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ON THE METRIC THEORY OF DIOPHANTINE APPROXIMATION

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A conjecture of Duffin and Schaeffer states that

$$\sum_{n=2}^{\infty} \alpha_n \varphi(n) n^{-1} = +\infty$$

is a necessary and sufficient condition that for almost all real x there are infinitely many positive integers n which satisfy $|x - a/n| < \alpha_n n^{-1}$ with $(a, n) = 1$. The necessity of the condition is well known. We prove that the condition is also sufficient if $\alpha_n = O(n^{-1})$.

1. Introduction. Let $\{\alpha_n\}$, $n = 2, 3, 4, \cdots$, be a sequence of real numbers satisfying $0 \leqq \alpha_n \leqq 1/2$. We consider the problem of determining a sufficient condition on the sequence $\{\alpha_n\}$ so that for almost all real x the inequality

$$\left| x - \frac{a}{n} \right| < \frac{\alpha_n}{n} \tag{1.1}$$

holds for infinitely many pairs of relatively prime integers a and n. We note that there is no loss of generality if we restrict x to the interval $I = [0, 1]$. Let λ be Lebesgue measure on I and define

$$E_n = \bigcup_{\substack{a=1 \\ (a,n)=1}}^{n} \left(\frac{a - \alpha_n}{n}, \frac{a + \alpha_n}{n} \right),$$

where (a, n) denotes the greatest common divisor of a and n. Then our problem is to determine a sufficient condition on $\{\alpha_n\}$ so that

$$\lim_{N \to \infty} \lambda \left\{ \bigcup_{n=N}^{\infty} E_n \right\} = 1 \, . \tag{1.2}$$

It is clear that $\lambda(E_n) = 2\alpha_n \varphi(n)/n$ where φ is Euler's function. Thus by the Borel-Cantelli lemma,

$$\sum_{n=2}^{\infty} \lambda(E_n) = 2 \sum_{n=2}^{\infty} \frac{\alpha_n \varphi(n)}{n} = +\infty \tag{1.3}$$

is a *necessary* condition for (1.2) It has been conjectured by Duffin and Schaeffer [4] that (1.3) is also a sufficient condition for (1.2), but this has never been proved. Khintchine [7] showed that if $n\alpha_n$ is a decreasing function of n then (1.3) implies (1.2). (Actually, Khintchine's result is usually stated in a different but equivalent form.) Duffin and Schaeffer [4] improved Khintchine's theorem by showing that if

$$\sum_{n=2}^{N} \frac{\alpha_n \varphi(n)}{n} \geqq c \sum_{n=2}^{N} \alpha_n$$

for some constant $c > 0$ and for arbitrarily large values of N then (1.3) implies (1.2). More recently Erdös [5] proved the following special case of the Duffin-Schaeffer conjecture:

ERDÖS' THEOREM. *If* $\alpha_n = 0$ *or* ε/n *for all* n *and some* $\varepsilon > 0$, *then* (1.3) *implies* (1.2).

In the present paper we generalize Erdös' theorem by proving

THEOREM 1. *If* $\alpha_n = O(n^{-1})$ *then* (1.3) *implies* (1.2).

If the sets E_n were pairwise independent, that is if $\lambda(E_n \cap E_m) = \lambda(E_n)\lambda(E_m)$ for all $n \neq m$, then (1.3) would imply (1.2) by the "divergence part" of the Borel-Cantelli lemma, (Chung [3], Theorem 4.3.2). In general the sets E_n are not pairwise independent. However, by using some weaker bound on $\lambda(E_n \cap E_m)$ we can still deduce the desired result. This is also the approach used in [4] and [5]. We give a simpler treatment of this part of the problem by employing a theorem of Gallagher. Let Z denote a finite subset of $\{2, 3, 4, \cdots\}$ and define $\Lambda(Z)$ by

$$\Lambda(Z) = \sum_{n \in Z} \lambda(E_n) . \tag{1.5}$$

Then we obtain Theorem 1 from

THEOREM 2. *Suppose there exists an integer* $K \geqq 2$ *and a real number* $\eta > 0$ *such that the following condition holds: every finite subset* Z *of* $\{K, K+1, K+2, \cdots\}$ *with* $0 \leqq \Lambda(Z) \leqq \eta$ *also satisfies*

$$\sum_{n \in Z} \sum_{\substack{m \in Z \\ n \neq m}} \lambda(E_n \cap E_m) \leqq \Lambda(Z) .$$

Then (1.3) *implies* (1.2).

Proof. We assume that (1.3) holds. By a result of Gallagher [6], the value of $\lim_{N\to\infty} \lambda\{\bigcup_{n=N}^{\infty} E_n\}$ is either zero or one. We suppose that

$$\lim_{N\to\infty} \lambda\left\{\bigcup_{n=N}^{\infty} E_n\right\} = 0 . \tag{1.6}$$

If $\limsup_{n\to\infty} \lambda(E_n) = \xi > 0$ then $\lambda\{\bigcup_{n=N}^{\infty} E_n\} \geqq \xi$ for all N, which contradicts (1.6). Thus we may assume that

$$\lim_{n\to\infty} \lambda(E_n) = 0 . \tag{1.7}$$

Now choose M so large that

$$\lambda\left\{\bigcup_{n=M}^{\infty} E_n\right\} \leqq \frac{1}{4}\eta .$$

Let $J = \max\{K, M\}$. From (1.3) and (1.7) it follows that there exists a finite subset Z of $\{J, J+1, J+2, \cdots\}$ such that

$$\frac{2}{3}\eta \leqq A(Z) \leqq \eta .$$

But then by a simple sieve argument

$$\begin{aligned}\frac{1}{4}\eta &\geqq \lambda\left\{\bigcup_{n\in Z} E_n\right\}\\ &\geqq \sum_{n\in Z}\lambda(E_n) - \frac{1}{2}\sum_{n\in Z}\sum_{\substack{m\in Z\\ n\neq m}}\lambda(E_n\cap E_m)\\ &\geqq A(Z) - \frac{1}{2}A(Z)\\ &\geqq \frac{1}{3}\eta ,\end{aligned}$$

which is impossible.

The remainder of our paper will consist of showing that if $\alpha_n = O(n^{-1})$ then the hypotheses in Theorem 2 are satisfied. In fact we will prove the following result, which gives a stronger estimate than we require.

THEOREM 3. *If $\alpha_n \leqq Cn^{-1}$ for all n and some $C > 0$ then there exists a real number $\eta_0 > 0$ such that the following condition holds: if Z is a finite subset of $\{2, 3, 4, \cdots\}$ with $0 < A(Z) \leqq \eta_0$, then*

$$\sum_{n\in Z}\sum_{\substack{m\in Z\\ n\neq m}}\lambda(E_n\cap E_m) \ll A(Z)^2(\log\log\{A(Z)^{-1}\})^2 . \tag{1.8}$$

Here, and elsewhere in this paper, the constant implied by $\ll$ is absolute.

Our proof of Theorem 3 is modeled after Erdös' proof in [5]. In §2 we give several lemmas for later use. We then split the sum on the left of (1.8) into three parts which are estimated in §§3 and 4. It is in §4 that the main difficulty occurs. Indeed it is only there that we require the hypothesis $\alpha_n \leqq Cn^{-1}$.

We remark that Catlin [1, 2] has recently found a connection between (1.1) and the problem of approximating almost all x by fractions a/n which are not necessarily reduced. Thus our results also have implications for this problem. We note, however, that the proof of Theorem 3 in [1] contains a serious error.

2. Preliminary lemmas. Throughout the remainder of this paper p will denote a prime. Thus $\sum_{p|n}$ is a sum over prime divisors of n and $\pi(x) = \sum_{p \leqq x} 1$ is the number of primes not exceeding x. For each integer $n \geqq 2$ we define $g(n)$ to be the smallest positive integer v such that

$$\sum_{\substack{p|n \\ p>v}} \frac{1}{p} < 1 .$$

If $g(n) = v$ then

$$\begin{aligned} \prod_{\substack{p|n \\ p\leqq v}} \left(1 - \frac{1}{p}\right) &= \frac{\varphi(n)}{n} \prod_{\substack{p|n \\ p>v}} \left(1 - \frac{1}{p}\right)^{-1} \\ &\leqq \frac{\varphi(n)}{n} \exp\left\{\sum_{\substack{p|n \\ p>v}} \frac{1}{p} + \sum_p \sum_{j=2}^{\infty} j^{-1}p^{-j}\right\} \qquad (2.1) \\ &\ll \frac{\varphi(n)}{n} . \end{aligned}$$

It follows from the theorem of Mertens that

$$1 \ll \frac{\varphi(n)}{n} \log(1 + v) . \qquad (2.2)$$

Next let $\xi > 0$, $x > 0$ and let v be a positive integer. We define $N(\xi, v, x)$ to be the number of integers $n \leqq x$ which satisfy

$$\sum_{\substack{p|n \\ p\geqq v}} \frac{1}{p} \geqq \xi . \qquad (2.3)$$

We then have the following estimate of Erdös [5].

LEMMA 4. *For any* $\varepsilon > 0$ *and* $\xi > 0$ *there exists a positive integer* $v_0 = v_0(\xi, \varepsilon)$ *such that for all* $x > 0$ *and all* $v \geqq v_0$,

$$N(\xi, v, x) \leqq x \exp\{-v^{\beta(1-\varepsilon)}\} \qquad (2.4)$$

where $\log \beta = \xi$.

Proof. We may assume that $0 < \varepsilon < (1 - e^{-\xi})$. Let

$$p_1 < p_2 < \cdots < p_M$$

be the set of all primes in $[v, w]$, where $w = v^{\beta(1-\varepsilon/3)}$. If v is sufficiently large then $M \geqq \pi(w) - \pi(v) \geqq v^{\beta(1-2\varepsilon/3)}$.

We split the integers $n \leqq x$ which satisfy (2.3) into two classes. In the first class are integers n with M prime factors in the interval $[v, \exp(w)]$. The number of such integers is clearly less than

$$x\left(\sum_{v \leqq p \leqq \exp(w)} \frac{1}{p}\right)^M / M! \leqq x(c_1 \log w)^M / M!$$

for some constant $c_1 > 0$. Using Stirling's formula this is easily seen to be

$$\ll x \exp(-M) \ll x \exp\{-v^{\beta(1-2\varepsilon/3)}\} \tag{2.5}$$

for sufficiently large v.

Next we observe that

$$\begin{aligned}\sum_{j=1}^{M} \frac{1}{p_j} = \sum_{v \leqq p \leqq w} \frac{1}{p} &= \log\left(\frac{\log w}{\log v}\right) + o(1) \\ &= \xi + \log(1 - \varepsilon/3) + o(1) \\ &\leqq \xi - \varepsilon/3\end{aligned} \tag{2.6}$$

for sufficiently large v. The integers $n \leqq x$ which satisfy (2.3) and which have fewer than M prime factors in $[v, \exp(w)]$ must therefore satisfy

$$\frac{3}{\varepsilon} \sum_{\substack{p \mid n \\ p > \exp w}} \frac{1}{p} \geqq 1 .$$

The number of such integers n is

$$\begin{aligned}&\leqq \frac{3}{\varepsilon} \sum_{n \leqq x} \sum_{\substack{p \mid n \\ p > \exp w}} \frac{1}{p} = \frac{3}{\varepsilon} \sum_{p > \exp w} \frac{1}{p}\left[\frac{x}{p}\right] \\ &\ll \frac{x}{\varepsilon} \sum_{p > \exp w} \frac{1}{p^2} \ll \frac{x}{\varepsilon} \exp(-w) .\end{aligned} \tag{2.7}$$

The bound (2.4) now follows from (2.5) and (2.7).

We now suppose that $g(n) = u \leqq v$. For each $\xi > 0$ we split the divisors d of n into two classes, $A_n(\xi, v)$ and $B_n(\xi, v)$. We say that d is in $A_n(\xi, v)$ if

$$\sum_{\substack{p \mid d \\ p \geqq v}} \frac{1}{p} \geqq \xi . \tag{2.8}$$

The class $B_n(\xi, v)$ consists of divisors which do not satisfy (2.8).

LEMMA 5. *For any* $\varepsilon > 0$ *and any* $\xi > 0$ *there exists a positive integer* $v_0 = v_0(\xi, \varepsilon)$ *such that if* $g(n) = u \leqq v$ *and* $v \geqq v_0$ *then*

$$\sum_{d \in A_n(\xi, v)} \frac{1}{d} \leqq (\log(1 + u)) \exp\{-v^{\beta(1-\varepsilon)}\} \tag{2.9}$$

where $\log \beta = \xi$.

Proof. Let v, w and M be as in the proof of Lemma 4. For any collection $\mathscr{P}$ of M primes in $[v, \infty)$ we have

$$\sum_{p \in \mathscr{P}} \frac{1}{p} \leqq \sum_{j=1}^{M} \frac{1}{p_j} \leqq \xi - \varepsilon/3$$

for sufficiently large v, as in (2.6). Thus if $d \in A_n(\xi, v)$ then d must have at least M prime factors in $[v, \infty)$. Let $q_1, q_2, \cdots, q_J$ be the prime factors of n which are greater than or equal to v. If $J \leqq M$ then $A_n(\xi, v)$ is empty. Otherwise

$$\sum_{d \in A_n(\xi, v)} \frac{1}{d} \leqq \Big(\sum_{d|n} \frac{1}{d}\Big)\Big(\sum_{j=1}^{J} \frac{1}{q_j}\Big)^M / M! \ .$$

Since $g(n) = u \leqq v$ we have

$$\Big(\sum_{j=1}^{J} \frac{1}{q_j}\Big)^M / M! \leqq (M!)^{-1} \ll \exp\{-v^{\beta(1-2\varepsilon/3)}\} \ .$$

Also,

$$\sum_{d|n} \frac{1}{d} \leqq \prod_{p \leqq u} \Big(1 - \frac{1}{p}\Big)^{-1} \prod_{\substack{p|n \\ p>u}} \Big(1 - \frac{1}{p}\Big)^{-1} \ll \log(1 + u)$$

by the theorem of Mertens.

Let $\sum_{m(v)}$ denote a sum over integers m which satisfy $g(m) = v$.

LEMMA 6. *Let* $\varepsilon > 0$. *Then there exists a constant* $v_0 = v_0(\varepsilon)$ *such that the following inequalities hold: if* $x > 0$ *and* $y \geqq 2$, *if* $g(n) = u \leqq v$ *and* $v \geqq v_0$, *then*

$$\sum_{\substack{m(v) \\ (n,m)x < m < (n,m)xy}} m^{-1} \leqq (\log 1 + u)(\log y) \exp\{-v^{\beta(1-\varepsilon)}\} \tag{2.10}$$

and

$$\sum_{\substack{m(v) \\ (n,m)^{-1}x < m < (n,m)^{-1}xy}} m^{-1} \leqq (\log 1 + u)(\log y) \exp\{-v^{\beta(1-\varepsilon)}\} \ , \tag{2.11}$$

where $\beta = e^{1/2}$.

Proof. The proofs of the two inequalities are virtually identical, so we prove only (2.10). We have

$$\sum_{\substack{m(v) \\ (n,m)x<m<(n,m)xy}} m^{-1} = \sum_{d|n} \sum_{\substack{m(v) \\ (n,m)=d \\ dx<m<dxy}} m^{-1} .$$

If $(n, m) = d$ we write $m = dm'$. Then by Lemma 5 with $\xi = 1/2$,

$$\begin{aligned} \sum_{d \in A_n(1/2,v)} \sum_{\substack{m(v) \\ (n,m)=d \\ dx<m<dxy}} m^{-1} &\leqq \sum_{d \in A_n(1/2,v)} d^{-1} \sum_{\substack{m' \\ x<m'<xy}} (m')^{-1} \\ &\leqq (\log 1 + u)(\log y) \exp\{-v^{\beta(1-\varepsilon/2)}\} , \end{aligned}$$

for sufficiently large v.

If $d \in B_n(1/2, v)$ then

$$\begin{aligned} 1 \leqq \sum_{\substack{p|m \\ p\geqq v}} p^{-1} &\leqq \sum_{\substack{p|d \\ p\geqq v}} p^{-1} + \sum_{\substack{p|m' \\ p\geqq v}} p^{-1} \\ &< \frac{1}{2} + \sum_{\substack{p|m' \\ p\geqq v}} p^{-1} , \end{aligned}$$

and so

$$\sum_{\substack{p|m' \\ p|\geqq v}} p^{-1} > \frac{1}{2} . \tag{2.12}$$

By Lemma 4

$$\sum_{\substack{m' \\ n<m'\leqq 2x}} (m')^{-1} \leqq x^{-1} N\left(\frac{1}{2}, v, 2x\right) \leqq 2 \exp\{-v^{\beta(1-\varepsilon/2)}\} \tag{2.13}$$

for sufficiently large v, where the sum on the left of (2.13) is over m' satisfying (2.12). Hence

$$\begin{aligned} \sum_{d \in B(1/2,v)} \sum_{\substack{m(v) \\ (n,m)=d \\ dx<m<dxy}} m^{-1} &\leqq \sum_{d \in B_n(1/2,v)} d^{-1} \sum_{\substack{m' \\ x<m'<xy}} (m')^{-1} \\ &\leqq \sum_{d \in B_n(1/2,v)} d^{-1}(\log y) \exp\{-v^{\beta(1-\varepsilon/2)}\} \\ &\ll \log(1 + u)(\log y) \exp\{-v^{\beta(1-\varepsilon/2)}\} \end{aligned}$$

for sufficiently large v.

3. **First estimates.** In this section we begin our proof of Theorem

3. For $n \neq m$ we define

$$\delta = \delta(n, m) = 2 \min \left\{\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right\},$$

$$\Delta = \Delta(n, m) = 2 \max \left\{\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right\},$$

and

$$t = t(n, m) = \max \{g(n), g(m)\}.$$

We write $\sum_{a=1}^{n*}$ and $\sum_{b=1}^{m*}$ for sums over integers prime to n and m respectively. Thus

$$\begin{aligned} \lambda(E_n \cap E_m) &= \sum_{a=1}^{n*} \sum_{b=1}^{m*} \lambda\left\{\left(\frac{a-\alpha_n}{n}, \frac{a+\alpha_n}{n}\right) \cap \left(\frac{b-\alpha_m}{m}, \frac{b+\alpha_m}{m}\right)\right\} \\ &\leqq \delta(n, m) \sum_{\substack{a=1 \\ |a/n-b/m|<\Delta(n,m)}}^{n*} \sum_{b=1}^{m*} 1 \\ &= \delta \sum_{\substack{a=1 \\ |am-bn|<nm\Delta}}^{n*} \sum_{b=1}^{m*} 1 . \end{aligned} \tag{3.1}$$

For each integer u we define $H(u)$ to be the number of pairs $\{a, b\}$ which satisfy

$$\begin{aligned} &am - bn = u, \quad 1 \leqq a \leqq n, \quad (a, n) = 1, \\ &1 \leqq b \leqq m, \quad (b, m) = 1 . \end{aligned}$$

From (3.1) it follows that

$$\lambda(E_n \cap E_m) \leqq \delta \sum_{\substack{u \\ |u|<nm\Delta}} H(u) . \tag{3.2}$$

Let $d = (n, m)$. It is clear that $H(0) = 0$ and if $d \nmid u$ then $H(u) = 0$. Thus in estimating the right hand side of (3.2) we may assume that

$$d < nm\Delta \tag{3.3}$$

and restrict ourselves to integers u which are divisible by d. We write $|u| = dd_u u_1$, where the prime divisors of d_u also divide d and $(d, u_1) = 1$. Obviously this decomposition is unique. It is shown in [5] that if either $(u_1, nmd^{-1}) > 1$ or $(d_u, nmd^{-2}) > 1$ then $H(u) = 0$. Hence we may further restrict ourselves to integers u which satisfy

$$(u_1, nmd^{-1}) = (d_u, nmd^{-2}) = 1 . \tag{3.4}$$

For such u we have the estimate

$$
\begin{aligned}
H(u) &\leqq d \prod_{\substack{p \mid d \\ p \nmid d_u nmd^{-2}}} \left(1 - \frac{2}{p}\right) \prod_{\substack{p \mid d \\ p \mid d_u nmd^{-2}}} \left(1 - \frac{1}{p}\right) \\
&\leqq \varphi(d) \prod_{\substack{p \mid d \\ p \nmid nmd^{-2}}} \left(1 - \frac{1}{p}\right) \prod_{p \mid d_u} \left(1 - \frac{1}{p}\right)^{-1}
\end{aligned} \tag{3.5}
$$

from [5].

Next let $\mathscr{P}_0$ be the set of primes p which divide d but do not divide nmd^{-2}. We split $\mathscr{P}_0$ into disjoint subsets $\mathscr{P}_1$ and $\mathscr{P}_2$ consisting of primes satisfying $p \leqq t$ and $p > t$ respectively. Let $\mathscr{S}_j$ be the set of positive integers whose prime divisors are in $\mathscr{P}_j$, for $j = 0, 1, 2$. From (3.4) we may assume that $d_u \in \mathscr{S}_0$ and hence that d_u is uniquely represented as $d_u = s_1 s_2$ with $s_1 \in \mathscr{S}_1$ and $s_2 \in \mathscr{S}_2$. Thus

$$
\begin{aligned}
H(u) &\leqq \varphi(d) \prod_{p \in \mathscr{P}_0} \left(1 - \frac{1}{p}\right) \prod_{p \mid s_1 s_2} \left(1 - \frac{1}{p}\right)^{-1} \\
&\leqq \varphi(d) \prod_{p \in \mathscr{P}_1} \left(1 - \frac{1}{p}\right) \prod_{p \mid s_1} \left(1 - \frac{1}{p}\right)^{-1} .
\end{aligned} \tag{3.6}
$$

Now $|u| = dd_u u_1 = ds_1 s_2 u_1$ where the set of primes which divide s_1, s_2, and u_1 are all distinct. Therefore if we set $k = s_2 u_1$ then k is relatively prime to

$$
Q = \prod_{\substack{p \mid nmd^{-1} \\ p \leqq t}} p
$$

by (3.4) and the definition of $\mathscr{P}_2$. From (3.2) and (3.6) we obtain

$$
\begin{aligned}
\lambda(E_n \cap E_m) &\leqq \delta \sum_{\substack{u \\ |u| < nm\Delta}} H(u) \\
&= \delta \sum_{s_1 \in \mathscr{S}_1} \sum_{1 \leqq k \leqq (nm\Delta / ds_1)}^{*} \{H(-ds_1 k) + H(ds_1 k)\} \\
&\leqq 2\delta\varphi(d) \prod_{p \in \mathscr{P}_1} \left(1 - \frac{1}{p}\right) \left\{ \sum_{s_1 \in \mathscr{S}_1} \prod_{p \mid s_1} \left(1 - \frac{1}{p}\right)^{-1} \sum_{1 \leqq k \leqq (nm\Delta / ds_1)}^{*} 1 \right\} ,
\end{aligned} \tag{3.7}
$$

where $(k, Q) = 1$ in the sum $\sum^*$.

By the prime number theorem there exists an absolute constant b such that

$$
\pi(y) \log 2y + \log\log y \leqq y \log 3 \tag{3.8}
$$

for all $y \geqq b$. Throughout the remainder of this section we shall assume that

$$
t = t(n, m) \geqq b \quad \text{and} \quad nm\Delta \geqq 3^t d . \tag{3.9}
$$

Then by the sieve of Erathosthenes

$$\sum_{1\leqq k\leqq(nm\Delta/ds_1)}^{*} 1 \leqq \frac{nm\Delta}{ds_1}\prod_{p\mid Q}\left(1-\frac{1}{p}\right)+2^{\pi(t)} .$$

If $s_1 \leqq t^{\pi(t)}$ then using (3.8) and (3.9) we have

$$\begin{aligned} 2^{\pi(t)} &\leqq 3^t t^{-\pi(t)} \log^{-1} t \\ &\ll \frac{nm\Delta}{ds_1}\prod_{p\mid Q}\left(1-\frac{1}{p}\right). \end{aligned}$$

It follows that if we sum over $s_1 \leqq t^{\pi(t)}$ on the right hand side of (3.7) we obtain the upper bound

$$\begin{aligned} &2\delta\varphi(d)\prod_{p\in\mathscr{P}_1}\left(1-\frac{1}{p}\right)\sum_{s_1\leqq t^{\pi(t)}} s_1\varphi(s_1)^{-1}\sum_{1\leqq k\leqq(nm\Delta/ds_1)}^{*} 1 \\ &\ll \delta\frac{\varphi(d)}{d}nm\Delta\prod_{p\mid Q}\left(1-\frac{1}{p}\right)\prod_{p\in\mathscr{P}_1}\left(1-\frac{1}{p}\right)\sum_{s_1\in\mathscr{P}_1}\varphi(s_1)^{-1} \\ &\ll \alpha_n\alpha_m\frac{\varphi(d)}{d}\prod_{p\mid nmd^{-1}}\left(1-\frac{1}{p}\right)\prod_{p\in\mathscr{P}_1}\left(1+\frac{1}{p(p-1)}\right) \\ &\ll \frac{\alpha_n\varphi(n)}{n}\frac{\alpha_m\varphi(m)}{m} \ll \lambda(E_n)\lambda(E_m) . \end{aligned} \tag{3.10}$$

Now if $s_1 > t^{\pi(t)}$ we easily see that for some prime $p \in \mathscr{P}_1$ and some integer $\gamma \geqq 2$ we must have $p^\gamma \mid s_1$, $p^\gamma > t$ $p \leqq t$. By considering the cases where γ is even or odd it follows that s_1 is divisible by a square greater than $t^{2/3}$. Thus summing over $s_1 > t^{\pi(t)}$ in (3.7) we obtain

$$\begin{aligned} &2\delta\varphi(d)\prod_{p\in\mathscr{P}_1}\left(1-\frac{1}{p}\right)\sum_{t^{\pi(t)}<s_1} s_1\varphi(s_1)^{-1}\sum_{1\leqq k\leqq(nm\Delta/ds_1)}^{*} 1 \\ &\leqq 2\delta\varphi(d)\sum_{t^{\pi(t)}<s_1}\sum_{1\leqq k\leqq(nm\Delta/ds_1)} 1 \\ &\leqq 2\delta\varphi(d)\sum_{r=[t^{1/3}]}^{\infty}\sum_{\substack{1\leqq j\leqq(nm\Delta/d)\\ r^2\mid j}} 1 \\ &\leqq 2\delta\varphi(d)\frac{nm\Delta}{d}\sum_{r=[t^{1/3}]}^{\infty} r^{-2} \\ &\ll \alpha_n\alpha_m t^{-1/3} \ll \left(\alpha_n\frac{\varphi(n)}{n}\right)\left(\alpha_m\frac{\varphi(m)}{m}\right)t^{-1/3}\log^2 t \\ &\ll \lambda(E_n)\lambda(E_m) . \end{aligned} \tag{3.11}$$

Putting the estimates in (3.10) and (3.11) together, it follows that

$$\lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m) \tag{3.12}$$

for all pairs $\{n, m\}$, $n \neq m$, which satisfy (3.9).

4. **Second estimates.** Let Z be a finite subset of $\{2, 3, 4, \cdots\}$ with $\Lambda(Z)$ defined by (1.5). We choose ε in Lemma 6 so that $e^{1/2}(1-\varepsilon) = 3/2$. This determines an absolute constant v_0 such that (2.10) and (2.11) hold for all $v \geqq v_0$. We then define η_0 by

$$\eta_0 = \exp\{-\max(b, C, v_0)\} \tag{4.1}$$

and assume that $0 < \Lambda(Z) \leqq \eta_0$.

Next we write

$$\sum_{\substack{n \in Z \\ n \neq m}} \sum_{m \in Z} \lambda(E_n \cap E_m) = S_1 + S_2$$

where S_1 is the sum over pairs $\{n, m\}$ which satisfy (3.9) and S_2 is the sum over the remaining pairs $\{n, m\}$ which do not satisfy (3.9). We apply (3.12) to obtain the estimate

$$S_1 \ll \sum_{n \in Z} \sum_{m \in Z} \lambda(E_n)\lambda(E_m) = \Lambda(Z)^2 . \tag{4.2}$$

Thus it remains only to bound S_2.

From (2.2) and (3.7) we have

$$\begin{aligned} &\lambda(E_n \cap E_m) \\ &\leqq 2\delta\varphi(d)\frac{nm\Delta}{d}\prod_{p_1 \in \mathscr{P}_1}\left(1 - \frac{1}{p_1}\right)\sum_{s_1 \in \mathscr{S}_1}\varphi(s_1)^{-1} \\ &\ll \alpha_n\alpha_m \ll \log^2(1+t)\lambda(E_n)\lambda(E_m) . \end{aligned} \tag{4.3}$$

Hence if we set $L = -\log\{\Lambda(Z)\}$ and sum over pairs $\{n, m\}$ which satisfy $t < L$ we obtain

$$\begin{aligned} &\sum_{\substack{n \in Z \\ n \neq m \\ t<L}} \sum_{m \in Z} \lambda(E_n \cap E_m) \\ &\ll \Lambda(Z)^2(\log\log\{\Lambda(Z)^{-1}\})^2 . \end{aligned} \tag{4.4}$$

Now for any pair $\{n, m\}$ in the sum S_2 we have either $t < b$ or $nm\Delta < 3^t d$, where $d = (n, m)$. But from (4.1) we have $b \leqq L$ so that terms for which $t < b$ are already included in (4.4). Therefore the only sum which we need to bound is

$$S_3 = \sum_{\substack{n \in Z \\ n \neq m}} \sum_{m \in Z} \lambda(E_n \cap E_m)$$

where each pair $\{n, m\}$ satisfies $t \geqq L$ and (using (3.3))

$$d < nm\Delta < 3^t d . \tag{4.5}$$

We have

$$S_3 \ll \sum_{v=L}^{\infty} \sum_{u=1}^{v} \Big\{ \sum_{\substack{n^{(u)} \\ (n,m)<nm\Delta<3^v(n,m)}} \sum_{m^{(v)}} \alpha_n \alpha_m \Big\}$$

$$\ll \sum_{v=L}^{\infty} \log(1+v) \sum_{u=1}^{v} \Big\{ \sum_{n^{(u)}} \lambda(E_n) \sum_{\substack{m^{(v)} \\ (n,m)<nm\Delta<3^v(n,m)}} \alpha_m \Big\}, \tag{4.6}$$

where we have used (2.2) and (4.3). Our objective it to establish

$$\sum_{\substack{m^{(v)} \\ (n,m)<nm\Delta<3^v(n,m)}} \alpha_m \ll Cv(\log 1+v)\exp\{-v^{3/2}\} \tag{4.7}$$

for the sums on the right of (4.6), that is for fixed n, $g(n) = u \leqq v$ and $v \geqq v_0$. To accomplish this we consider two cases.

If $\alpha_m/m \leqq \alpha_n/n$ then the condition $(n, m) < nm\Delta < 3^v(n, m)$ becomes $(n, m) < 2m\alpha_n < 3^v(n, m)$. Clearly we may assume that $\alpha_n > 0$ so that by (2.10) we have

$$\sum_{\substack{m^{(v)} \\ (n,m)<2m\alpha_n<3^v(n,m)}} \alpha_m \leqq C \sum_{\substack{m^{(v)} \\ (n,m)<2m\alpha_n<3(n,m^v)}} m^{-1}$$
$$\ll Cv(\log 1+v)\exp\{-v^{3/2}\}.$$

If $\alpha_n/n < \alpha_m/m$ then the condition becomes

$$(n, m) < 2n\alpha_m < 3^v(n, m). \tag{4.8}$$

Since $\alpha_k \leqq Ck^{-1}$ we may partition Z into disjoint classes W_j, $j = 0, 1, 2, \cdots$, defined by

$$W_j = \{k \in Z: C2^{-j-1} < k\alpha_k \leqq C2^{-j}\}.$$

If $m \in W_j$ and m satisfies (4.8) then we have

$$2^{-1}(n, m) < nm^{-1}C2^{-j} < 3^v(n, m)$$

and so

$$C2^{-j}n3^{-v}(n, m)^{-1} < m < C2^{1-j}n(n, m)^{-1}. \tag{4.9}$$

Therefore we may apply (2.11) with $x = C2^{-j}n3^{-v}$ and $y = 2(3^v)$ to obtain

$$\sum_{\substack{m^{(v)} \\ (n,m)<2n\alpha_m<3^v(n,m)}} \alpha_m \leqq C \sum_{j=0}^{\infty} 2^{-j} \sum_{\substack{m^{(v)} \\ m \in W_j}}{}^{*} m^{-1}$$
$$\ll Cv(\log 1+v)\exp\{-v^{3/2}\},$$

where $\sum^*$ indicates a sum over m's which satisfy (4.9). This proves (4.7).

By using (4.1), (4.6), and (4.7) we find that

$$\begin{aligned}(4.10)\qquad S_3 &\ll C\sum_{v=L}^{\infty} v(\log 1+v)^2 \exp\{-v^{3/2}\}\sum_{u=1}^{\infty}\sum_{n(u)}\lambda(E_n)\\ &\ll \exp\{-L\}\Lambda(Z) = \Lambda(Z)^2 .\end{aligned}$$

The three upper bounds (4.2), (4.4), and (4.10) now establish (1.8) and so complete the proofs of Theorem 3 and Theorem 1.

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