PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No. 1, 1978

POTENTIAL OPERATORS AND EQUIMEASURABILITY

A. N. AL-HUSSAINI

W. Rudin proved the following.

THEOREM 1.1. **Assume $0 < p < \infty$, $p \neq 2, 4, 6, \cdots$. Let n be a positive integer. If $f_i \in L_p(\mu)$, $g_i \in L_p(\nu)$ for $1 \leqq i \leqq n$ and**

$$\int_X |1 + z_1 f_1 + \cdots + z_n f_n|^p d\mu = \int_Y |1 + z_1 g_1 + \cdots + z_n g_n|^p d\nu$$

for all $(z_1, \cdots, z_n) \in C^n$, then $(f_1, \cdots, f_n)$ and $(g_1, \cdots, g_n)$ are equimeasurable. Here as usual $L_p(\mu)$ and $L_p(\nu)$ stand for pth power integrable functions defined on finite measure spaces (X, X, μ) and (Y, Y, ν) respectively. $\mathscr{C}$ is the field of complex numbers.

The purpose of this paper is to provide a new setting for Rudin's result by recasting it and its extension to real valued functions into the framework of the theory of potential operators as formulated by K. Yosida.

We begin by outlining the theory of potential operators ([8], [9]). K-I. Sato's 1970 paper [6] contains extensive material on the subject.

Let T_t be a strongly continuous semigroup of linear operators on a Banach space X, satisfying $\sup_t \| T_t \| < +\infty$, with infinitesimal generator $A = s \lim_{t \to 0} (T_t f - f)/t$ where as usual s denotes the strong limit, and resolvent $J_\lambda = (\lambda - A)^{-1}$, $\lambda > 0$. K. Yosida defined the potential operator V as follows:

$$Vf = s \lim_{\lambda \to 0} J_\lambda f \ ,$$

if the strong limit exists for a dense set in X. This is one way to unify the potential operator concept for a large class of Markov processes, which includes Brownian motion, stable processes and of course transient Markov processes.

Motivated by an application to equimeasurability in Section (3), we specialize to potential operators induced by Markov processes. Thus let S be a locally compact, noncompact, Hausdorff space with countable basis. By $C_0(S)$, $C_k(S)$ we denote the spaces of real valued functions which vanish at infinity, and those with compact support respectively. Let T_t be a strongly continuous semigroup of positive linear operators on $C_0(S)$ with $\| T_t \| \leqq 1$. To this semigroup there corresponds a right continuous Markov process $\{X_t\}$ on S with transition $P_t(x, A)$, such that:

$$T_t f(x) = \int P_t(x, dy) f(y) \ .$$

We will not distinguish between T_t and its corresponding Markov process.

If $\int_0^{+\infty} P_t(x, K)dt < +\infty$ for all $x \in S$, all compact sets K, we call the Markov process transient. If $\int_0^{+\infty} P_t(x, \mathcal{O})dt = +\infty$ for all $x \in S$, and all open sets $\mathcal{O}$, we call the Markov process recurrent. It will be called null recurrent if it is recurrent and $\lim_{t\to\infty} P_t(x, K) = 0$ for all $x \in S$, all compact sets K. V the potential operator introduced earlier in this section exists for transient and null recurrent Markov processes. (Theorem 3.1, Theorem 3.2 [6].) We will make use of the following theorem, whose proof is included in Theorem 2.3 of [6].

THEOREM 1.2. *If a semigroup T_t admits a potential operator V, then $\mathscr{D}(V) = \mathscr{R}(A)$, $\mathscr{R}(V) = \mathscr{D}(A)$, and $V = -A^{-1}$. Here $\mathscr{D}$ and $\mathscr{R}$ stand for domain and range respectively.*

2. More about Markov processes. Concepts treated in this section are mainly those necessitated by the section to follow. Let $\{X_t\}$ be a Markov process on S (as in § 1), with transition $P_t(x, A)$ which as assumed to exist. As is well known,

$$J_\lambda f = \int_0^{+\infty} e^{-\lambda t} T_t f dt\ , \quad \text{where} \quad T_t f = \int P_t(x, dy) f(y)\ .$$

Set $P_{t,\lambda}(x, A) = e^{-\lambda t} P_t(x, A)$. Then there is a Markov process $\{X_{t,\lambda}\}$, a sub-Markov process of $\{X_t\}$, corresponding $P_{t,\lambda}(x, A)$. Thus V may be viewed as a limit of potentials $\int_0^{+\infty} T_{t,\lambda} f dt$ where $T_{t,\lambda}$ is the semigroup associated with $\{X_{t,\lambda}\}$. Also we may note that the infinitesimal generator A_λ of $T_{t,\lambda}$ is given by $A_\lambda = A - \lambda$, (see [2]). By Theorem 1.2, there is a $\phi \in \mathscr{D}(V)$ for every $f \in \mathscr{D}(A)$, such that $f = V\phi$ or alternatively, by making use of $\{X_{t,\lambda}\}$, $\lambda > 0$, a sub-Markov process of $\{X_t\}$; $A_\lambda f = -(-A_\lambda f)$ we obtain $f = V_\lambda(-A_\lambda f)$, using ([2], pp. 24) where V_λ is the potential operator of $\{X_{t,\lambda}\}$. Taking limit as $\lambda \to 0$, $f = V(-Af) = V\phi$.

Some known examples of V are [1], [3], [4], [6].

EXAMPLE (1). One-dimensional Brownian motion:

$$Vf(x) = c\int |x - y| f(y) dy\ .$$

EXAMPLE (2). One-dimensional stable process of order $0 < \alpha < 1$, $1 < \alpha < 2$. Here $Vf = c\int |x - y|^{\alpha-1} f(y) dy$.

EXAMPLE (3). n-dimensional stable process.

$$Vf(x) = c\int|x-y|^{\alpha-n}f(y)dy\ , \quad 0<\alpha<2\ ,\ 0<\alpha<n\ .$$

The constants c are not the same from line to line. Example (1) could be included in Example (2) if we allow $\alpha = 2$. Example (3), with additional probabilistic arguments will be used (§ 3) to give an alternative proof to Theorem (1) of [5] concerning equimeasurability of C^n valued random functions. W. Rudin proves his theorem by transforms and techniques based on complex variable theory.

By utilizing Examples (1) and (2), we will give an $\mathscr{R}^n$ version of the theorem cited above.

The reader may observe that Vf's in the preceding examples are none other than Riesz potentials. Within general theory of Markov processes Vf is just $\iint P_t(x, y)f(y)dtdy$ the Green's potential, if the Markov process is transient ($p_t(x, y)$ is the transition density). However if the Markov process is null recurrent and transition density $p_t(x, y)$ exists then Vf is the limit of potentials of some sub-Markov processes. More precisely $Vf = \lim_{\lambda\to 0} V_\lambda f$, where V_λ, $\lambda > 0$ is the Green potential corresponding to $e^{-\lambda t}p_t(x, y)$.

One-dimensional stable process $0<\alpha<2$, require special attention. V (the set of all functions in $C_k(S)$ having null integrals) is dense in $C_0(S)$ ([4], [6]). Another important matter is the *core* of V. A set M is called core for V [6] if $M\subset \mathscr{D}(V)$ and if the smallest closed extension of $V|M$ coincides with V. If M is a core for V, then M is dense in $\mathscr{D}(V)$, and $V|M$ determines the semigroup. Also $V(M)$ is dense in $\mathscr{R}(V)$. In [6], see also [4], it is shown that $C_k(S)\cap\mathscr{D}(V) = M_0$ is a core for V.

To extend this to $C_k^{+\infty}(R^N)$, the set of all infinitely differentiable functions on R^N, (Theorem 2.3), we borrow (for convenience) some results from [7].

We say a function ϕ is α order homogeneous on R^N, outside a compact set, if there is a $b>0$ such that

$$\phi(\lambda x) = \lambda^\alpha\phi(x) \quad \text{for} \quad |x|\geqq b\ , \quad \lambda\geqq 1\ .$$

Define $\tilde{\phi}(x) = (|x|/b)^\alpha\phi(bx/|x|)$ and note $\phi(x) = \tilde{\phi}(x)$ for $|x|\geqq b$.

THEOREM 2.1. *If X_t is a transient process with stationary independent increment, with right continuous paths and $E|X_t|^\alpha < +\infty$ for a real number $\alpha>0$. Let ϕ_i, $1\leqq i\leqq l$, be an arbitrary number of continuous functions on R^N such that ϕ_i is α_i order homogeneous outside a compact set, $0<\alpha_i\leqq\alpha$ and such that the set $\{\phi_i\colon 1\leqq i\leqq l\}$ is linearly independent. Given real numbers a_i, $1\leqq i\leqq l$, let $\mathscr{M}$ be the set of function $f\in C_k^{+\infty}(R^N)$ such that*

$$\int_{R^N} f(x)dx = 0, \int_{R^N} f((x)\phi_i(x)dx = a_i\,, \quad 1 \leqq i \leqq l\,.$$

Then $\mathscr{M}$ *is a core of the potential* V.

Proof. See Theorem 5.1 [7].

For the next theorem, we assume for some t the distribution of X_t has a nontrivial absolutely continuous part, and that X_t is recurrent stationary independent process with right continuous paths, in which case $N = 1$ or 2.

THEOREM 2.2. *If* $E|X_t| < \infty$, *then the set* $f \in C_k^\infty(R^N)$ *satisfying*

$$\int_{R^N} f(x)dx = \int_{R^N} f(x)x_i dx = 0 \quad \textit{for} \quad 1 \leqq i \leqq N$$

is a core of the potential operator V.

Proof. See Theorem 6.1 [7].

For the next theorem $N = 1$ or 2 only.

THEOREM 2.3. *For a stable process* X_t *with index* $0 < \alpha < 2$, $(\alpha \neq 1, N = 1)$; $C_k^\infty \cap \mathscr{D}(V)$ *is a core of the potential operator* V.

Here and elsewhere, it is understood that we are dealing with right continuous path versions.

Proof. We divide the proof into 2 cases:

Transient Case. $(0 < \alpha < 1, N = 1)$ or $(0 < \alpha < 2, N = 2)$. Let $\varepsilon > 0$ be such that $\alpha - \varepsilon > 0$ and take $\phi(x) = \lambda^{\alpha-\varepsilon-1}x$, $x \in R^N$. The proof is completed by Theorem 2.1. It may be remarked that the introduction of the ε above was to insure $E|X|^{\alpha-\varepsilon} < +\infty$, by moment properties of stable processes.

Recurrent Case. $(1<\alpha<2, N=1)$. Here again $E|X_t| < +\infty$ by moment properties of stable processes, since $1 < \alpha$. The rest follows from Theorem 2.2.

We did not consider $\alpha = 1$, $N = 1$. Instead we follow Example (1), a detour.

3. Equimeasurability. Let $(\Omega_1, \mathscr{A}_1, P_1)$ and $(\Omega_2, \mathscr{A}_2, P_2)$ be two probability spaces. For R^n or C^n valued measurable functions F, G defined on Ω_1, Ω_2 respectively, set:

$$\mu(B) = P_1F^{-1}(B) \quad B \in R^n \quad \text{or} \quad C^n$$
$$\nu(B) = P_2G^{-1}(B) \quad B \in R^n \quad \text{or} \quad C^n$$

so that μ and ν are the distribution functions of F and G under P_1 and P_2 respectively. Changes in n, R, C, F or G, result perhaps in different μ and ν; but to simplify our notation we will continue using the same symbols μ and ν in all cates. If

$$\int\left|1 + \sum_{i=1}^{n} a_i f_i\right|^p dP_1 = \int\left|1 + \sum_{i=1}^{n} a_i g_i\right|^p dP_2\,,$$

$0 < p < +\infty$, $p \neq$ even integer, for all complex numbers $a_1, a_2, \cdots, a_n$, then [5], $F = (f_1, f_2, \cdots, f_n)$ and $G = (g_1, g_2, \cdots, g_n)$ have the same distribution, that is $\mu = \nu$.

In Theorem 3.2 of this section, we give a probabilistic proof of this result. (Real versions will be given in Theorem 3.1. In § 4 we comment on further results.) The proof here is, more or less, Markov-potential theoretic.

THEOREM 3.1. *Let $F = (f_1, f_2, \cdots, f_n)$, $G = (g_1, g_2, \cdots, g_n)$ be two R^n valued measurable functions defined on Ω_1, Ω_2 respectively. If*

$$\int\left|1 + \sum_{i=1}^{n} a_i f_i\right|^p dP_1 = \int\left|1 + \sum_{i=1}^{n} a_i g_i\right|^p dP_2\,,$$

$0 < p < +\infty$, $p \neq$ even integer, for all reals $a_1, a_2, \cdots, a_n$. Then $\mu = \nu$.

Proof. Let $n = 1$. Passing from P_1, P_2 to μ, ν respectively,

$$\int|1 + ay|^p d\mu(y) = \int|1 + ay|^p d\nu(y)\,,$$

a is real which implies $\int|x - y|^p d\mu(y) = \int|x - y|^p d\nu(y)$ for x real. Depending on p we have the following cases:

(a) $0 < p < 1$: So we may take $p = \alpha - 1$, $1 < \alpha < 2$. Thus for $f = V\phi$ in Section 2 and Example (2),

$$\begin{aligned}\int f(y)d\mu(y) &= C\iint|x - y|^{\alpha-1}\phi(x)dxd\mu(y)\\ &= C\iint|x - y|^{\alpha-1}\phi(x)d\mu(y)dx\\ &= C\iint|x - y|^{\alpha-1}\phi(x)d\nu(y)dx\\ &= \int f(y)d\nu(y)\end{aligned}$$

for every $f \in \mathscr{D}(A)$, proving $\mu = \nu$ by denseness of $\mathscr{D}(A)$.

(b) $p = 1$: Here we use Example (1), and follow the procedure of the case just preceded.

(c) $1 < p < 2$: So that $p = \alpha + 1$, $0 < \alpha < 1$. For $f \in C_k^\infty(S)$

$$\int Vf(y)d\mu(y) = \iint |x - y|^{\alpha-1} f(x)dxd\mu(y)\ , \quad 0 < \alpha < 1\ ,$$

by Example (2). Integrating twice by parts relative to x, the right hand side is now equal to $\iint |x - y|^{\alpha+1} f''(x)dxd\mu(y)$. Repeating the same operations for ν, $\int Vf(y)d\mu(y) = \int Vf(y)d\nu(y)$, implying $\mu = \nu$.

(d) $2 < p < +\infty$, $p \neq$ even integer: Here we take an even number of derivatives or use integration by parts and reduce this case to one of the preceding cases.

Finally, if $n > 1$ and

$$\int \left|1 + \sum_1^n a_i x_i\right|^p d\mu(x) = \int |1 + \sum a_i x_i|^p d\nu(x)\ ,$$

then it is obvious that $\int \left|1 + a \sum_{i=1}^n a_i x_i\right|^p d\mu = \int \left|1 + a \sum_{i=1}^n a_i x_i\right|^p d\nu$ which by $n = 1$ case implies that $\sum_{i=1}^n a_i f_i$ and $\sum_{i=1}^n a_i g_i$ have the same distribution for all reals $a_1, a_2, \cdots, a_n$. Thus F and G have the distribution; that is $\mu = \nu$.

THEOREM 3.2. *Let F and G be C^n valued measurable functions on Ω_1, Ω_2 respectively. If*

$$\int \left|1 + \sum_{i=1}^n c_i f_i\right|^p dP_1 = \int \left|1 + \sum_{i=1}^n c_i g_i\right|^p dP_2\ ,$$

$0 < p < +\infty$ and $p \neq$ even integer, for all complex numbers $c_1, c_2, \cdots, c_n$. Then $\mu = \nu$.

Proof. We identify C with R^2, and start as before with $n = 1$. Thus $\int |x - y|^p d\mu(x) = \int |x - y|^p d\nu(y)$ with $x = x_1 + ix_2$, $y = y_1 + iy_2$. Recalling [1] that the Markov process at hand has Green's potential, we have for $f \in C_k^{+\infty}(S)$

$$\iint |x - y|^p f(x)dxd\mu(y) = \iint |x - y|^p f(x)dxd\nu(x)\ ,$$

using the hypothesis. There is no problem of integrability, since the hypothesis implies continuity in x. Integrate by parts twice relative to x_1, repeat the same for x_2, and add up. Performing this operation an appropriate number of times

$$\iint |x - y|^{p-2k} g(x) dx d\mu(y) = \iint |x - y|^{p-2k} g(x) dx d\nu(y)$$

and in such a way that $-2 < p - 2k < 0$ or

$$\alpha - 2 = p - 2k \quad \text{and} \quad 0 < \alpha < 2 .$$

Example (3) completes the proof for $n = 1$. If $n > 1$ the proof is similar to that of $n > 1$ in Theorem 3.1.

4. Remarks. (1) Theorems 3.1 and 3.2 may be extended to $-\infty < p < 0$ in the sense of analytic continuation of α, a procedure well known in Riesz potential theory.

(2) As an application of Theorem 3.1 one may prove a real version of Theorem 2 of [5].

(3) The case $p = 1$ of Theorem 3.1 can be proved in an elementary fashion. Consider for an arbitrary positive x,

$$\int |x + \alpha y| d\mu(y) = \int |x + \alpha y| d\nu(y) .$$

Differentiate relative to α at $\alpha = 0$, obtain by Lebesgue's dominated convergence theorem $\int y d\mu(y) = \int y d\nu(y)$. Combine this with the hypothesis $\int (z - y)^+ d\mu(y) = \int (z - y)^+ d\nu(y)$. Integrate both sides then differentiate relative to z, and obtain $\mu = \nu$.

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Received May 20, 1977.

University of Alberta
Edmonton, Alberta, Canada
T6G 2G1