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SEMISIMPLE NIL ALGEBRAS OF TYPE δ

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We prove that a finite dimensional semisimple nil algebra over a field F which satisfies the identity $(1+\delta)z(x\circ y)+(1-\delta)(x\circ y)z = x(y\circ z)+y(x\circ z)$, where $\delta \in F$ and $\delta \neq -1/2$, is anti-commutative. This result permits a further reduction in the problem of classifying those varieties of power-associative algebras over F having the property that squares of ideals are ideals and for which the nil algebras are not pathological.

1. Introduction. Recently we gave a survey (see [1]) of those varieties $\mathscr{V}_F$ of power-associative algebras over a field F which satisfy the following condition:

(1.1) For each $A \in \mathscr{V}_F$ and ideal I of A, I^2 is also an ideal of A. It is well-known that the varieties of alternative and Lie algebras have this property. On the other hand, there are some basic structural differences between these two varieties of algebras. In particular, while every Lie algebra is a nil algebra, in the alternative theory, nil algebras are usually regarded as pathological in the radical sense. Consequently, in our classification [1] we had to account for those varieties $\mathscr{V}_F$ which satisfy in addition to (1.1) the condition.

(1.2) $\mathscr{V}_F$ contains a nonzero semi-simple nil algebra, and we showed that the classification of such varieties was largely reduced to the problem of classifying algebras of type δ. The algebras of type δ satisfy, among other identities, the relation

$$(1.3)\qquad (1+\delta)z(x\circ y)+(1-\delta)(x\circ y)z = x(y\circ z)+y(x\circ z)$$

where $\delta \in F$ and $x\circ y = 1/2(xy+yx)$.

We are now able to prove the following result:

(1.4) THEOREM. *If A is a semisimple nil algebra satisfying the identity* (1.3) *with $\delta \neq -1/2$, then A is anti-commutative.*

As we shall point out later, this allows us to complete the classification of semisimple algebras of type $\delta \neq -1/2$.

2. Preliminaries. Throughout this paper all algebras are assumed to be finite dimensional over a fixed, but arbitrary, field F of characteristic not two. For an algebra A and element $a \in A$, the right multiplication R_a is defined as the map $R_a\colon x \to xa$, $x \in A$. Furthermore, when n is a positive integer, A^n will denote the linear span of all products of n elements of A. The associator (a, b, c), for $a, b, c \in A$, is defined by $(a, b, c) = (ab)c - a(bc)$. We shall always assume the algebras to be power-associative, so that the identity $0 = (x, x, x)$ holds, as well as its linearization

$$0 = \Sigma(\sigma(x), \sigma(y), \sigma(z)) , \tag{2.1}$$

where the summation is taken over all $\sigma \in S_3$, the symmetric group on three letters.

For the basic notions of nil, solvable, and nilpotent algebras we refer the reader to Schafer [2], with the understanding that in this paper whenever an algebra is called semisimple that means that it has no nonzero solvable ideal.

Given an algebra A, we may form the commutative algebra A^+ by replacing the product ab of A with the symmetrized product $1/2(ab + ba)$. Clearly, A is power-associative or nil if and only if A^+ is. Moreover, if $\lambda \neq 1/2$ is any scalar from F, we may replace the product ab of A with the twisted product $\lambda ab + (1 - \lambda)ba$ to get an algebra A^λ, which is called quasi-equivalent to A (see [2]). It is well-known that power-associativity, ideals, and semisimplicity are preserved under quasi-equivalence. Furthermore, $A^+ = (A^\lambda)^+$ for every $\lambda \neq 1/2$.

3. The structure of certain commutative algebras.

We shall show in next section that symmetrizing the product of an algebra which satisfies (1.3) yields a commutative algebra in which

$$0 = (y, z, wx) + (z, y, wx) + (w, x, yz) + (x, w, yz) \tag{3.1}$$

is an identity. The purpose of this section is to get some useful preliminary results on such algebras. Consequently, we shall assume throughout this section that A is a commutative algebra in which the relation (3.1) holds.

It is easy to see that (3.1) is equivalent to the following identity in right multiplications:

$$0 = R_x R_y R_z + R_x R_z R_y + R_{yz} R_x + R_{(yz)x} - 4R_x R_{yz} . \tag{3.2}$$

If B is any subalgebra of A then we shall denote by B^* the subalgebra of $\mathrm{Hom}\,(A, A)$ which is generated by the set $\{R_b \mid b \in B\}$. We suppose further that B is a solvable subalgebra, and our object is to prove that B^* is a nilpotent algebra. As $B^2 \neq B$ (in the case $B \neq 0$) we may find a subspace C of B of codimension one in B such that $C \supseteq B^2$. Clearly C is an ideal of B and for any (fixed) $w \notin C$, $w \in B$, we have the decomposition $B = C + Fw$. We set $T = B^*C^* + C^*$;

the following sequence of lemmas is about T.

(3.3) LEMMA. *T is a left ideal of B^*.*

Proof. Obvious.

(3.4) LEMMA. *If $c \in C$ then $R_cR_xR_x \in T$ for all $x \in B$.*

Proof. Follows immediately from (3.2) and $B^2 \subseteqq C$.

(3.5) LEMMA. *If $c, d \in C$ then $R_cR_dR_x \in T$ for all $x \in B$.*

Proof. Follows immediately from (3.2), $B^2 \subseteqq C$.

(3.6) LEMMA. *If $c \in C$ then $R_cR_xR_xR_x \in T$ for all $x \in B$.*

Proof. From (3.2) we have $0 = 2R_xR_xR_x + R_{x^2}R_x + R_{x^3} - 4R_xR_{x^2}$. Multiplying this equation by R_c and using the fact that C contains x^2 and x^3, it then follows from Lemma (3.5) that $R_cR_xR_xR_x \in T$.

(3.7) LEMMA. *If $c, d \in C$ then $R_dR_xR_cR_x \in T$ for all $x \in B$.*

Proof. Using (3.2), $0 = R_xR_cR_x + R_xR_xR_c + R_{cx}R_x + R_{(cx)x} - 4R_xR_{cx}$, thus $0 = R_dR_xR_cR_x + R_dR_xR_xR_c + R_dR_{cx}R_x + R_dR_{(cx)x} - 4R_dR_xR_{cx}$. From Lemma (3.3) and Lemma (3.5) it follows from this equation that $R_dR_xR_cR_x \in T$.

(3.8) LEMMA. *$R_xR_xR_xR_xR_x \in T$ for all $x \in B$.*

Proof. Using (3.2) we have $2R_xR_xR_x = -R_{x^2}R_x - R_{x^3} + 4R_xR_{x^2}$. Multiplying this equation of the right by R_xR_x yields $2R_xR_xR_xR_xR_x = -R_{x^2}R_xR_xR_x - R_{x^3}R_xR_x + 4R_xR_{x^2}R_xR_x$. Then using Lemma (3.4) and Lemma (3.3), we conclude that $2R_xR_xR_xR_xR_x \equiv -R_{x^2}R_xR_xR_x \pmod{T}$. Now applying the identity (3.2) to the factor $R_xR_xR_x$, we find that $2R_xR_xR_xR_xR_x \equiv -R_{x^2}(-1/2R_{x^2}R_x - 1/2R_{x^3} + 2R_xR_{x^2}) \equiv 0 \pmod{T}$ because of Lemma (3.5). Thus $R_xR_xR_xR_xR_x \in T$.

(3.9) LEMMA. *If $c \in C$, then $R_cR_xR_xR_xR_x \in T$ for all $x \in B$.*

Proof. Using (3.2), $R_cR_xR_xR_xR_x = R_cR_x(R_xR_xR_x) = R_cR_x(-1/2R_{x^2}R_x - 1/2R_{x^3} + 2R_xR_{x^2}) \in T$ because of Lemma (3.7).

(3.10) LEMMA. *If $c, d \in C$, then $R_cR_xR_xR_dR_x \in T$ for all $x \in B$.*

Proof. Using (3.2), $R_cR_xR_xR_dR_x = R_c(R_xR_xR_d)R_x = R_c(-R_xR_dR_x - R_{dx}R_x - R_{(dx)x} + 4R_xR_{dx})R_x = -R_cR_xR_dR_xR_x - R_cR_{dx}R_xR_x - R_cR_{(dx)x}R_x + 4R_cR_xR_{dx}R_x \in T$, because of Lemmas (3.3), (3.4), (3.5), and (3.7).

(3.11) LEMMA. *If $c \in C$, then $R_xR_xR_xR_cR_x \in T$ for all $x \in B$.*

Proof. From (3.2), $R_xR_xR_xR_cR_x = (R_xR_xR_x)R_cR_x = (-1/2R_{x^2}R_x - 1/2R_{x^3} + 2R_xR_{x^2})R_cR_x = -1/2R_{x^2}R_xR_cR_x - 1/2R_{x^3}R_cR_x + 2R_xR_{x^2}R_cR_x \in T$, because of Lemmas (3.7), (3.5), and (3.3).

Now we can prove

(3.12) THEOREM. *Let A be a commutative algebra satisfying the identity* (3.1) *and let B be a solvable subalgebra of A. Then B^* is nilpotent.*

Proof. We use induction on the dimension of B, the result being trivial for $\dim B = 0$. For $\dim B \geqq 1$, set $B = C + Fw$ and $T = B^*C^* + C^*$ as before. We claim that

$$U = R_pR_qR_rR_sR_t \in T \quad \text{for all} \quad p, q, r, s, t \in T\,. \tag{3.13}$$

To show this, we may suppose p, q, r, s, t are either in C or are $= w$. If $t \in C$ then $U \in T$ by Lemma (3.3). Thus we suppose $U = R_pR_qR_rR_sR_w$. Now if both r and s belong to C then $U \in T$ because of Lemma (3.5) and Lemma (3.3). Hence we may assume either $r = s = w$ or exactly one of r and s belong to C. In the latter case, if $r \in C$ and $s = w$ then $U \in T$ by Lemmas (3.4) and (3.3). Thus we may assume either $U = R_pR_qR_wR_wR_w$ or $U = R_pR_qR_wR_sR_w$, where $s \in C$. In the first of these cases, if $q \in C$, then $U \in T$ because of Lemmas (3.6) and (3.3). On the other hand, if $q = w$, then $U \in T$ because of Lemmas (3.8) and (3.9). This leaves the case $U = R_pR_qR_wR_sR_w$, where $s \in C$. Now if $q \in C$, then $U \in T$ because of Lemmas (3.7) and (3.3). Thus we suppose $U = R_pR_wR_wR_sR_w$, where $s \in C$. However then $U \in T$ because of Lemmas (3.10) and (3.11).

Having verified (3.13), it follows from Lemma (3.3) that $(B^*)^5 \subseteqq B^*C^* + C^*$; whence $(B^*)^6 \subseteqq B^*C^*$. Using induction on k, these last two relations imply that $(B^*)^{5k+1} \subseteqq B^*(C^*)^k$. As $\dim C < \dim B$, it follows from the induction hypothesis that $C^{*k} = 0$ for some k; hence B^* is nilpotent.

(3.14) COROLLARY. *A commutative solvable algebra A satisfying* (3.1) *is nilpotent.*

Proof. This follows from choosing $B = A$ in the previous theorem.

(3.15) COROLLARY. *If x is a nilpotent element of a commutative algebra satisfying the identity* (3.1), *then R_x is nilpotent.*

(3.16) THEOREM. *If A is a commutative nil algebra satisfying* (3.1) *then A is nilpotent.*

Proof. We use induction on the dimension of A. If $A^2 \neq A$ then the solvability of A/A^2 and A^2 imply A is solvable; hence A is nilpotent because of Corollary (3.14). Thus we may suppose $A = A^2$.

For $u \in A$ we let $u^\delta = \{t \in A \mid ut = 0\}$. Using (3.1), we find that if $t \in u^\delta$, $a \in A$ then $0 = (t, u, a^2) + (u, t, a^2) + 2(a, a, tu) = (a^2)(R_uR_t + R_tR_u)$. However, as $A = A^2$ and A is commutative, A is spanned by $\{a^2 \mid a \in A\}$. Thus

$$R_uR_t + R_tR_u = 0 \quad \text{for all} \quad t \in u^\delta\,. \tag{3.17}$$

From this relation it follows that u^δ is a subalgebra of A. Indeed, is $s, t \in u^\delta$ then $u(st) = (s)R_tR_u = -(s)R_uR_t = -(su)t = 0$; whence $st \in u^\delta$.

Now among all the nonzero elements of A we choose u so that u^δ is maximal with respect to inclusion. We shall first consider the case that $u^\delta \neq A$. Here the induction hypothesis tells us that u^δ is solvable: hence by Theorem (3.12) we have that $(u^\delta)^*$ is nilpotent. By a well-known argument (see [2, p. 96]), this implies there exists an element $x \notin u^\delta$ such that $xu^\delta \subseteq u^\delta$. We claim that

$$u^\delta = (uR_x^k)^\delta \quad \text{for all } k. \tag{3.18}$$

To prove this, we use induction on k. Suppose $k = 1$; let $t \in u^\delta$. Then $(ux)t = (x)R_uR_t = -(x)R_tR_u = -(xt)u = 0$. Thus $u^\delta \subseteq (ux)^\delta$. As $x \notin u^\delta$, $ux \neq 0$; hence from our choice of u, $u^\delta = (ux)^\delta$. This verifies (3.18) in the case $k = 1$. We show next that $u^\delta = (uR_x^{k+1})^\delta$. Let $t \in u^\delta$ and $v = uR_x^k$. Then $(vx)t = (x)R_vR_t$. The induction hypothesis says that $t \in v^\delta$; hence by (3.17), $(vx)t = xR_vR_t = -xR_tR_v = -(xt)v$. However, $xt \in u^\delta = v^\delta$. Thus $(vx)t = 0$ and $t \in (vx)^\delta = (uR_x^{k+1})^\delta$. This shows that $u^\delta \subseteq (uR_x^{k+1})^\delta$. Moreover, as $xu \neq 0$ and $u^\delta = (uR_x^k)^\delta$, $uR_x^{k+1} \neq 0$. Hence from the maximality of u it follows that $u^\delta = (uR_x^{k+1})^\delta$, which completes the induction proof of (3.18). Now as R_x is nilpotent (see Corollary (3.15)), it follows from (3.18) that $u^\delta = 0^\delta = A$, a contradiction.

We have shown that u^δ cannot be a proper subalgebra of A. Therefore $u^\delta = A$, which implies $uA = 0$. Then Fu is a nonzero ideal of A, and by the induction hypothesis, A/Fu is nilpotent. As $(Fu)^2 = 0$, this implies A is solvable; hence A is nilpotent.

4. Main results.

(4.1) LEMMA. *An algebra A satisfying the identity*

$$(1 + \delta)z(x \circ y) + (1 - \delta)(x \circ y)z = x(y \circ z) + y(x \circ z)$$

is quasi-equivalent to an algebra satisfying

$$z(x \circ y) + (x \circ y)z = x(y \circ z) + y(x \circ z) \tag{4.2}$$

unless $\delta = -1/2$.

Proof. Let A^λ be quasi-equivalent to A via the product $x \otimes y = \lambda xy + (1 - \lambda)yx$, where $\lambda \neq 1/2$. As $y \otimes x = (1 - \lambda)xy + \lambda yx$, we find

$$xy = \mu x \otimes y + (1 - \mu)y \otimes x \tag{4.3}$$

where $\mu = \lambda(2\lambda - 1)^{-1}$. Substituting (4.3) into (1.3) and keeping in mind the fact that $(A^\lambda)^+ = A^+$, we find

$$\begin{aligned} &[(1 + \delta)\mu + (1 - \delta)(1 - \mu)]z \otimes (x \circ y) \\ &\quad + [(1 + \delta)(1 - \mu) + (1 - \delta)\mu](x \circ y) \otimes z \\ &= \mu x \otimes (y \circ z) + \mu y \otimes (x \circ z) + (1 - \mu)G(x, y, z) \end{aligned} \tag{4.4}$$

where $G(x, y, z) = (y \circ z) \otimes x + (x \circ z) \otimes y$. However, from (2.1) we have

$$G(x, y, z) = -(x \circ y) \otimes z + z \otimes (x \circ y) + y \otimes (x \circ z) + x \otimes (y \circ z)\,. \tag{4.5}$$

Substituting (4.5) into (4.4) yields

$$(1 + \omega)z \otimes (x \circ y) + (1 - \omega)(x \circ y) \otimes z = x \otimes (y \circ z) + y \otimes (x \circ z) \tag{4.6}$$

where $\omega = -1 + \mu - \delta + 2\delta\mu$. Evidently, as long as $\delta \neq -1/2$, we may choose λ so that $\omega = 0$. However, when $\omega = 0$, (4.6) is simply (4.2).

(4.7) THEOREM. *If A satisfies* (4.2) *then*

$$z(x \circ y) = z \circ (x \circ y) - [x, y, z] - [y, x, z] \tag{4.8}$$

and

$$[w, x, y \circ z] + [x, w, y \circ z] + [y, z, w \circ x] + [z, y, w \circ x] = 0 \tag{4.9}$$

for all $w, x, y, z \in A$, *where* $[a, b, c] = (a \circ b) \circ c - a \circ (b \circ c)$.

Proof. Interchanging y and z in (4.2) and then subtracting the resulting identity from (4.2) yields

$$y(x \circ z) - z(x \circ y) = 2[y, x, z]\,. \tag{4.10}$$

Interchanging x and y in (4.10) we have

$$x(y \circ z) - z(x \circ y) = 2[x, y, z]\,. \tag{4.11}$$

Adding (4.10) and (4.11) we have

$$x(y \circ z) + y(x \circ z) - 2z(x \circ y) = 2([y, x, z] + [x, y, z])\,. \tag{4.12}$$

Now comparison of (4.12) and (4.2) yields (4.8).

As $a \circ a = a^2$, setting $z = x \circ y$ in (4.8) yields

$$0 = [x, y, x \circ y] + [y, x, x \circ y]\,. \tag{4.13}$$

Moreover, as A^+ is power-associative, $0 = [x, x, x \circ x]$. Linearizing this we find $0=[y, x, x \circ x]+[x, y, x \circ x]+2[x, x, x \circ y]$. Another linearization gives $0 = [y, y, x \circ x] + 2[y, x, x \circ y] + [y, y, x \circ x] + 2[x, y, x \circ y] + 2[y, x, x \circ y] + 2[x, y, x \circ y] + 2[x, x, y \circ y]$. Reducing this last relation by means of (4.13) we get

$$0 = [x, x, y \circ y] + [y, y, x \circ x]\,. \tag{4.14}$$

Linearizing (4.14) by replacing x with $x + w$ and y with $y + z$ yields the desired identity (4.9). The reader should note that (4.9) is similar to (3.1) in the commutative case.

(4.15) THEOREM. *If A is a semisimple nil algebra satisfying the identity*

$$(1 + \delta)z(x \circ y) + (1 - \delta)(x \circ y)z = x(y \circ z) + y(x \circ z)\,,$$

where $\delta \neq -1/2$, then A is anti-commutative.

Proof. It suffices to prove A is quasi-equivalent to an algebra which is anti-commutative. Thus, without loss of generality, and on account of Lemma (4.1), Theorem (4.7), and Theorem (3.16), we may assume that

$$A^+ \text{ is nilpotent} \tag{4.16}$$

and

$$z(x \circ y) = z \circ (x \circ y) - [x, y, z] - [y, x, z] \tag{4.17}$$

where $[a, b, c] = (a \circ b) \circ c - a \circ (b \circ c)$.

As A^+ is nilpotent, there is an integer n such that $A^{[n]} \neq 0$ and $A^{[n+1]} = 0$, where $A^{[k]}$ denotes the linear span in the algebra A^+ of all symmetrised products of n elements. We show $n = 1$.

If $n > 1$, choose $u \in A^{[n]}$. As $A^{[n]} = A^{[n-1]} \circ A + A^{[n-2]} \circ A^{[2]} + \cdots$, u is a linear combination of terms of the type $x \circ y$, where $x \in A^{[n-k]}$ and $y \in A^{[k]}$. Using the relation (4.17), we see that $z(x \circ y) \in A^{[n+1]} = 0$ for all $z \in A$. Thus $0 = Au$; hence $AA^{[n]} = 0$. This shows $A^{[n]}$ is a left ideal of A. As $A^{[n]} \circ A = 0$, it follows that $A^{[n]}$ is a 2-sided ideal of A. However, $A^{[n]}A^{[n]} \subseteqq AA^{[n]} = 0$. Thus $A^{[n]}$ is a solvable ideal of the semisimple algebra A. Consequently, $A^{[n]} = 0$, contrary to the choice of n. Thus $n = 1$ and $A \circ A = 0$, which means A is anti-commutative.

In our paper [1] we introduced the class of algebras of type δ. These satisfy the identity $(1 + \delta)z(x \circ y) + (1 - \delta)(x \circ y)z = x(y \circ z) + y(x \circ z)$, as well as

$$(4.19) \quad \begin{aligned}(x_1x_2)x_3 = {} & \alpha_1(x_3x_1)x_2 + \alpha_2(x_1x_3)x_2 + \alpha_3x_2(x_3x_1) + \alpha_4x_2(x_1x_3) \\ & + \alpha_5(x_3x_2)x_1 + \alpha_6(x_2x_3)x_1 + \alpha_7x_1(x_3x_2) + \alpha_8x_1(x_2x_3)\end{aligned}$$

and

$$(4.20) \quad \begin{aligned}x_3(x_1x_2) = {} & \beta_1(x_3x_1)x_2 + \beta_2(x_1x_3)x_2 + \beta_3x_2(x_3x_1) + \beta_4x_2(x_1x_3) \\ & + \beta_5(x_3x_2)x_1 + \beta_6(x_2x_3)x_1 + \beta_7x_1(x_3x_2) + \beta_8x_1(x_2x_3)\end{aligned}$$

where the coefficients $\alpha_1, \cdots, \beta_8$ belong to the field F and satisfy the relations

$$(4.21) \quad \begin{aligned}1 &= -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 \\ &= \beta_1 - \beta_2 - \beta_3 + \beta_4 = -\beta_5 + \beta_6 + \beta_7 - \beta_8 \\ &= \alpha_1 + \alpha_2 + \cdots + \alpha_8 = \beta_1 + \beta_2 + \cdots + \beta_8 .\end{aligned}$$

It was proved in [1] that if A is a semisimple algebra of type $\delta \neq -1/2$ then A is the direct sum $A = A_1 + A_0$, where A_1 is a direct sum of fields and A_0 is a semi-simple nil algebra. Now in view of Theorem (4.15), we know that A_0 is anti-commutative. However, in the anti-commutative case, the identity (4.19) reduces to the Jacobi identity on account of (4.21). Therefore, we have the following result:

(4.22) THEOREM. *A semisimple algebra of type $\delta \neq -1/2$ is a direct sum of fields and a semisimple Lie algebra.*

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