 is helpful in showing the validity of some of these translations.

4.3. Denote by $\mathscr{N}$ the collection of all sequences of neighbornets of X. Then X is

(i) stratifiable iff some $\mathscr{S} \in \mathscr{N}$ is strongly co-basic [10].

(ii) a σ-space iff some $\mathscr{S} \in \mathscr{N}$ is doubly co-basic (infinitely co-basic) [11].

(iii) semi-stratifiable iff some $\mathscr{S} \in \mathscr{N}$ is co-basic [5, 13]

(iv) semi-developable iff some $\mathscr{S} \in \mathscr{N}$ is both basic and co-basic [1].

(v) first countable iff some $\mathscr{S} \in \mathscr{N}$ is basic.

(vi) a γ-space iff some $\mathscr{S} \in \mathscr{N}$ is doubly basic (strongly basic) [12, 15].

(vii) nonarchimedeanly quasi-metrizable iff some $\mathscr{S} \in \mathscr{N}$ is infinitely basic [7].

REMARKS. 1° In (i) through (vii) above, we can choose the sequences in question to be decreasing and to consist of open neighbornets.

2° For all the generalized metric spaces mentioned above, we have adopted the "separation-axiom free" versions of the definitions.

3° If a σ-space is defined without assuming the sets of the (σ-locally finite) network to be closed, we need some extra assumption (such as regularity) for (ii) to hold.

4° To prove (iv), use Lemma 3.2 together with the fact that for any relation R, the relation $R \cup R^{-1}$ is symmetric.

5° The term "stratifiable space" is due to C. Borges ([2]); in [4], these spaces are called M_3-spaces. Semi-stratifiable spaces are called pseudostratifiable in [13].

(b) Neighbornets and covers of semi-stratifiable spaces.

We start this section by indicating how neighbornets of semi-stratifiable spaces can be "approximated" by unsymmetric neighbornets. Then we study certain partitions connected with neighbornets and we use the properties of these partitions to obtain characterizations for some important subclasses of the class of semi-stratifiable spaces. In the end of the section we study the properties of unsymmetric neighbornets of semi-stratifiable spaces satisfying some covering axioms.

Our first result deals with developable spaces (X is *developable* if X has open covers $\mathcal{O}_n$, $n \in N$, such that $\langle S\mathcal{O}_n \rangle$ is a basic sequence for X; the sequence $\langle \mathcal{O}_n \rangle$ is then called a development of X).

THEOREM 4.4. *Let U be a neighbornet of a developable space X. Then there exists an unsymmetric neighbornet V of X such that $V \subset U^2$.*

Proof. Let $\langle \mathcal{O}_n \rangle$ be a development of X such that for each $n \in N$, the cover $\mathcal{O}_{n+1}$ is a refinement of the cover $\mathcal{O}_n$. Let $H_0 = \varnothing$ and for each $n \in N$, let $H_n = \{x \in X \mid St(x, \mathcal{O}_n) \subset U\{x\}\}$; then we have $H_n \uparrow X$ (that is, $H_n \subset H_{n+1}$ for each n and $X = \bigcup_{n \in N} H_n$). For each $x \in X$, let $k(x)$ be the least natural number k such that $x \in \bar{H}_k$. Well-order the family $\bigcup_{n \in N} \mathcal{O}_n$ and for each $x \in X$, let O_x be the least member of the subfamily $(\mathcal{O}_{k(x)})_x$. We define the neighbornet V of X by setting $V\{x\} = O_x - \bar{H}_{k(x)-1}$ for each $x \in X$.

For all $x \in X$ and $y \in X$, if $x \in V\{y\}$ and $y \in V\{x\}$, then $k(x) = k(y)$; thus we see easily that the neighbornet V is unsymmetric. To complete the proof, let $x \in X$ and let $m = k(x)$. Since $x \in \bar{H}_m$, there is a point $y \in H_m \cap O_x \cap U\{x\}$. We have $U\{y\} \subset U^2\{x\}$ and $V\{x\} \subset O_x \subset St(y, \mathcal{O}_m)$; since $y \in H_m$, it follows that we have $V\{x\} \subset U^2\{x\}$. We have shown that $V\{x\} \subset U^2\{x\}$ for each $x \in X$; thus we have $V \subset U^2$.

We now give an example to show that the result of Theorem 4.4 does not remain valid if we replace "developable" by "semi-stratifiable" in the theorem; the example depends upon the following simple result.

LEMMA 4.5. *Let S be a dense subset of a semi-stratifiable space*

X and let V be a neighbornet of X. Then the set $\{x \in X | x \notin \overline{S \cap V^{-1}\{x\}}\}$ *is of the first category in X.*

Proof. Denote the set in question by L. Let $\langle U_n \rangle$ be a cobasic sequence of open neighbornets of X. We may assume that for each $n \in N$, $U_n \subset V$ (if this were not the case, we could use the neighbornets $U_n \cap \dot{V}$ in room of U_n). For each $n \in N$, set $K_n = \{x \in X | U_n^{-1}\{x\} \subset X - S\}$. It follows from Lemma 2.4 that these sets K_n are closed and as they are disjoint from the dense set S, they are nowhere dense. It is easily seen that $L \subset \bigcup_{n \in N} K_n$; hence L is of the first category in X.

EXAMPLE 4.6. Let X denote the set of all real numbers, Q the set of all rational numbers and D the set of all irrational numbers. For each $x \in D$, let $\langle g_n(x) \rangle$ be a sequence of rational numbers converging to x (in the Euclidean topology), and let $A(x) = \{g_n(x) | n \in N\}$. For all $x \in D$ and $n \in N$, let $B_n(x) = \{y \in X | |y - x| < 1/n\}$ and let $W_n(x) = B_n(x) - A(x)$. We equip X with the topology in which the points of Q are isolated and for each $x \in D$, $\langle W_n(x) \rangle$ is a base for η_x.

To be able to prove that X provides the desired counterexample, we have to choose the sequences $\langle g_n(x) \rangle$, $x \in D$, in a more specific way but before doing so we show that, even in the general case, X is a semi-stratifiable space. We do this by showing that X has a σ-closure-preserving base. For all rational numbers s and t such that $s < t$, let $(s, t) = \{x \in X | s < x < t\}$ and let $\mathscr{B}_{s,t} = \{(s, t) - A(x) | x \in D \cap (s, t)\}$. Evidently these families $\mathscr{B}_{s,t}$ are closure-preserving. For every $x \in D$, the sets $(s, t) - A(x)$, where $s \in Q$, $t \in Q$ and $x \in (s, t)$, form a base for η_x. Thus if we let $\mathscr{B}_{s,s} = \{\{s\}\}$ for each $s \in Q$, then the family $\mathscr{B} = \bigcup \{\mathscr{B}_{s,t} | s \in Q, t \in Q \text{ and } s \leqq t\}$ is a σ-closure-preserving base for X. By observing that the sets of the family $\mathscr{B}$ are closed, as well as open, we see that X is a regular space. It follows by results in [4 and 5] that X is semi-stratifiable.

Let $S = \{d_n | n \in N\}$ be a countable and dense subset of D. To choose the sequences $\langle g_n(x) \rangle$, $x \in D$, we start by choosing for each $n \in N$ an element r_n from the set $Q \cap B_n(d_n)$. Then we choose the element $g_n(x)$ of Q, for $n \in N$ and $x \in D$, in the following way: if $x \in B_n(d_n)$, then we take $g_n(x)$ to be r_n; otherwise we take $g_n(x)$ to be some element of the set $Q \cap B_n(x)$. Then, for each $x \in D$, the sequence $\langle g_n(x) \rangle$ has the required property, since $|x - g_n(x)| < 2/n$ for every $n \in N$. From now on, we assume that the topology of X has been defined by using these special sequences $\langle g_n(x) \rangle$, $x \in D$.

We define a neighbornet U of X by setting $U\{g\} = \{g\}$ for each $g \in Q$, $U\{x\} = X - A(x)$ for each $x \in D - S$ and $U\{d_n\} = W_n(d_n)$ for each $n \in N$. It is easily seen that we have $r_n \notin U^2\{d_n\}$ for each $n \in N$.

To show that the conclusion of Theorem 4.4 is not valid for X and U, assume on the contrary that there exists an unsymmetric neighbornet V of X such that we have $V \subset U^2$. Denote by H the set $\{x \in D \mid x \in \overline{S \cap V^{-1}\{x\}}\}$. Note that the topology which D inherits from X is the same as the Euclidean topology on D; hence the subspace D of X is of the second category in itself. By using the result of Lemma 4.5 on the subspace D and on its dense subset S, we see that the set H is uncountable. As the neighbornet V is unsymmetric, we have $x \in \overline{\{y \in S \mid V\{y\} = V\{x\}\}}$ for each $x \in H$; as the set S is countable, it follows that there exists $z \in S$ such that the set $H' = \{x \in H \mid V\{x\} = V\{z\}\}$ is uncountable. By using the second countability of the subspace D, we see that there exist $x,\ x' \in H' - S$ and $n \in N$ such that $x \neq x'$ and $x' \in W_n(x) \subset V\{z\}$. As we have $x \neq x'$, we see that x' does not belong to the Euclidean closure of the set $A(x)$; it follows that there exists $k \in N$ such that $B_k(x') \subset W_n(x)$. Since $x' \in H$, we have $x' \in \overline{\{y \in S \mid V\{y\} = V\{x'\}\}}$ and since $x' \notin S$, we see that there exists $i \geqq 2k$ such that $d_i \in B_{2k}(x')$ and $V\{d_i\} = V\{x'\}$. Then we have that $B_i(d_i) \subset B_k(x')$ and hence that $r_i \in B_k(x')$. It follows that we have $r_i \in W_n(x) \subset V\{z\}$. On the other hand, we have $V\{d_i\} = V\{x'\} = V\{z\}$ and it follows that $V\{z\} \subset U^2\{d_i\} \subset X - \{r_i\}$. We have reached a contradiction and thus we have shown that there is no unsymmetric neighbornet contained in the neighbornet U^2.

After this negative result we show that by making the conclusion of Theorem 4.4 slightly weaker, we can generalize the result from developable spaces to semi-stratifiable spaces.

THEOREM 4.7. *Let U be a neighbornet of a semi-stratifiable space X. Then there exists an unsymmetric neighbornet V of X such that $V \subset U^3$.*

Proof. Let $\langle V_n \rangle$ be a decreasing co-basic sequence for X such that for each $n \in N$, V_n is an open neighbornet contained in U. Let $H_0 = \varnothing$ and for each $n \in N$, let $H_n = \{x \in X \mid V_n^{-1}\{x\} \subset U\{x\}\}$; then we have $H_n \uparrow X$. For each $x \in X$, let $k(x)$ be the least natural number k such that $x \in \bar{H}_k$. Well-order X and define a mapping $\varphi: X \to X$ by letting each φx be the least element of the subset $V_{k(x)}^{-1}\{x\}$ of X. We define the neighbornet V of X by setting $V\{x\} = V_{k(x)}\{\varphi x\} - \bar{H}_{k(x)-1}$ for each $x \in X$.

The neighbornet V is unsymmetric (compare with the proof of Theorem 4.4). To complete the proof, let $x \in X$ and let $m = k(x)$. Since $x \in \bar{H}_m$, there is a point $y \in H_m \cap V_m\{\varphi x\} \cap U\{x\}$. We have $\varphi x \in V_m^{-1}\{y\}$ and $y \in H_m$ and it follows that $\varphi x \in U\{y\}$; we have thus

$\varphi x \in U^2\{x\}$. It follows that we have $V\{x\} \subset V_m\{\varphi x\} \subset U^3\{x\}$. We have shown that $V\{x\} \subset U^3\{x\}$ for each $x \in X$; thus we have $V \subset U^3$.

We now turn to consider the relationships existing between neighbornets and families of subsets of semi-stratifiable spaces. Our first result deals with partitions (i.e., disjoint covers).

THEOREM 4.8. *Let $\mathscr{P}$ be a partition of a semi-stratifiable space X and let P be the equivalence relation determined by $\mathscr{P}$. Then the following conditions are mutually equivalent:*

(i) *$\mathscr{P}$ has a σ-discrete and closed refinement.*

(ii) *$\mathscr{P}$ has a σ-closure-preserving and closed refinement.*

(iii) *There exists a transitive neighbornet V of X such that $V \cap V^{-1} \subset P$.*

(iv) *There exists a neighbornet U of X such that $U \cap U^{-1}$ is an equivalence relation and $U \cap U^{-1} \subset P$.*

Proof. (i) $\Rightarrow$ (ii) is trivially true.

(ii) $\Rightarrow$ (iii): Let $\mathscr{F} = \bigcup_{n\in N} \mathscr{F}_n$ be a closed refinement of $\mathscr{P}$ such that for each $n \in N$, the family $\mathscr{F}_n$ is closure-preserving. For each $x \in X$, denote by $k(x)$ the least number k such that $x \in \bigcup \mathscr{F}_k$. Define a relation V on X by setting $V\{x\} = X - \bigcup\{F \in \bigcup_{n \leqq k(x)} \mathscr{F}_n \mid x \notin F\}$ for each $x \in X$. Since each of the families $\mathscr{F}_n$, $n \in N$, is closure-preserving and closed, we see that V is a neighbornet. To show that V is transitive, let x and y be points of X such that $y \in V\{x\}$. Then it follows from the definition of the set $V\{x\}$ that we have $(\mathscr{F}_n)_y \subset (\mathscr{F}_n)_x$ for each $n \leqq k(x)$; in particular, we have $(\mathscr{F}_n)_y = \varnothing$ for each $n < k(x)$ and thus we have $k(y) \geqq k(x)$. From the foregoing it follows that $V\{y\} \subset V\{x\}$. We have shown that the neighbornet V is transitive. To show that $V \cap V^{-1} \subset P$, let x be a point of X. Let F be a set of the nonempty family $(\mathscr{F}_{k(x)})_x$. From the foregoing it follows that we have $(\mathscr{F}_{k(x)})_x = (\mathscr{F}_{k(x)})_y$ for each $y \in V \cap V^{-1}\{x\}$; we have thus $V \cap V^{-1}\{x\} \subset F$. The set F is contained in some set of the family $\mathscr{P}$ and, since $x \in F$ and $(\mathscr{P})_x = \{P\{x\}\}$, it follows that $F \subset P\{x\}$. By the foregoing, we have that $V \cap V^{-1}\{x\} \subset P\{x\}$. We have shown that $V \cap V^{-1} \subset P$.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): Assume that X has a neighbornet U with the properties mentioned in (iv). Denote by R the equivalence relation $U \cap U^{-1}$. Let $\langle V_n \rangle$ be a co-basic sequence for X such that for each $n \in N$, V_n is an open neighbornet contained in U. For every $n \in N$, denote by H_n the set $\{x \in X \mid V_n^{-1}\{x\} \subset U\{x\}\}$ and denote by $\mathscr{F}_n$ the family $\{H_n \cap R\{x\} \mid x \in X\}$. By Lemma 3.9, each of the families $\mathscr{F}_n$ is closed and discrete. As we have $R\{x\} \subset P\{x\}$ and $P\{x\} \in \mathscr{P}$ for each $x \in X$, we see that every set of the family $\mathscr{F} = \bigcup_{n \in N} \mathscr{F}_n$ is contained in some set belonging to the partition $\mathscr{P}$. Since the sequence $\langle V_n \rangle$ is co-basic, we have that $\bigcup_{n \in N} H_n = X$ and it follows that the family $\mathscr{F}$ covers X. We have shown that the σ-discrete and closed family $\mathscr{F}$ is a refinement of the partition $\mathscr{P}$.

REMARKS. 1° From Corollary 3.8 it follows that every open cover of a topological space is refined by a partition $\mathscr{P}$ satisfying condition (iv) of the theorem above; hence it follows from the theorem that every semi-stratifiable space is subparacompact (this result is due to G. Creede [5] and to Ya. Kofner [15]).

2° By using Theorem 3.14 we can express condition (iii) above in terms of interior-preserving and open (or closure-preserving and closed) families. (Note that for any family $\mathscr{L}$ of subsets of X, we have $D\mathscr{L} \cap D^{-}\mathscr{L}\{x\} = \{y \in X \mid (\mathscr{L})_y = (\mathscr{L})_x\}$ for each $x \in X$.)

3° It is easily seen that if a partition $\mathscr{P}$ satisfies condition (ii) of Theorem 4.8, then $\bigcup \mathscr{P}'$ is an F_σ-set for each $\mathscr{P}' \subset \mathscr{P}$.

We now use Theorem 4.8 to obtain characterizations for some spaces contained in the class of semi-stratifiable spaces.

COROLLARY 4.9. *The following conditions are mutually equivalent for a space X:*

(i) *X is semi-stratifiable and X has an antisymmetric neighbornet.*

(ii) *X is semi-stratifiable and X has a neighbornet that is both antisymmetric and transitive.*

(iii) *The family consisting of all singleton subsets of X is σ-discrete and closed.*

Proof. If X satisfies condition (iii), then X is a σ-space and thus semi-stratifiable. The rest of the proof follows directly from Theorem 4.8 since a relation T on X is antisymmetric iff $T \cap T^{-1}$ is the identity relation on X.

If X has an antisymmetric neighbornet V, then for any neighbornet U of X, $U \cap V$ is an antisymmetric, and thus unsymmetric, neighbornet contained in U. It is well known that there exist non-developable spaces that satisfy condition (iii) of the above corollary; thus we see that the property of developable spaces mentioned in Theorem 4.4 does not characterize developability in the class of semi-stratifiable spaces.

Next we use sequences of unsymmetric neighbornets to characterize developable spaces and σ-spaces; the following result is useful in proving these characterizations.

COROLLARY 4.10. *Let V be an unsymmetric neighbornet of a semi-stratifiable space X. Then there exists a σ-discrete and closed cover $\mathscr{F}$ of X such that for each $F \in \mathscr{F}$, we have $V\{x\} = VF$ for every $x \in F$.*

Proof. By Theorem 4.8, the partition determined by the equivalence relation $V \cap V^{-1} = \{(x, y) \in X \times X \mid V\{x\} = V\{y\}\}$ has a σ-discrete and closed refinement.

THEOREM 4.11. *Let X be a semi-stratifiable space. Then*

(i) *X is developable iff X has a basic sequence by unsymmetric neighbornets.*

(ii) *X is a σ-space iff X has unsymmetric neighbornets V_n, $n \in N$, such that the sequence $\langle V_n \cap V_n^{-1} \rangle$ is basic.*

Proof. (i): The necessity of the condition in (i) follows directly from Corollary 3.8. To prove the sufficiency, assume that X has a basic sequence $\langle V_n \rangle$ such that V_n is an unsymmetric neighbornet for each $n \in N$. By using Corollary 4.10, we see that for every $n \in N$ there exists a closed cover $\mathscr{F}_n = \bigcup_{k \in N} \mathscr{F}_{n,k}$ of X such that each of the families $\mathscr{F}_{n,k}$ is discrete and such that we have $V_n\{x\} = VF$ whenever $x \in F \in \mathscr{F}_n$. For all $n \in N$ and $k \in N$, denote by $K_{n,k}$ the set $\bigcup \mathscr{F}_{n,k}$ and by $\mathscr{O}_{n,k}$ the family $\{X - K_{n,k}\} \cup \{\dot{V}_n F - (K_{n,k} - F) \mid F \in \mathscr{F}_{n,k}\}$. The reader may verify that these families $\mathscr{O}_{n,k}$ are open covers of X and that the family $\{St(x, \mathscr{O}_{n,k}) \mid n \in N \text{ and } k \in N\}$ is a base for η_x for each $x \in X$.

(ii): Necessity. If X has a σ-locally finite and closed network, then, since every locally finite family is closure-preserving, it follows from Theorem 3.14 that X has a co-basic sequence $\langle V_n \rangle$, where each V_n is a transitive, and thus unsymmetric, neighbornet of X. As the sequence $\langle V_n \rangle$ is co-basic, the sequence $\langle V_n \cap V_n^{-1} \rangle$ is basic.

Sufficiency. Suppose that X has unsymmetric neighbornets V_n, $n \in N$, such that the sequence $\langle V_n \cap V_n^{-1} \rangle$ is basic. By Corollary 4.10, there exist σ-discrete closed covers $\mathscr{F}_n$, $n \in N$, of X such that for every $n \in N$, we have $V_n\{x\} = V_n F$ whenever $x \in F \in \mathscr{F}_n$. The family $\bigcup_{n \in N} \mathscr{F}_n$ is a σ-discrete and closed network for X.

We note that the characterizations (i) and (ii) above can also be proved by using results given in [12] and [11], respectively (the characterizations for σ-spaces given in [11] are stated for regular spaces; however, it can be shown that the assumption of regularity can be dispensed with if we define σ-spaces by using closed networks).

In the remainder of this section we study the relationships that exist between the properties of open covers and the properties of neighbornets in a semi-stratifiable space that satisfies certain covering properties. The covering properties that we are going to consider are metacompactness, orthocompactness and the Lindelöf-property (a space is *orthocompact* if every open cover of the space has an interior-preserving open refinement).

THEOREM 4.12. *Let U be an unsymmetric neighbornet of an orthocompact (metacompact) semi-stratifiable space X. Then there is an interior-preserving (point-finite) open cover $\mathscr{V}$ of X such that $D\mathscr{V} \subset U$.*

Proof. By Corollary 4.10, there is a closed cover $\mathscr{F} = \bigcup_{n \in N} \mathscr{F}_n$ of X such that the families $\mathscr{F}_n$, $n \in N$, are discrete and such that we have $U\{x\} = UF$ whenever $x \in F \in \mathscr{F}$. For every $n \in N$, let $K_n = \bigcup \mathscr{F}_n$ and let $O_n(F) = \dot{U}F - (K_n - F)$ for each $F \in \mathscr{F}_n$; then the family $\mathscr{O}_n = \{X - K_n\} \cup \{O_n(F) \mid F \in F_n\}$ is an open cover of X. Note that if we have $x \in F \in \mathscr{F}_n$, then $O_n(F)$ is the only set of the cover $\mathscr{O}_n$ that contains x; thut we have in this case that $St(x, \mathscr{O}_n) = O_n(F) \subset UF = U\{x\}$.

For every $n \in N$, let $\mathscr{V}_n$ be an interior-preserving (point-finite) open refinement of $\mathscr{O}_n$. Let $H_0 = \emptyset$ and for each $n \in N$, let $H_n = \bigcup_{k \leqq n} K_k$. For every $n \in N$, the family $\mathscr{V}'_n = \{V - H_{n-1} \mid V \in \mathscr{V}_n\}$ is interior-preserving (point-finite); as $H_n \uparrow X$, we see that the family $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}'_n$ has this same property. $\mathscr{V}$ is evidently open and as we have $\mathscr{V}_1 \subset \mathscr{V}$, $\mathscr{V}$ covers X. To show that $D\mathscr{V} \subset U$, let $x \in X$. Denote by k the least of the numbers n such that $x \in K_n$. Then it follows from $x \notin H_{k-1}$ that $\bigcap (\mathscr{V}'_k)_x \subset \bigcap (\mathscr{V}_k)_x$. As $\mathscr{V}_k$ is a refinement of $\mathscr{O}_k$, we have $\bigcap (\mathscr{V}_k)_x \subset St(x, \mathscr{O}_k)$. We have $x \in F$ for some $F \in F_k$ and it follows, as noted above, that $St(x, \mathscr{O}_k) \subset U\{x\}$. As we have $D\mathscr{V}\{x\} = \bigcap (\mathscr{V})_x$ and $\bigcap (\mathscr{V})_x \subset \bigcap (\mathscr{V}'_k)_x$, it follows from the foregoing that $D\mathscr{V}\{x\} \subset U\{x\}$. We have shown that $D\mathscr{V} \subset U$.

The part of the above theorem outside the parenthesis can also be stated in the following way: in an orthocompact semi-stratifiable space, every unsymmetric neighbornet contains a transitive neighbornet.

In this theorem, one cannot replace "interior-preserving" by "locally finite" even if X were paracompact. To see this, take X to be the closed unit interval in the Euclidean topology and let $U = D\mathscr{U}$, where $\mathscr{U}$ is the disjoint (and thus interior-preserving) family $\{(1/(n+1), 1/n) \mid n \in N\}$ of open subsets of X.

The following is an immediate consequence of Theorems 3.7 and 4.12.

COROLLARY 4.13. *Let U be a neighbornet of an orthocompact (metacompact) semi-stratifiable space X. Then there is an interior-preserving (point-finite) open cover $\mathscr{V}$ of X such that $D\mathscr{V} \subset U^3$.*

It follows from Theorem 4.4 that for a developable space X, we can replace U^3 by U^2 in the corollary.

The result of Corollary 4.13 can be used in the study of quasi-uniformities of semi-stratifiable spaces. To see this, note that a normal neighbornet O of X contains, for each $n \in N$, V^n for some neighbornet V of X; in particular, we have $U^3 \subset O$ for some neighbornet U. Thus it follows from Corollary 4.13 that every orthocompact semi-stratifiable space is a transitive space. To interpret the parenthesized part of this corollary in terms of quasi-uniformities, note that the collection $\{D\mathscr{V} \mid \mathscr{V}$ is a point-finite open cover of $X\}$ forms a base for a quasi-uniformity on X. This quasi-uniformity is called the *point-finite covering quasi-uniformity* of X ([8]), and it follows from Corollary 4.13 that for a metacompact semi-stratifiable space this quasi-uniformity coincides with the fine quasi-uniformity of the space.

Next we show that in a metacompact semi-stratifiable space every interior-preserving family of open subsets is generated by some point-finite family of open subsets of the space. We need the following notation: when $\mathscr{L}$ is a family of sets, the symbol $\mathscr{L}^S$ is used to denote the family formed by all possible unions of sets from $\mathscr{L}$.

COROLLARY 4.14. *Let $\mathscr{U}$ be an interior-preserving and open family of subsets of a metacompact semi-stratifiable space X. Then there is a point-finite and open cover $\mathscr{V}$ of X such that $\mathscr{U} \subset \mathscr{V}^S$.*

Proof. By using Theorem 4.12 (or Corollary 4.13) we see that there exists a point-finite open cover $\mathscr{O}$ of X such that $D\mathscr{O}$ is contained in the transitive neighbornet $D\mathscr{U}$. Let $\mathscr{V} = \{\bigcap(\mathscr{O})_x \mid x \in X\}$; then $\mathscr{V}$ is a point-finite open cover of X. To show that $\mathscr{U} \subset \mathscr{V}^S$, let $U \in \mathscr{U}$. For every $y \in U$ we have $\bigcap(\mathscr{O})_y = D\mathscr{O}\{y\} \subset D\mathscr{U}\{y\} = \bigcap(\mathscr{U})_y \subset U$. It follows that $U = \bigcup\{\bigcap(\mathscr{O})_y \mid y \in U\}$; thus we have $U \in \mathscr{V}^S$.

By considering the proof above, we see that the conclusion of Corollary 4.14 holds in precisely those spaces X in which the point-finite covering quasi-uniformity of X coincides with the fine transitive quasi-uniformity of X. It can be shown that all such spaces are hereditarily metacompact. The Sorgenfrey line provides us with an example of a hereditarily paracompact space for which the conclusion of Corollary 4.14 fails to hold.

In the last result of this section we give several characterizations of the Lindelöf-property for semi-stratifiable spaces in terms of neighbornets.

THEOREM 4.15. *The following conditions are mutually equivalent for a semi-stratifiable space* X:

(i) X *is a Lindelöf-space.*

(ii) *If* U *is a neighbornet of* X *such that* $U \cap U^{-1}$ *is an equivalence relation, then the family* $\mathscr{A}(U \cap U^{-1})$ *is countable.*

(iii) *If* R *is an unsymmetric relation on* X *such that either* R *or* R^{-1} *is a neighbornet of* X, *then the family* $\mathscr{A}R$ *is countable.*

(iv) *If* V *is a transitive neighbornet of* X, *then the family* $\mathscr{A}V$ *is countable.*

(v) *If* V *is a transitive neighbornet of* X, *then the family* $\mathscr{A}V^{-1}$ *is countable.*

Proof. (i) $\Rightarrow$ (ii): Assume that X is a Lindelöf-space and let U be a neighbornet of X such that $U \cap U^{-1}$ is an equivalence relation. It follows from Theorem 4.8 that the family $\mathscr{A}(U \cap U^{-1})$ has a σ-discrete and closed refinement. As X is a Lindelöf-space, every σ-discrete family of subsets of X is countable. Since the disjoint family $\mathscr{A}(U \cap U^{-1})$ has a countable refinement, the family itself is countable.

(ii) $\Rightarrow$ (iii): Assume that condition (ii) holds and let R be an unsymmetric relation on X such that either R or R^{-1} is a neighbornet. We have $R \cap R^{-1}\{x\} = \{y \in X \mid R\{y\} = R\{x\}\}$ for each $x \in X$ and it follows that the family $\mathscr{A}R$ is countable if the family $\mathscr{A}(R \cap R^{-1})$ has this property. As $R \cap R^{-1}$ is an equivalence relation and as either R or R^{-1} is a neighbornet, it follows from the assumption we have made that $\mathscr{A}(R \cap R^{-1})$ is a countably family.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (v): This follows directly by observing that for a transitive and reflexive relation T on X and for points x and y of X, we have $T\{x\} = T\{y\}$ iff $T^{-1}\{x\} = T^{-1}\{y\}$.

(v) $\Rightarrow$ (i): Assume that (v) holds. To show that X is Lindelöf, it is enough, by a result in [5], to show that every discrete family of closed subsets of X is countable. Let $\mathscr{F}$ be discrete family of closed subsets of X. Denote by V the transitive neighbornet $D^{-}\mathscr{F}$ of X. By assumption, the family $\mathscr{A}V^{-1}$ is countable. For each $x \in X$, we have $V^{-1}\{x\} = D^{-}(^{-}\mathscr{F})\{x\} = D\mathscr{F}\{x\} = \bigcap \mathscr{F}_x$; hence we have $V^{-1}\{x\} = F$ whenever $x \in F \in \mathscr{F}$. It follows that $\mathscr{F} \subset \mathscr{A} V^{-1}$; the family $\mathscr{F}$ is thus countable.

For an unsymmetric neighbornet U of a semi-stratifiable Lindelöf-space, the family $\mathscr{A} U^{-1}$ does not necessarily have to be countable. To see this, let X be the closed interval $[0, 2]$ in the Euclidean topology and let $\langle g_n \rangle$ be an enumeration of the rational numbers in the interval $(1, 2]$. We define a neighbornet U of X by setting $U\{x\} = (1, 2]$ for each $x > 1$, $U\{0\} = X$ and $U\{1/n\} = X$ for each $n \in N$, and, finally, $U\{x\} = (1/(n+1), 1/n) \cup (1, g_n)$ for each $n \in N$ and each $x \in (1/(n+1), 1/n)$. Then it is not difficult to show that U is an unsymmetric neighbornet of X such that the family $\mathscr{A} U^{-1}$ is uncountable.

(c) Stratifiable spaces.

In this last section we give some characterizations for stratifiable spaces. For the proof of these characterizations we need the following result on strongly basic sequences.

LEMMA 4.16. *Let $\langle R_n \rangle$ be a decreasing and strongly basic sequence of relations on X and let $k \in N$. Then the sequence $\langle R_n^k \rangle$ is strongly basic.*

Proof. We use induction on the number k. For $k=1$, the result is true by assumption. Suppose that it has been proved for $k = m$. To show that the result is true for $k = m + 1$, let $x \in X$ and let $O \in \eta_x$. As the sequence $\langle R_n \rangle$ is strongly basic, there exist $U \in \eta_x$ and $n \in N$ such that $R_n U \subset O$. By using the induction assumption, we can fine $V \in \eta_x$ and $l \geqq n$ such that $R_l^m V \subset U$. Then we have $R_l^{m+1} V = R_l(R_l^m V) \subset R_n U \subset O$. It follows from the foregoing that the sequence $\langle R_n^{m+1} \rangle$ is strongly basic.

As we have $(R^k)^{-1} = (R^{-1})^k$ for any relation R and for any $k \in N$, we see that the result of the lemma above remains true if we replace "basic" by "co-basic" in the lemma.

We need some terminology to state the following theorem. A family $\mathscr{N}$ of subsets of X is a *quasi-base* for X if for every $x \in X$, the family $\{N \in \mathscr{N} \mid x \in N^0\}$ is a base for η_x. A topological space is an M_2-space [4] if the space has a σ-closure-preserving quasi-base and is regular (in this paper we do not assume that a regular space is always a T_1-space).

THEOREM 4.17. *The following conditions are mutually equivalent for a space X:*

(i) *X is stratifiable.*

(ii) *X has a sequence* $\langle \mathscr{V}_n \rangle$ *of point-finite open covers such that the sequence* $\langle D\mathscr{V}_n \rangle$ *is strongly co-basic.*

(iii) *X is a* M_2*-space.*

Proof. (i) $\Rightarrow$ (ii): Suppose that X is stratifiable and let $\langle U_n \rangle$ be a decreasing and strongly co-basic sequence of neighbornets of X. Since every stratifiable space is paracompact (see [4]), Corollary 4.13 implies that there exists a sequence $\langle \mathscr{V}_n \rangle$ of point-finite open covers of X such that we have $D\mathscr{V}_n \subset U_n^3$ for each $n \in N$. By using the remark following Lemma 4.16, we see that the sequence $\langle U_n^3 \rangle$, and hence also the sequence $\langle D\mathscr{V}_n \rangle$ (note that $R \subset S$ iff $R^{-1} \subset S^{-1}$, for any relations R and S), is strongly co-basic.

(ii) $\Rightarrow$ (iii): Let $\langle \mathscr{V}_n \rangle$ be a sequence of point-finite open covers of X such that the sequence $\langle D\mathscr{V}_n \rangle$ is strongly co-basic. For each $n \in N$, let $V_n = D\mathscr{V}_n$ and let $\mathscr{N}_n = \{V_n^{-1}A \mid A \subset X\}$. By Theorem 3.14, each V_n is a transitive neighbornet; it follows from Corollary 3.15 that the families $\mathscr{N}_n$ are closure-preserving and closed. To show that the family $\mathscr{N} = \bigcup_{n \in N} \mathscr{N}_n$ is a quasi-base for X, let $x \in X$ and $O \in \eta_x$. Since the sequence $\langle V_n \rangle$ is strongly co-basic, there exists $n \in N$ and $U \in \eta_x$ such that $V_n^{-1}U \subset O$. Since $x \in \operatorname{Int}(V_n^{-1}U)$, we have shown that $\mathscr{N}$ is a quasi-base for X. As $\mathscr{N}$ is a closed quasi-base, X is regular, and thus X is an M_2-space.

(iii) $\Rightarrow$ (i): This is well known (see [4]).

In [4], J. Ceder raised the questions whether $M_3 \Rightarrow M_2$ and $M_2 \Rightarrow M_1$. Theorem 4.17 answers the first question[1] but it leaves the second one unanswered. For partial answers to the second question, see [23], [3], and [19].

Next we characterize a subclass of stratifiable spaces by imposing a restriction on the cardinalities of "generators" of a quasi-base:

THEOREM 4.18. *A topological space X is Lindelöf and stratifiable iff there exist countable and closure-preserving families* $\mathscr{F}_n$, $n \in N$, *of closed subsets of X such that the family* $\bigcup_{n \in N} (\mathscr{F}_n^S)$ *is a quasi-base for X.*

Proof. Necessity. Assume that X is Lindelöf and stratifiable. By Theorem 4.17, there is a sequence $\langle \mathscr{V}_n \rangle$ of point-finite open covers of X such that the sequence $\langle D\mathscr{V}_n \rangle$ is strongly co-basic. For each $n \in N$, let $V_n = D\mathscr{V}_n$ and $\mathscr{F}_n = \{V_n^{-1}\{x\} \mid x \in X\}$. Then we have $\mathscr{F}_n^S = \{V_n^{-1}A \mid A \subset X\}$ for each $n \in N$ and it follows that the family

[1] While preparing this paper for publication, the author learned that G. Gruenhage had also proved that M_3 implies M_2.

$\bigcup_{n\in N}(\mathscr{F}_n^S)$ is a quasi-base for X (compare with the proof of (ii) $\Rightarrow$ (iii) in Theorem 4.17). From Corollary 3.15 and Theorem 4.15, we see that the families $\mathscr{F}_n$ are closure-preserving, closed and countable.

Sufficiency. Suppose that there are families $\mathscr{F}_n$, $n\in N$, of subsets of X with the properties mentioned in the theorem. Then the countable family $\bigcup_{n\in N}\mathscr{F}_n$ is a network for X and it follows that X is a Lindelöf space. For every $n\in N$, it follows by the corresponding properties of the family $\mathscr{F}_n$ that the family $\mathscr{F}_n^S$ is closure-preserving and closed (compare with the proof of Corollary 3.15). The space X has thus a σ-closure-preserving closed quasi-base. It follows that X is an M_2-space.

In our final theorem we characterize pseudometrizable spaces in terms of strongly co-basic sequences. To prove the theorem, we need A. H. Stone's result that all pseudometrizable spaces are paracompact (the result is proved in [24] only for metrizable spaces but the same proof works for pseudometrizable spaces if we do not require paracompact spaces to be T_2). We also need the following modification of A. H. Frink's metrization theorem ([9], Theorem 3): X is pseudometrizable if X has a sequence $\langle U_n\rangle$ of neighbornets such that the sequence $\langle U_n\circ U_n^{-1}\circ U_n\rangle$ is basic.

THEOREM 4.19. *The following conditions are mutually equivalent for a space X:*

(i) *X is pseudometrizable.*

(ii) *X has a sequence $\langle \mathscr{V}_n\rangle$ of locally finite open covers such that the sequence $\langle D\mathscr{V}_n\rangle$ is strongly co-basic.*

(iii) *X has a strongly co-basic sequence formed by cushioned neighbornets of X.*

Proof. (i) $\Rightarrow$ (ii): Assume that X is pseudometrizable and let d be a pseudometric on X such that d induces the topology of X. For each $n\in N$, let $\mathscr{V}_n$ be a locally finite open refinement of the cover of X consisting of all the open d-spheres of radius $1/n$. Then it follows from the properties of d that the sequence $\langle S\mathscr{V}_n\rangle$ is doubly basic. For all $x\in X$ and $n\in N$ we have $(D\mathscr{V}_n)^{-1}(S\mathscr{V}_n\{x\})\subset S^2\mathscr{V}_n\{x\}$ and it follows that the sequence $\langle D\mathscr{V}_n\rangle$ is strongly co-basic.

(ii) $\Rightarrow$ (iii): Assume that X has locally finite open covers V_n, $n\in N$, such that the sequence $\langle D\mathscr{V}_n\rangle$ is strongly co-basic. For each $n\in N$, denote by W_n the neighbornet associated with the indexed family $\{\overline{D\mathscr{V}_n\{x\}}\mid x\in X\}$. Then we have $W_n^{-1}O=(D\mathscr{V}_n)^{-1}O$ for each $n\in N$ and for each open set $O\subset X$; it follows that the sequence $\langle W_n\rangle$ is strongly co-basic. The neighbornets W_n, $n\in N$, are cushioned since it follows from the local finiteness of the families $\mathscr{V}_n$ that the families $\{\overline{D\mathscr{V}_n\{x\}}\,|\,x\in X\}$ are locally finite and thus closure-preserving.

(iii) $\Rightarrow$ (i): Let $\langle W_n\rangle$ be a strongly co-basic sequence for X such that each W_n is a cushioned neighbornet of X. It follows from Lemma 2.3 that for each $n\in N$, the symmetric relation $U_n = \bigcap_{k\leq n}(W_k\cap W_k^{-1})$ is a neighbornet of X. The sequence $\langle U_n\rangle$ is strongly co-basic and as the relations U_n are symmetric, it follows that this sequence is strongly basic. By using Lemma 4.16, we see that the sequence $\langle U_n^3\rangle$ is strongly basic. For each $n\in N$, we have $U_n^{-1}=U_n$ and hence $U_n\circ U_n^{-1}\circ U_n = U_n^3$. It follows that the sequence $\langle U_n\circ U_n^{-1}\circ U_n\rangle$ is basic. We have shown that X satisfies the condition of Frink's theorem; X is thus a pseudometrizable space.

By the last part of the proof above, it is evident that the result of Theorem 4.19 remains true if we replace "strongly co-basic" by "strongly basic" in condition (iii) of the theorem. Note that it follows further, by using Lemmas 3.2 and 3.3, that a space X is pseudometrizable iff X has a sequence $\langle\mathscr{L}_n\rangle$ of semi-open covers such that the sequence $\langle S\mathscr{L}_n\rangle$ is strongly basic.

We close this paper with a few comments on the construction of closure-preserving families. The result of Theorem 4.17 above, as well as numerous other results (see e.g., [6] and [16]), show that closure-preserving families or their "complements", interior-preserving families, can be constructed from families without these properties. However, when these constructions are studied more closely, it becomes apparent that all of them involve families having certain finiteness or order properties (point-finite families were used in the proof of Theorem 4.17; Lemma 2.6 of [16] deals with certain generalizations of monotone families). Thus it appears that there does not as yet exist any technique (apart from the trivial one mentioned after Corollary 3.15) for constructing closure- or interior-preserving families from families (or collections of families) which do not fulfill explicit or implicit conditions assuring the existence of finite or monotone subfamilies. Some such technique would be most welcome to those who work on problems connected with covering axioms, generalized metric spaces or quasi-uniformities, because it is only for restricted classes of spaces that we can expect results analogous to Corollary 4.14 to hold. As normal sequences of open covers can be used to construct σ-discrete refinements for the members of the sequence (see [24] and [25]), it is not unfounded to conjecture that normal sequences of neighbornets could be used in a similar way to construct interior- and closure-preserving families; however, the example given in [14] shows that the analogy cannot be pushed very far. Anyway, we hope that neighbornets will aid in finding these constructions, in case they exist.

Acknowledgments. The author is indebted to Peter Fletcher and William Lindgren for the inspiration that their work on quasi-uniformities has given him, for the warm interest that they have taken in his work on semi-stratifiable spaces and for the unselfish help they have given him in the writing of this paper.

The author is grateful for having seen [16] prior to publication.

References

1. C. C. Alexander, *Semi-developable spaces and quotient images of metric spaces*, Pacific J. Math., **37** (1971), 277-293.
2. C. J. R. Borges, *Stratifiable spaces*, Pacific J. Math., **17** (1966), 1-16.
3. C. J. R. Borges and D. J. Lutzer, *Characterizations and Mappings of M_i-Spaces, R. F. Dickman and P. Fletcher, eds.*, Topology Conference, Virginia Polytechnic Institute and State University, March 22-24, 1973 (Springer-Verlag, Berlin, 1974), 34-40.
4. J. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105-125.
5. G. Creede, *Semi-Stratifiable Spaces, E. E. Grace, ed.*, Topology Conference, Arizona State University, 1967 (Temple, Arizona, 1968), 318-323.
6. W. M. Fleischman, *On Fundamental Open Coverings, D. R. Kurepa, ed.*, Proc. Int. Symp, on Topology and Its Applications, Herzeg-Novi, 25.-31.8, 1968, Yugoslavia (Savez Društava Matematičara, Fizičara I Astronoma, Belgrade, 1969), 154-155.
7. P. Fletcher and W. F. Lindgren, *Transitive quasi-uniformities*, J. Math. Anal. Appl., **39** (1972), 397-405.
8. ———, *Quasi-uniformities with a transitive base*, Pacific J. Math., **43** (1972), 619-631.
9. A. H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc., **43** (1937), 133-142.
10. R. W. Heath, *An Easier Proof that a Certain Countable Space in not Stratifiable, A. M. Carstens, ed.*, Proc. Washington State Univ. Conf. on General Topology, March 1970 (Pi Mu Epsilon Washington Alpha Center, 1970), 56-59.
11. R. W. Heath and R. E. Hodel, *Characterizations of σ-spaces*, Fund. Math., **77** (1973), 271-275.
12. R. E. Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J., **39** (1972), 253-263.
13. Ya. A. Kofner, *On a new class of spaces and some problems of symmetrizability theory*, Soviet Math. Dokl., **10** (1969), 845-848.
14. ———, *On Δ-metrizable spaces*, Math. Notes, **13** (1973), 168-174.
15. W. F. Lindgren and P. Fletcher, *Locally quasi-uniform spaces with countable bases*, Duke Math. J., **41** (1974), 231-240.
16. W. F. Lindgren and P. J. Nyikos, *Spaces with bases satisfying certain order and intersection properties*, preprint.
17. E. Michael, *Yet another note on paracompact spaces*, Proc. Amer. Math. Soc., **10** (1959), 309-314.
18. J. Nagata, *Modern General Topology*, North-Holland Publ. Co., Amsterdam, 1974.
19. ———, *On Hyman's M-Space, R. F. Dickman and P. Fletcher, eds.*, Topology Conference, Virginia Polytechnic Institute and State University, March 22-24, 1973, (Springer-Verlag, Berlin, 1974), 198-208.
20. W. V. Quine, *Set Theory and Its Logic*, The Belknap Press of Harvard University Press, Cambridge, Massachusetts, 1970.
21. H. Ribeiro, *Sur les espaces à métrique faible*, Portug. Math., **4** (1943), 21-40 and 65-68.

22. M. Sion and R. C. Willmott, *Hausdorff measures on abstract spaces*, Trans. Amer. Math. Soc., **123** (1966), 275–309.
23. F. G. Slaughter. Jr., *The closed image of a metrizable space is* M_1, Proc. Amer. Math. Soc., **37** (1973), 309–314.
24. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc., **54** (1948), 977–982.
25. J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton Univ. Press, Princeton, N.J., 1940.

Received May 16, 1977 and in revised form August 5, 1977.

VIRGINIA POLYTECHNIC INSTITUTE
BLACKSBURG, VA 24061