A PRODUCT INTEGRAL REPRESENTATION
FOR THE GENERALIZED INVERSE
OF CLOSED OPERATORS

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Suppose that z is in a complete, normed linear space E and that A is normally solvable - in the sense that A is a closed, linear operator which is densely defined and which has closed range. This paper is concerned with solving, in some sense, the operator equation $A(x) + z = 0$.

If x is a point in the domain of A and λ is a positive number such that $1/\lambda$ is not in the spectrum of A, then x satisfies $A(x) + z = 0$ only in case x solves $x = [1 - \lambda A]^{-1}(x + \lambda z)$. This suggests that there is the possibility of getting such a solution x by the following type of iteration process: W_0 is in E and, if p is positive integer, then $W_p = [1 - \lambda A]^{-1}(W_{p-1} + \lambda z)$. Another iteration which is intrinsically connected with this one is: M_0 is in E and, if p is a positive integer, then $M_p = [1 - \lambda A]^{-1} M_{p-1}$.

Studies by Martin [8], Groetsch [2], and Purdom [10] have already shown that an appropriate context for the study of iteration processes is the Stieltjes integral equation theory.

The papers of Martin and Groetsch were concerned with bounded linear operators; Purdom's hypothesis on A was that it should be linear, dissipative, and densely defined on a Banach space. Left undone in this latter paper were conditions which would imply convergence of the iteration scheme.

In what follows, and with the hypothesis that A is dissipative - so that no positive number is in the spectrum of A - conditions are given which imply that $\{M_p\}_{p=1}^{\infty}$ and $\{W_p\}_{p=1}^{\infty}$, as defined above, converge.

These results will use the following.

THEOREM [*Purdom*]. *If g is a number valued, nonincreasing function on a linearly ordered set $\{S, \geqq\}$ and $t \geqq s$ in that ordering then*

(a) $M(t, s)x = \prod_t^s [1 - dgA]^{-1}x$ *exists for each x in E,*

(b) *if x is in $D(A)$, then $M(t, s)x = x + (R)\int_t^s dgA(M(t, \cdot)x)$, and*

(c) *if x is in E, then $(L)\int_t^s dgM(\cdot, s)\,x$ is in $D(A)$ and*

$$M(t, s)x = x + A\left((L)\int_t^s dgM(\cdot, s)x\right).$$

Moreover, if z is in E then

(d) $W(t, s)x = \prod_t^s [1 - dg(A + z)]^{-1}x$ *exists for each* x *in* E, *and*

(e) $W(t, s)x = M(t, s)x + (R)\int_t^s dgM(t, \cdot)z$.

DEFINITIONS AND REMARKS. (1) The summation integrals in the above theorem are right or left Stieltjes integrals. Their definition can be found in Purdom's paper, but the central role they have held in the Stieltjes integral equation theory can be seen in an address by J. S. Mac Nerney [7].

(2) The function $(A + z)$ is an affine operator given by $(A + z)(x) = A(x) + z$ for each x in E. And, while the W generated as in (d) satisfies a nonhomogeneous integral equation, it is the variation of parameters formula in (e) that is important here.

(3) Identifying the linearly ordered set S as the nonnegative integers and g as the number sequence $g(p) = -\sum_{i=0}^p \lambda_i$, where $\{\lambda_i\}_{i=0}^\infty$ is a sequence of nonnegative numbers, then the product integral $M(n, 0)x = \prod_n^0 [1 - dgA]^{-1}(x)$ is the iteration process

$$M_0 = x \quad \text{and} \quad M_n = [1 - \lambda_n A]^{-1}M_{n-1}$$

or, what is the same,

$$M_n = \prod_{p=1}^n [1 - \lambda_p A]^{-1}(x) = [1 - \lambda_n A]^{-1} \cdots [1 - \lambda_2 A]^{-1}[1 - \lambda_1 A]^{-1}(x) .$$

In case $\lambda_p = \lambda$ for all p then $M(n, 0)$ x is $[1 - \lambda A]^{-n}(x)$ and the integral equation

$$M(n, 0)x = x + (R)\int_n^0 dgA(M(n, \cdot)x)$$

asserts that

$$[1 - \lambda A]^{-n}(x) = x + \sum_{p=1}^n \lambda A[1 - \lambda A]^{-p}(x) .$$

Also, the product integral $\prod_n^0 [1 - dg(A + z)]^{-1}(x)$ reduces to

$$[1 - \lambda(A + z)]^{-n}(x) .$$

And, the variation of parameters formula replaces the two iteration processes:

$$[1 - \lambda(A + z)]^{-n}(x) = [1 - \lambda A]^{-n}(x) + \sum_{p=1}^n \lambda[1 - \lambda A]^{-p}(z) .$$

(4) In this paper, as in [10], the statement that A is dissipative means that if $c > 0$ then the range of $1 - cA$ is all of E and, if x is in $D(A)$, then $|x| \leq |[1 - cA](x)|$. Also $N(A)$ and $R(A)$ denote the null space of A and the range of A.

II. **Results.** In what follows, we assume that A is linear, dissipative, and densely defined on the Banach space E, that S is a linearly ordered set containing an element denoted by 0, that g is nonincreasing and unbounded on S, and that M and W are defined by $M(t, s)x = \prod_t^s [1 - dgA]^{-1}x$ and, if z is in E, then $W(t, s)x = \prod_t^s [1 - dg(A + z)]^{-1}x$ for $t \geq s$ and x in E.

THEOREM 1. *Suppose that $K < 0$ and that if $c > 0$ then $[1 - cK]|x| \leq |[1 - cA]x|$ for each x in $D(A)$. It follows that $\overline{R(A)} = E$ and, if z is in $R(A)$, then $\lim_{t\to\infty} W(t, 0)x = -A^{-1}(z)$.*

Indication of proof. If $\{\alpha_p\}_{p=1}^\infty$ is a sequence of negative numbers then $\prod_{p=1}^n [1 - \alpha_p] - 1 \geq -\sum_{p=1}^n \alpha_p$. Thus, for each $t \geq 0$ and x in E,

$$|M(t, 0)x| \leq |x|/[1 + (g(t) - g(0))K]$$

and, hence, $\lim_{t\to\infty} M(t, 0)x = 0$. It follows that $\overline{R(A)} = E$ for, if x is in E, then

$$\left| x + A\left((L) \int_t^0 dgM(\cdot, 0)x\right)\right| = |M(t, 0)x| .$$

To get the representation for A^{-1}, let z be in $R(A)$ and u be the point such that $Au = z$. By the variation of parameters formula

$$W(t, 0)x = M(t, 0)x + (R) \int_t^0 dgM(t, \cdot)z$$
$$= M(t, 0)x + M(t, 0)u - u .$$

Thus $|-A^{-1}(z) - W(t, 0)x| \leq (|x| + |A^{-1}(z)|)/[1 + (g(t) - g(0))K]$. Since g is unbounded on S, $\lim_{t\to\infty} W(t, 0)x = -A^{-1}(z)$ for each x in E. Also, the convergence is uniform in the operator topology.

COROLLARY 1. *In addition to the supposition of the above theorem, suppose that $\{\lambda_p\}_{p=1}^\infty$ is a sequence of positive numbers such that $\sum_{p=1}^\infty \lambda_p = \infty$ and z is in $R(A)$. Then $A^{-1}(z) = -\sum_{p=1}^\infty \lambda_p \prod_{i=1}^p [1 - \lambda_i A]^{-1}z$.*

Indication of proof. If $A(u) = z$ then

$$\left| u + \sum_{p=1}^n \lambda_p \prod_{i=1}^p [1 - \lambda_i A]^{-1}A(u)\right| = \left|\prod_{i=1}^n [1 - \lambda_i A]^{-1}u\right|$$
$$\leq |u|/\left[1 - K\cdot\sum_{i=1}^n \lambda_i\right] .$$

LEMMA 1. *If E is a reflexive Banach space then $E = \overline{R(A)} \oplus N(A)$.*

Indication of proof. (An outline of this proof can be found in [11, pages 215-218], see also [3, Lemma 4.2].) Each of the following holds: $N(A) = \{x: \lim_{t\to\infty} [1 - tA]^{-1}x = x\}$, $\overline{R(A)} = \{x: \lim_{t\to\infty} [1 - tA]^{-1}x = 0\}$, $N(A) \cap \overline{R(A)} = 0$, and, since bounded sequences have subsequences that converge weakly in reflexive spaces, $\lim_{t\to\infty} [1 - tA]^{-1}x$ exists for each x in E and defines a norm 1 projection with $R(P) = N(A)$ and $N(P) = \overline{R(A)}$. Thus $E = \overline{R(A)} \oplus N(A)$.

REMARK. The above decomposition does not hold in a general Banach space; see the example of [4, p. 520]. Rather, $\overline{R(A)} \oplus N(A) = \{x: \lim_{t\to\infty} [1 - tA]^{-1}x \text{ exists}\}$.

LEMMA 2. *Suppose that E is a Hilbert space and that A has closed range. There is a positive number m such that if y is in $D(A) \cap R(A)$ and $c > 0$, then*

$$|[1 - cA]y| \geq \sqrt{1 + c^2 m^2}\, |y|\,.$$

Indication of proof. If y is in $D(A)$ then $|[1 - cA]y|^2 \geq |y|^2 + c^2 |A(y)|^2$. Also, A is one-to-one from $D(A) \cap R(A)$ onto $R(A)$. Since A is closed, then the restriction of A to $D(A) \cap R(A)$ is closed, one-to-one, and has range the closed set $R(A)$. Hence, it has a bounded inverse; and this provides a positive number m such that if $y \in D(A) \cap R(A)$ then $|A(y)| \geq m \cdot |y|$. This inequality, together with the inequality in the first sentence of this argument, establishes the lemma.

THEOREM 2. *Suppose that E is a Hilbert space, A has closed range, and $\int_0^\infty (dg)^2 = \infty$. It follows that $M(t, 0)$ converges uniformly in the operator topology and has limit the projection onto $N(A)$ along $R(A)$.*

Indication of proof. If $c > 0$ then $[1 - cA]^{-1}n = n$ for each n in $N(A)$. Also, since $[1 - cA]$ maps $D(A) \cap R(A)$ onto $R(A)$, then $[1 - cA]^{-1}$ is a bounded, linear function from $R(A)$ onto $D(A) \cap R(A)$. Since $R(A)$ is closed then $M(t, 0) = \prod_t^0 [1 - dgA]^{-1}$ maps $R(A)$ into $R(A)$. Furthermore, if m is as in the previous lemma, $r \in R(A)$, and $t = s_0 \geq s_1 \geq \cdots \geq s_n = 0$ then

$$\left| \prod_{p=1}^n [1 - (g(s_p) - g(s_{p-1}))A]^{-1}r \right|$$
$$\leq |r| \Big/ \sqrt{\prod_{p=1}^n [1 + m^2(g(s_p) - g(s_{p-1}))^2]}\,.$$

But,

$$\prod_{p=1}^{n} [1 + m^2 (g(s_p) - g(s_{p-1}))^2] \geqq 1 + m^2 \sum_{p=1}^{n} ((g(s_p) - g(s_{p-1})))^2$$

Thus

$$|M(t, 0)r| \leqq |r| \Big/ \sqrt{1 + m^2 \int_0^t (dg)^2} \ .$$

It follows that if z is in E, n is in $N(A)$, r is in $R(A)$, $z = n + r$, and $t > 0$ then

$$|M(t, 0)z - n| \leqq |r| \Big/ \sqrt{1 + m^2 \int_0^t (dg)^2} \ .$$

COROLLARY 2. *With the suppositions on E and A, let $\{\lambda_p\}_{p=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{p=1}^{\infty} \lambda_p^2 = \infty$. Then $\prod_{p=1}^{\infty} [1 - \lambda_p A]^{-1}$ is the projection of E onto $N(A)$ along $R(A)$.*

REMARK. Since there are dissipative operators A for which solutions of $y' = Ay$ do not have asymptotic limits, one would not expect to get the results of the previous theorem if the condition that $\int_0^{\infty} (dg)^2 = \infty$ is replaced by g is unbounded. In fact, if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}$, and $Q = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}$ then $A = iP - iQ$ and

$$[1 - \lambda A]^{-1} = (1 - \lambda i)^{-1} P + (1 + \lambda i)^{-1} Q \ .$$

Then $N(A) = \{0\}$ but, by [5, p. 27],

$$\left| \prod_{p=1}^{\infty} [1 - A/p]^{-1}(x) \right| = \pi \sqrt{(|Px|^2 + |Q(x)|^2)}/\sinh(\pi) \ .$$

REMARK. The next theorem is concerned with the condition that $\lim_{t \to \infty} W(t, 0)x$ exists for each x in E. Purdom has shown in Theorem 7 of [10] that statement (c) of the following theorem implies that $E = \overline{R(A)} \oplus N(A)$. Also, he shows that $P(x) = \lim_{t \to \infty} M(t, 0)x$ defines a norm one projection onto $N(A)$ such that $1 - P$ is a projection Q onto $\overline{R(A)}$. In the setting described here, the generalized inverse is determined by the equations

$$A^+Ax = x - Px \ , \qquad AA^+Ax = Ax \ ,$$
$$AA^+y = Qy \ , \quad \text{and} \quad A^+AA^+y = A^+y$$

for all x in $D(A)$ and y in $D(A^+)$. See also [9].

THEOREM 3. *Suppose that E is a Banach space and z is in E. Any two of the following statements implies the third:*
    (a) *z is in $R(A)$,*
    (b) *if x is in E, then $\lim_{t \to \infty} W(t, 0)x$ exists, and*

(c) *if x is in E, then $\lim_{t \to \infty} M(t, 0)x$ exists.*
Moreover, in case (a) *and* (b) *hold and x is in E, then*

$$\lim_{t \to \infty} W(t, 0)x = P(x) - A^+(z) \,.$$

Indication of proof. In case (c) holds, then the equivalence of (a) and (b) is contained in Theorem 8 of [10]. The value of this limit follows from the variation of parameters formula: If z is in $R(A)$ then $z = AA^+z$ and

$$W(t, 0)x = M(t, 0)x + M(t, 0)A^+(z) - A^+(z) \,.$$

Since $PA^+(z) = 0$, then $\lim_{t \to \infty} W(t, 0)x = P(x) - A^+(z)$.

It remains only to show that (a) and (b) imply (c). To do this, let $u = A^+(z)$.   Then $\lim_{t \to \infty} M(t, 0)u$ exists for

$$W(t, 0)(0) = M(t, 0)(0) + (R)\int_t^0 dg M(t, \cdot)z$$

$$= M(t, 0)u - u \,.$$

By (b), $\lim_{t \to \infty} M(t, 0)u$ exists. Now let x be in E. Applying this result and (b) again to the equation $W(t, 0)x = M(t, 0)x + M(t, 0)u - u$ establishes (c).

The next theorem and corollary provide a connection with recent work of Lovelady in [6]. Suppose E is a Banach space and let T be the semigroup generated by A and given by the formula

$$T(t)(x) = \prod_t^0 [1 - d\xi A]^{-1}(x) \,.$$

THEOREM 4.   *These are equivalent:*
(a) $E = \overline{R(A)} \oplus N(A)$,
(b) *if x is in E, then $\lim_{t \to \infty} [1 - tA]^{-1}(x)$ exists, and*
(c) *if x is in E, then $\lim_{t \to \infty} \left[1/t \int_0^t T(\xi)x d\xi \right]$ exists.*

Indication of proof. The equivalence of (a) and (b) uses the techniques of Lemma 1. The sets $N(A)$ and $\overline{R(A)}$ are characterized by $\{x: \lim_{t \to \infty} [1 - tA]^{-1}(x) = x\}$ and $\{x: \lim_{t \to \infty} [1 - tA]^{-1}x = 0\}$, respectively. If (a) holds and x is in E, then let n be in $N(A)$, r be in $\overline{R(A)}$, and $x = n + r$. Thus $\lim_{t \to \infty} [1 - tA]^{-1}(x)$ exists and is n. And (a) implies (b). On the other hand, suppose (b) holds and $P(x)$ is defined by $\lim_{t \to \infty} [1 - tA]^{-1}(x)$. Then P is a projection with $R(P) = N(A)$ and $N(P) = \overline{R(A)}$. Hence, $E = \overline{R(A)} \oplus N(A)$.

To establish the equivalence of (a) and (c), we first suppose (a) holds. Let n be in $N(A)$ and $c > 0$. Then $[1 - cA]^{-1}n = n$ and $T(c)n = n$, so that

$$\lim_{t \to \infty} \left[\frac{1}{t} \int_0^t T(\xi) n d\xi \right] = n \ .$$

Now let r be in $\overline{R(A)}$ and u be in $D(A)$. Then

$(*)$ $\qquad \left| \frac{1}{t} \int_0^t T(\xi) r d\xi \right| \leqq |r - A(u)| + \frac{1}{t} |T(t)u - u| \ .$

Hence,

$$\lim_{t \to \infty} \left[\frac{1}{t} \int_0^t T(\xi) r d\xi \right] = 0 \ .$$

Thus $\lim_{t \to \infty} \left[1/t \int_0^t T(\xi) x d\xi \right]$ exists for each x in E and we have that (c) holds.

To show that (c) implies (a), suppose (c) holds and $P(x) = \lim_{t \to \infty} \left[1/t \int_0^t T(\xi) x d\xi \right]$ for each x in E. Lovelady shows in [6, Theorem 2] that P is a projection. It is clear that $N(A)$ lies in $R(P)$. Also, since, for $t > 0$,

$$\left[A \left(\frac{1}{t} \int_0^t T(\xi) x d\xi \right) \right] = \frac{1}{t} [T(t)x - x]$$

and since A is closed, then $A(P(x)) = 0$. Thus P is a projection onto $N(A)$. It remains to see that $N(P) = \overline{R(A)}$. By the inequality $(*)$ above, we have $\overline{R(A)} \subset N(P)$. For any x in E, Lovelady shows in [6, p. 156] that

$$-\frac{1}{t} \int_0^t \xi T(t - \xi) x d\xi$$

is in $D(A)$ and

$$x - A \left(-\frac{1}{t} \int_0^t \xi T(t - \xi) x d\xi \right) = -\frac{1}{t} \int_0^t T(\xi) x d\xi \ .$$

Thus, if x is in $N(P)$ then x is in $\overline{R(A)}$. This completes the argument that $N(P) = \overline{R(A)}$ and $R(P) = N(A)$. Hence, $E = \overline{R(A)} \oplus N(A)$.

COROLLARY 3. *If* $E = \overline{R(A)} \oplus N(A)$, z *is in* $R(A)$, *and* x *is in* E *then each of the following limits exists and is* $P(x) - A^+(z)$:

$$\lim_{t \to \infty} ([1 - tA]^{-1}x + t[1 - tA]^{-1}z)$$

and

$$\lim_{t \to \infty} \left(\frac{1}{t} \int_0^t T(\xi) x d\xi + \frac{1}{t} \int_0^t \xi T(t - \xi) z d\xi \right) \ .$$

Indication of proof. Using the previous theorem and Theorems 5 and 6 of [6], we have that each of the limits exists. The previous theorem shows that each of $\lim_{t \to \infty} [1 - tA]^{-1}(x)$ and

$$\lim_{t \to \infty} \left[\frac{1}{t} \int_0^t T(\xi)x d\xi \right]$$

is $P(x)$ (where, as before, P is the projection onto $N(A)$). But if z is in $R(A)$ then $AA^+(z) = z$ and

$$t[1 - tA]^{-1}AA^+z = [1 - tA]^{-1}A^+(z) - A^+(z) .$$

Thus

$$\lim_{t \to \infty} (t[1 - tA]^{-1}z) = -A^+(z) .$$

To get the value of the final limit, it remains only to evaluate $\lim_{t \to \infty} \left[1/t \int_0^t \xi T(t - \xi)z d\xi \right]$. The calculus gives

$$\frac{1}{t} \int_0^t \xi T(t - \xi)AA^+(z)d\xi = \frac{1}{t} \int_0^t (t - \xi)T(\xi)AA^+(z)d\xi$$

$$= \frac{1}{t} \int_0^t \left(\int_0^\xi T(\sigma)AA^+(z)d\sigma \right)d\xi = \frac{1}{t} \int_0^t [T(\xi)A^+(z) - A^+(z)]d\xi$$

$$= \frac{1}{t} \int_0^t T(\xi)A^+(z)d\xi - A^+(z) .$$

Again, this has limit $PA^+(z) - A^+(z) = -A^+(z)$.

III. **Applications.** In what follows, the results and techniques of what has come before are used to obtain a generalized inverse in a Hilbert space for a densely defined, closed, linear operator B having closed range. This generalized inverse is determined by these equations: $BB^+B = B$, $B^+BB^+ = B^+$, $BB^+ = Q$, and $B^+B = 1 - P$ where Q is the orthogonal projection onto $R(B)$ and P is the orthogonal projection onto $N(B)$. The essence of the first theorem is the simple realization that if $A = -B^*B$, then a result of von Neumann [11, p. 200] gives that A is dissipative. But, for such an A, not so much will be required for g; compare the following with Lemma 2.

LEMMA 3. *Suppose that E is a Hilbert space, and B is a densely defined, closed, linear operator with closed range. There is a positive number m such that if y is in $D(B^*B) \cap R(B^*)$ then $|[1 + cB^*B]y| \geq (1 + cm)|y|$, for each positive number c.*

Indication of proof. If y is in $D(B^*B)$, then $|[1 + cB^*B]y|^2 \geq$

$|y|^2 + 2c|By|^2 + c^2|B^*By|^2$. Also, B is one-to-one from $D(B) \cap R(B^*)$ onto $R(B)$. Since B is closed, then the restriction of B to $D(B) \cap R(B^*)$ has a bounded inverse and this provides a number m_1 such that if y is in $D(B) \cap R(B^*)$ then $|B(y)| \geqq m_1|y|$. Now, B^*B is a self adjoint operator and, thus, a closed, linear operator with dense domain. Also, $R(B^*B) = R(B^*)$ and this is closed. So, the method for obtaining m_1 for B is applicable for B^*B. Thus, there is a positive number m_2 such that if y is in $D(B^*B) \cap R(B^*)$ then $|B^*B(y)| \geqq m_2|y|$.

Finally, if $m = \min\{m_1^2, m_2\}$ and y is in $D(B^*B) \cap R(B^*)$ then $|[1 + cB^*B]y| \geqq (1 + cm)|y|$.

THEOREM 5. *Suppose that E is a Hilbert space, z is in E, g is nonincreasing and unbounded and*

$$W(t, 0)x = \prod_t^0 [1 + dg(B^*B + z)]^{-1}x$$

for each $t \geqq 0$ and x in E. Then these are equivalent:
    (a) *z is in $R(B^*)$, and*
    (b) *if x is in E, then $\lim_{t \to \infty} W(t, 0)x$ exists.*
Moreover, in case (b) holds and x is in E, then $\lim_{t \to \infty} W(t, 0)x = P(x) + B^+(u)$ where $z = B^*(u)$.

Indication of proof. If $t > 0$ and x is in E, let $M(t, 0)x = \prod_t^0 [1 + dgB^*B]^{-1}x$. As in Theorem 2, one can show that $\lim_{t \to \infty} M(t, 0)x$ exists. In fact, if z is in E, n is in $N(B)$, r is in $R(B^*)$, $z = n + r$, m is as in Lemma 3, and $t > 0$ then

$$|M(t, 0)z - n| \leqq |r|/[1 + m(g(0) - g(t))].$$

The equivalence of (a) and (b) now follows as a corollary to Theorem 3.

To evaluate the limit, let $z = B^*(u) = B^*BB^+(u)$. If x is in E, then

$$W(t, 0)x = M(t, 0)x - (R)\int_t^0 dgM(t, \cdot)B^*BB^+(u)$$
$$= M(t, 0)x - M(t, 0)B^+(u) + B^+(u).$$

As in Theorem 3, this has limit $P(x) + B^+(u)$.

Each of the following corollaries holds in case E is a Hilbert space.

COROLLARY 3. *If u is in $D(B^*)$ and $\lambda > 0$ then*

$$B^+(u) = \lambda \sum_{p=1}^{\infty} [1 + \lambda B^*B]^{-p} B^*(u) .$$

COROLLARY 4. *If* $\{\lambda_p\}_{p=1}^{\infty}$ *is a positive number sequence,* $\sum_{p=1}^{\infty} \lambda_p = \infty$, $z = B^*(u)$, *and* $\{x_p\}_{p=1}^{\infty}$ *is given by the iteration:* x_0 *is in* E *and* $x_n = [1 + \lambda_n B^*B]^{-1}(x_{n-1} + \lambda_n z)$ *then* $\lim_{n \to \infty} x_n = P(x_0) + B^+(u)$.

COROLLARY 5 [1, p. 24]. *Let* y *be a solution of* $y'(t) + B^*B(y(t)) = B^*(u)$. *Then* $\lim_{t \to \infty} y(t) = P(y(0)) + B^+(u)$.

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