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ON THE STRUCTURE OF $B_\infty(F)$, F A STABLE SPACE

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For a given space F, let $F \to E_\infty(F) \to B_\infty(F)$ be the classifying fibration for fibre homotopy equivalence classes of fibrations with fibre F. The usual theoretical construction of this fibration offers little insight into its structure homotopically. Below we study this structure under the hypothesis that F has homotopy concentrated in a stable range. As an application of this study, for F a stable two stage Postnikov system determined by a Steenrod operation, we obtain explicit descriptions of the spaces $E_\infty(F)$ and $B_\infty(F)$.

Our study is based on two observations about these spaces for such stable F. Firstly, $B_\infty(F)$ "splits" between $E_\infty(F)$ and $B(F)$ [12]. Secondly, the finite Postnikov decomposition of F will be shown to translate into a finite filtration of $E_\infty(F)$. The homotopy spectral sequence of this filtration is relatively accessible.

In the first section we review certain constructions that will be needed in the sequel.

The second section is devoted to setting up the spectral sequence mentioned above. Restricting attention to two and three stage Postnikov systems, we are able to use this spectral sequence to obtain information about the homotopy of $B_\infty(F)$.

Finally, in the third section we consider stable two stage systems where it is possible to give a fairly complete geometric description of $E_\infty(F)$ and $B_\infty(F)$. Again, when the system is determined by a Steenrod operation the description is made precise. We show that $E_\infty(F)$ is a certain fibre product of spaces of type $L(\pi, n)$ [9] and $B_\infty(F)$ is essentially the total space of a fibration over $E_\infty(F)$ determined by a single k-invariant for which a formula is given.

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1. Preliminaries.

We assume that all of the constructions and computations below take place in an appropriate topological setting (e.g., [6]). We will be interested in fibrations with a given cross-section. James [4] would consider the first constructions to be a fragment of ex-Postnikov theory.

NOTATION 1.1. (a) Let $\mathscr{E} = (E, p, B)$ be a fibration with fibre F. For a given space X we let $E^X = \{f: X \to E \mid pf = \text{const.}\}$. $\mathscr{E}^X = (E^X, \tilde{p}, B)$ is a fibration with fibre $\operatorname{map}(X, F)$ and with $\tilde{p}$ defined by the formula $\tilde{p}(f) = p(f(x))$.

(b) Let $s\colon B \to E$ be a cross-section for $\mathscr{E}$ and let x_0 be a fixed point of X. For each $b \in B$ we have a preferred base point $s(b) \in p^{-1}(b)$. Let $E_s^X = \{f \in E^X \mid f(x_0) = s(b) \text{ where } pf(x_0) = b\}$. $\mathscr{E}_s^X = (E_s^X, \tilde{p}, B)$ is a fibration with fibre the base pointed maps of X to F. $\mathscr{E}_s^X$ has a preferred cross-section of constant maps.

Given a map we may replace it by a fibration using a well known construction. We now wish to observe that this may also be carried out in the setting of ex-homotopy theory.

Suppose we are given a commutative diagram of fibrations. (Not necessarily with the same fibre.)

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"] & \bar{E} \arrow[d, "\bar{p}"] \\ B \arrow[r, "f"] & \bar{B} \end{tikzcd}$$

Define $E_{\bar{f}} = \{(e, h) \in E \times \bar{E}^I \mid \bar{f}(e) = h(0)\}$ and let $\hat{p}\colon E_{\bar{f}} \to B$ be given by the formula $\hat{p}(e, h) = p(e)$.

LEMMA 1.2. *$\mathscr{E}_{\bar{f}} = (E_{\bar{f}}, \hat{p}, B)$ is a fibration fibre homotopy equivalent to $\mathscr{E}$.*

The proof is straightforward as is the proof of the lemma that follows the next definition.

DEFINITION 1.3. Suppose we are given cross-sections s and $\bar{s}$ of $\mathscr{E}$ and $\bar{\mathscr{E}}$ respectively. Define $E'_{\bar{f}} = \{(e, h) \in E_{\bar{f}} \mid \bar{s}fp(e) = h(1)\}$ and let p' be the restriction of $\hat{p}$ to $E'_{\bar{f}}$.

LEMMA 1.4. *$\mathscr{E}'_{\bar{f}} = (E'_{\bar{f}}, p', B)$ is a fibration with cross-section s' and the fibre of $\mathscr{E}'_{\bar{f}}$ is the fibre of the map $\bar{f}\colon F \to \bar{F}$.*

Note that we have the following commutative diagram of fibrations with cross-section.

$$\begin{tikzcd} E'_{\bar{f}} \arrow[r, "i"] \arrow[d, "p'"] & E_{\bar{f}} \arrow[r, "\bar{f}"] \arrow[d, "\hat{p}"] & \bar{E} \arrow[d, "\bar{p}"] \\ B \arrow[r, equal] & B \arrow[r, "f"] & \bar{B} \end{tikzcd}$$

We will denote such diagrams by the notation

$$\mathscr{E}'_{\bar{f}} \xrightarrow{(i,1)} \mathscr{E}_{\bar{f}} \xrightarrow{(\bar{f},f)} \bar{\mathscr{E}} .$$

We will also write $\mathscr{E}$ (resp. $\mathscr{E}'$) interchangeably for $\mathscr{E}_{\bar{f}}$ (resp. $\mathscr{E}'_{\bar{f}}$) taking the appropriate meaning from context.

The following application of 1.4 is basic to much of what follows.

THEOREM 1.5. *Let F, the fibre of $\mathscr{E}$, be $(n-1)$-connected and let $t: F \to K(\pi, n)$ represent the fundamental class of $H^n(F, \pi)$ where $\pi_n(F) = \pi$.*

(a) *There exists a fibration $\mathscr{E}_\pi = (E_\pi, p_\pi, B)$ with cross-section and with fibre $K(\pi, n)$, and a map of fibrations $\mathscr{E} \xrightarrow{(\bar{t}, 1)} \mathscr{E}_\pi$ with $\bar{t} \mid F = t$.*

(b) *The fibre of $\mathscr{E}'$ in the sequence*

$$\mathscr{E}' \xrightarrow{(i, 1)} \mathscr{E} \xrightarrow{(\bar{t}, 1)} \mathscr{E}_\pi$$

is the result of killing the lowest nonzero homotopy group of F.

Proof. (See also [7].)

The existence of $\bar{t}$ is a corollary of 1.6 of [11]. In particular, $\mathscr{E}_\pi$ is the first stage of the twisted Postnikov decomposition of $\mathscr{E}$.

In applying 1.6 of [11] one notes that the existence of a cross-section implies that the twisted k-invariant $\tau(i) = 0$ and that it suffices to set $s_\pi = ts$. Note that $\mathscr{E}_\pi$ need not be a product. To see this one notes that the vanishing of the twisted k-invariant does not have the usual geometric meaning that the corresponding map of spaces is trivial (see 1.5 of [11]).

The Functors H and H_0

We now recall some facts about classifying spaces of fibre homotopy equivalence classes of fibrations.

Let $H(X, F)$ be the set of fibre homotopy equivalence classes of fibrations over X with fibre F. H has a classifying space $B_\infty(F)$ and universal fibration $\mathscr{E}_\infty(F) = (E_\infty(F), p_\infty, B_\infty(F))$ [1, 2].

One also has $H_0(X, F)$, the set of base cross-section preserving fibre homotopy equivalences. In [12] we established the following information about $H_0(-, F)$ (see also [3]).

THEOREM 1.6. (a) *$H_0(-, F)$ is classified by a space $B_\dagger(F)$.*

(b) *$B_\dagger(F) = E_\infty(F)$, the total space of the universal fibration over $B_\infty(F)$. Moreover, a classifying fibration $\mathscr{E}_+(F)$ for H_0 is given by pulling $\mathscr{E}_\infty(F)$ back along p_∞ (see [3]).*

(c) *If $\pi_i(F) = 0$ for $i < n$ and $i > 2n-3$ for some n (such an*

F is called stable) then the map $B_\dagger(F) \xrightarrow{p_\infty} B_\infty(F)$ admits a retraction $B_\infty(F) \xrightarrow{r} B_\dagger(F)$.

The following theorem follows at once from the proof of 1.6 c in [12] and should have been stated there (see 2.3 and 2.13 of [12]). Since stable spaces are associative H-spaces we may form principal F-bundles. Hence,

THEOREM 1.7. *Let F be stable and $B(F)$ be the classifying space for principle F-bundles. Let $B(F) \xrightarrow{\rho} B_\infty(F)$ classify $B(F)$ as a fibration. Then the fibre of the map r is $B(F)$ and $B(F) \xrightarrow{\rho} B_\infty(F) \xrightarrow{r} B_\dagger(F)$ is a universal fibration for $H_0(-, B(F))$.*

Since $rp_\infty \sim Id$ 1.7 implies:

COROLLARY 1.8. $\pi_n(B_\infty(F)) \cong \pi_n(B(F)) \oplus \pi_n(B_\dagger(F))$.

In fact, 1.8 holds when F is an associative H-space [12].

NOTATION 1.9. In all that follows we will be considering fibrations with 1-connected fibres F, F'. We will want to look at certain subspaces of various mapping spaces. We adopt the following notation:

(a) Let $\langle F', F\rangle$ denote the subspace of the space of maps of F' to F such that $f \in \langle F', F\rangle$ if and only if $f_*: \pi_n(F') \cong \pi_n(F)$ for all n such that both $\pi_n(F')$ and $\pi_n(F)$ are not trivial.

(b) If F' and F have specified base points then $\langle F', F\rangle_0 \subseteqq \langle F', F\rangle$ denotes the subspace of base point preserving maps.

(c) For a fibration $\mathscr{E} = (E, p, B)$ with fibre F, let $\mathscr{E}\langle F', F\rangle = (E\langle F', F\rangle, \tilde{p}, B)$ denote the fibration with fibre $\langle F', F\rangle$ and total space $E\langle F, F\rangle = \bigcup_{x \in B} \langle F', p^{-1}(x)\rangle \subseteqq E_F$.

(d) If $\mathscr{E}$ has a base section and F' a given base point, let $\mathscr{E}\langle F, F\rangle_0 = (E\langle F', F\rangle_0, \tilde{p}, B)$ denote the obvious fibration with fibre $\langle F', F\rangle_0$ and base section of constant maps.

(e) Finally if $F = F'$ we write $\mathscr{E}\langle F\rangle$ (resp. $\mathscr{E}\langle F\rangle_0$) for $\mathscr{E}\langle F, F\rangle$ (resp. $\mathscr{E}\langle F, F\rangle_0$).

We complete this section by recording two more or less well known facts.

THEOREM 1.10. [1], [3]. *Let $(E_\infty(F), p_\infty, B_\infty(F))$ and $(E_\dagger(F), p_\dagger, B_\dagger(F))$ be as 1.6 then $E_\infty\langle F\rangle$ and $E_\dagger\langle F\rangle_0$ have vanishing homotopy.*

1.11. Next, let (E, p, B) be a fibration, let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$ be given base points and let $\theta: \Omega B \times I \to B$ be the evaluation map

($\theta(l, t) = l(t)$). Let $\bar{\theta}\colon \Omega B \times I \to E$ be a lifting of θ such that $\bar{\theta} \mid \Omega\beta \times 0 = e_0$. Finally define $d\colon \Omega B \to F$ by $d(l) = \bar{\theta}(l, 1)$. d is called a geometric boundary map and is unique up to homotopy class if F is path connected.

LEMMA 1.12. *Suppose we are given the following homotopy commutative diagram of fibrations*

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' . \end{array}$$

Let $\Omega B \xrightarrow{d} F$ and $\Omega B' \xrightarrow{d'} F'$ be geometric boundary maps then the diagram

$$\begin{array}{ccc} \Omega B & \xrightarrow{d} & F \\ {\scriptstyle \Omega f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \Omega B' & \xrightarrow{d'} & F' \end{array}$$

homotopy commutes.

2. A Filtration of the space $B_\dagger(F)$. In this section we make use of 1.5 to give a filtration of the space $B_\dagger(F)$ by fibrations. We then analyze the associated homotopy spectral sequence, first formally then in the case F stable. In the stable case the E_1 and E_2 terms of the spectral sequence are seen to be computable in terms of more familiar objects.

2.1. Applying 1.5 to the fibration $\mathscr{E}_\dagger(F)$ yields the fibration $\mathscr{E}' = (E', p', B_\dagger(F))$ with fibre F'. Again F' is the result of killing the lowest homotopy group of F. Since $\mathscr{E}'$ is classified by a map $\rho\colon B_\dagger(F) \to B^\dagger(F')$ we have the following diagram.

$$\begin{array}{ccccc} F' & = & F' & \xrightarrow{i} & F \\ \downarrow & & \downarrow & & \downarrow \\ E_\dagger(F') & \longleftarrow & E' & \longrightarrow & E_\dagger(F) \\ \downarrow & & \downarrow & & \downarrow \\ B_\dagger(F') & \xleftarrow{\rho} & B_\dagger(F) & = & B_\dagger(F) \end{array} \tag{2.2}$$

Writing F' as F^1 and ρ as ρ^1 we may iterate this process giving the sequence of spaces.

$$(2.3)\qquad B_\dagger(F) \xrightarrow{\rho^1} B_\dagger(F^1) \xrightarrow{\rho^2} B_\dagger(F^2) \longrightarrow \cdots$$

and the associated sequence of fibres of opposite variance.

$$F \xleftarrow{i^1} F^1 \xleftarrow{i^2} F^2 \longleftarrow \cdots$$

It is now our intention to study the fibres of the maps ρ^i. Since our diagrams are already heavily adorned we will work with 2.2 itself but state conclusions in general.

We begin by noting that the map $F' \xrightarrow{i} F$ induces $i^*: \langle F, F\rangle_0 \to \langle F', F\rangle_0$ and $i_*: \langle F', F'\rangle_0 \to \langle F', F\rangle_0$. The following extension of 2.2 is basic to our analysis of the maps ρ^i.

$$(2.4)\qquad \begin{array}{ccccccc}
\langle F', F'\rangle_0 & = & \langle F', F'\rangle_0 & \xrightarrow{i_*} & \langle F', F\rangle_0 & \xleftarrow{i^*} & \langle F, F\rangle_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_\dagger\langle F'\rangle_0 & \longleftarrow & E'\langle F'\rangle_0 & \longrightarrow & E'\langle F', F\rangle_0 & \longleftarrow & E_\dagger\langle F\rangle_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_\dagger(F') & \xleftarrow{\rho} & B_\dagger(F) & = & B_\dagger(F) & = & B_\dagger(F)
\end{array}$$

We will also need the following observation.

LEMMA 2.5. *i_* induces an isomorphism in homotopy.*

Proof. Since F' is n-connected $\langle F', K(\pi, n)\rangle_0$ has vanishing homotopy.

Finally, letting $d(d')$ be a geometric boundary map for

$$\mathscr{E}_\dagger\langle F\rangle_0(\mathscr{E}_\dagger\langle F'\rangle_0)$$

we have the following:

THEOREM 2.6. *The following diagram of spaces is homotopy commutative.*

$$(2.7)\qquad \begin{array}{ccc}
\Omega B_\dagger(F') & \xleftarrow{\Omega\rho} & \Omega B_\dagger(F) \\
\big\downarrow{\scriptstyle i_*d'} & & \big\downarrow{\scriptstyle d} \\
\langle F', F\rangle_0 & \xleftarrow{i^*} & \langle F, F\rangle_0
\end{array}$$

Moreover, i_*d' and d induce isomorphisms in homotopy.

Proof. Homotopy commutativity follows from 2.4 and 1.12. d and d' induce isomorphisms by 1.10. Finally, i_* is an isomorphism by 2.5.

Using 2.7 we can now study the homotopy of T, the fibre of the map ρ.

$$\pi_0(T) \quad \text{and} \quad \pi_1(T)$$

(2.8) Since $B_\dagger(F)$ and $B_\dagger(F')$ are connected $\pi_0(T) = \operatorname{coker} \rho_* = \pi_1(B_+(F'))/\rho_*\pi_1(B_+(F))$. Moreover, $\pi_1(B_+(F)) = Eq[F]$ the group of base point preserving self-equivalences of F.

The map $Eq[F] \xrightarrow{\rho^*} Eq[F']$ has been studied in several places and in fact for the stable case a formula for the image of $Eq[F]$ in $Eq[F']$ can be found in [9]. We give an interpretation of this formula in a simple case.

Let F be a stable two stage Postnikov system determined by a cohomology operation $\phi \in H^{m+1}(K(\pi_1, n), \pi_2)$. $Eq[F'] = \operatorname{aut}(\pi_2)$ and the image of ρ in $\operatorname{aut}(\pi_2)$ are precisely those α in $\operatorname{aut}(\pi_2)$ for which there is a β in $\operatorname{aut}(\pi_1)$ with

$$\begin{array}{ccc} K(\pi_1, n) & \xrightarrow{\phi} & K(\pi_2, m+1) \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\alpha} \\ K(\pi_1, n) & \xrightarrow{\phi} & K(\pi_2, m+1) \end{array}$$

commutative.

One may state conditions when ρ_* is onto, (if π_1 and π_2 are prime cyclic) but in general, of course, this is not the case.

On the other hand, since T is the fibre of a fibration with path connected base and total space, all of the path components of T have the same homotopy type. Hence the higher homotopy of T is reflected in $\Omega(T)$ which is also the fibre of $i^*\colon \langle F, F\rangle_0 \to \langle F', F\rangle_0$.

The study of $\pi_1(T)$ requires similar consideration [5], [8], [9]. We will consider these groups again in §3 for some special cases.

$$\pi_n(T),\ n \geqq 2$$

NOTATION. We let F/F' denote the cofibre of the map $i\colon F' \to F$. For X a space we let $S(X)$ denote its reduced suspension. Finally, for base pointed spaces X and Y, let $[X, Y]$ denote the set of base pointed homotopy classes of maps of X to Y.

LEMMA 2.9. *If F is an H-space then*

$$\pi_n(T) = [S^{n-1}(F/F'), F]$$

for $n \geqq 2$.

Proof. By 2.6, it suffices to study $\pi_{n-1}(\Omega T)$ where ΩT is considered as the fibre of the map $i^*\colon \langle F, F\rangle_0 \to \langle F', F\rangle_0$. Since F is an H-space all components of spaces of maps into F have the same homotopy. Hence we may replace i^* by the map of components of maps base point homotopic to the constant map. The result is now a standard mapping space argument.

The Spectral Sequence

In order to set up the spectral sequence we extend the sequence.

$$F \xleftarrow{i^1} F^1 \xleftarrow{i^2} F^2 \longleftarrow \cdots$$

in a trivial way by setting $F^n = F$ for $n \leqq 0$. Also set

$$\bar{d}^n = i^1_* i^2_* \cdots i^*_n d^n\colon \Omega(B_\dagger(F^n)) \longrightarrow \langle F^n, F\rangle_0 .$$

We have the following formal display of spaces.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega(B_\dagger(F^{n-1})) \arrow[rr, "\Omega\rho^n"] \arrow[d, "\bar{d}^{n-1}"] & & \Omega(B_\dagger(F^n)) \arrow[rr, "\Omega\rho^{n+1}"] \arrow[d, "\bar{d}^n"] & & \Omega(B_\dagger(F^{n+1})) \arrow[r] \arrow[d, "\bar{d}^{n+1}"] & \cdots \\
{} \arrow[r] & \langle F^{n-1}, F\rangle_0 \arrow[rr, "i^{n*}"] & & \langle F^n, F\rangle_0 \arrow[rr, "i^{(n+1)*}"] \arrow[dl, dashed, "\partial^{n-1}"] & & \langle F^{n+1}, F\rangle_0 \arrow[r] \arrow[dl, dashed, "\partial^n"] & \cdots \\
& & \Omega(T^{n-1}) \arrow[ul, "t^{n-1}"] & & \Omega(T^n) \arrow[ul, "t^n"] & &
\end{tikzcd}$$

Before extracting our exact couple from 2.10 we establish some notation.

The general form of our bigraded exact couple will be

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

$$i_{p,q}\colon A_{p,q} \longrightarrow A_{p,q+1},\ j_{p,q}\colon A_{p,q} \longrightarrow E_{p-1,q-1},\ k_{p,q}\colon E_{p,q} \longrightarrow A_{p,q} .$$

For the pair (F^n, F^{n+1}) its Barratt-Puppe sequence is

$$F^{n+1} \xrightarrow{i^{n+1}} F^n \xrightarrow{\phi^n} F^n/F^{n+1} \xrightarrow{\Delta^n} S(F^{n+1}) .$$

Note, for example, that Δ^n induces

$$(S^{p-1}\Delta^n)^*\colon [S^{p-1}(S(F^{n+1})), F] \longrightarrow [S^{p-1}(F^n/F^{n+1}), F] .$$

THEOREM 2.11. *For F an H-space there is a bigraded exact couple (A, E, i, j, k) as above and such that*

1. $A^1_{p,q} = \pi_{p+1}(B_\dagger(F^q))$
2. $E^1_{p,q} = [S^p(F^q/F^{q+1}), F]$, $p \geqq 1$
3. $d^1_{p,q} = j^1_{p,q}k^1_{p,q} = (S^{p-1}\Delta^q)^*(S^p\phi^q)^*$.

Proof. The exact couple in question is induced by 2.10. Hence all that is left to prove is (3) which, like 2.8, is a standard mapping space argument. For example one identifies $j^1_{p,q}$ as induced by the pth suspension of the geometric boundary homomorphism then observe that $(S^{p-1}\Delta^q)^*$ is essentially an explicit choice for that map.

When F is a stable space 2.11 takes a more convenient form. Before stating the result we introduce some notation relating to the Postnikov decomposition of F. Let the rth nonzero homotopy group of F be denoted by π_{n_r}, where n_r is the dimension where it occurs. Let E_r denote the rth stage in a Postnikov decomposition of F and let $k_r \in H^{n_{r+1}+1}(E_r, \pi_{n_{r+1}})$ be the rth k-invariant of F. Thus we have the following decomposition of F.

(2.12)
$$\begin{tikzcd}
 & F \arrow[d] & \\
 & E_n \arrow[d] & \\
 & \cdot \arrow[d] & \\
K(\pi_{n_1}, n_1) \arrow[r, "i_2"] & E_2 \arrow[r, "k_2"] \arrow[d, "p_1"] & K(\pi_{n_2}, n_3 + 1) \\
K(\pi_{n_0}, n_0) \arrow[r, equal, "i_1"] & K(\pi_{n_0}, n_0) \arrow[r, "k_1"] & K(\pi_{n_1}, n_2 + 1) .
\end{tikzcd}$$

We set $\psi_r = [k_r i_r] \in H^{n_{r+1}+1}(K(\pi_{n_r}, n_r), \pi_{r+1})$.

Since we are in a stable range it makes sense to speak of $S^p\psi_r \in H^{n_{r+1}+p+1}(K(\pi_{n_r}, n_r + p), \pi_{r+1})$, the pth suspension of ψ_r. Note that p may take negative values. The following well known fact motivates our restatement of 2.11 for the stable case.

LEMMA 2.13. *For $p \leqq 2n_r - 2$ we have that*

$$S^{p-1}\psi_{r-1} \circ S^p\psi_r = 0 .$$

Proof. It suffices to consider the case $p = 0$. Other cases following by isomorphism under suspension. For $p = 0$ we have the diagram-

$$\begin{array}{ccccc} K(\pi_{n_r}, n_r) & \xrightarrow{i_r} & E_r & \xrightarrow{k_r} & K(\pi_{n_{r+1}}, n_{r+1} + 1) \\ & & \downarrow{\scriptstyle p_{r-1}} & & \\ K(\pi_{n_{r-1}}, n_{r-1}) & \xrightarrow{i_{r-1}} & E_{r-1} & \xrightarrow{k_{r-1}} & K(\pi_{n_r, n_r+1} \end{array}$$

now $S^{-1}(\psi_{r-1}) = [\Omega k_{r-1} \cdot \Omega i_{r-1}]$ and Ωk_{r-1} is the geometric boundary of the fibration $K(\pi_r, n_r) \xrightarrow{i_r} E_r \xrightarrow{p_{r-1}} E_{r-1}$ so $[i_r \Omega k_{r-1}] = 0$ and

$$S^1(\psi_{r-1}) \circ S^0(\psi_r) = [k_r i_r \Omega k_{r-1} \Omega i_{r-1}] = 0 .$$

We are now prepared to state the additional properties that our spectral sequence has in the stable case.

THEOREM 2.14. *Let F be stable (again, $\pi_i(F) = 0$ $n > i$ or $i > 2n - 3$) then*

(a) *The spectral sequence of* 2.11 *finitely converges.*

(b) $E^1_{p,q} = [K(\pi_{n_q}, n_q + p), F]$, $p \geqq 1$.

(c) $d^1_{p,q} = (S^{p-1}\psi_{q-1})^*$.

Proof. Firstly let $n_0, n_1, n_2 \cdots n_s$ be the dimensions in which $\pi_n(F) \neq 0$. Then $E^1_{p,q} = 0$ $p > 2n - 3$ or $q > s$ (2.11-2).

It is a simple matter to check that the maps

$$S^p(F^q/F^{q+1}) \xrightarrow{S^p(p_q)} S^p(K(\pi_{n_q}, n_q))$$

and

$$S^p(K(\pi_{n_q}, n_q + p)) \xrightarrow{\sigma} K(\pi_{n_q}, n_q + p)$$

are $2n - 3$ equivalences hence

$$(\sigma S^p(p_q))^*[K(\pi_{n_q}, n_q + p), F] \cong [S^p(F^q/F^{q+1}), F] .$$

If $\theta_q \in H^{n_q}(K(\pi_{n_q}, n_q), \pi_{n_q})$ is the fundamental class then $[k_{q-1}] = (p^*_{q-1})^{-1}\delta^*_q\theta_q$ where δ^* is the coboundary homomorphism in the sequence of the pair $(E_{n_q}, K(\pi_{n_q}, n_q))$. On the other hand if S^* is the cohomology suspension and $E_{n_q}/K(\pi_{n_q}, n_q) \xrightarrow{\Delta^q} S(K(\pi_{n_q}, n_q))$ is as in 2.10 then $\delta^*_q = \Delta^{q*}S^*$. Using this fact, (2) above, and 2.11.3, the remainder of the proof reduces to an examination of the appropriate diagrams.

APPLICATIONS 2.14. We now apply 2.13 to certain simple cases. We assume $n_0 > 1$.

1. $\underline{F = K(\pi_{n_0}, n_0)}$. This situation was studied in [11] where it was shown that $\overline{B_{\dagger}(F)}$ is a $K(\mathrm{Aut}\,(\pi_{n_0}), 1)$.

We rederive this result from 2.13 by noting that $E^1_{p,0} = [K(\pi_{n_0}, n_0+p), K(\pi_{n_0}, n_0)] = 0$ $p \geqq 1$. Also $E^1_{p,q} = 0$ trivially for $q \neq 0$. Thus we conclude that $\pi_n(B_\dagger(K(\pi_{n_0}, n_0)) = 0$ $n \geqq 2$. Finally, we know that $\pi_1(B_\dagger(F)) = E_q[f] = \text{aut}\,(\pi_{n_0})$ (2.8).

2. $\underline{F = (K(\pi_{n_0}, n_0), K(\pi_{n_1}, n_1), k_1)}$. Here $E^1_{p,q} = 0$ $q \neq 0, 1$. By elementary obstruction theory one shows

(a) $E^1_{p,1} = 0$, $p \geqq 1$.

(b) $E^1_{p,0} = [K(\pi_{n_0}), n_0 + p, K(\pi_{n_1}, n_1)]$, $p \geqq 1$.

Thus:

$$\pi_n(B_\dagger(F)) = H^{n_1+1-n}(K(\pi_{n_0}, n_0), \pi_{n_1})$$

for $n \geqq 2$.

We will discuss the actual structure of $B_\dagger(F)$ in §3.

3. $\underline{F = (K(\pi_{n_0}, n_0), K(\pi_{n_1}, n_1), K(\pi_{n_2}, n_2), k_1, k_2)}$.

As above: (a) $E^1_{p,q} = 0$, $q \neq 0, 1, 2$ (trivially)

(b) $E^1_{p,2} = 0$, $p \geqq 1$ (as in 1 above).

(c) $E^1_{p,1} = H^{n_2}(K(\pi_{n_1}, n_1 + p), \pi_{n_2})$, $p \geqq 1$.

$E^1_{p,0} = H^{n_2}(K(\pi_{n_0}, n_0 + p), \pi_{n_2}) p \geqq n_1 - n_0 + 1$ (as in 2 above).

(d) In general for $p \geqq 1$ there is an exact sequence

$$\xrightarrow{(S^{-1}\psi_2)^*} H^{n_2}(K(\pi_{n_0}, n_0 + p), \pi_{n_2}) \longrightarrow E^1_{p,1} \longrightarrow H^{n_1}(K(\pi_{n_0}, n_1 + p), \pi_{n_1}) \xrightarrow{(\psi_2)^*} .$$

We next consider the E^2-terms.

(e) For $p \geqq 1$ $E^2_{p,2} = \ker d^1_{p,2} = \ker (S^{p-1}\psi_1)^*$ and for $p \geqq n_2 - n_2 + 1$ $E^2_{p,1} = \text{coker}\,(S^p\psi_1)^*$ where

$$(S^{p-1}\psi_1)^*\colon H^{n_2}(K(\pi_{n_1}, n_1 + p), \pi_{n_2}) \longrightarrow H^{n_2}(K(\pi_{n_0}, n_0 + p - 1), \pi_{n_2}) .$$

Thus, for $p \geqq n_2 - n_1 + 1$, there is a short exact sequence

$$0 \longrightarrow \ker (S^{p-1}\psi_1)^* \longrightarrow \pi_p(B_\dagger(F)) \longrightarrow \text{coker}\,(S^p\psi_1)^* \longrightarrow 0 .$$

A more complex sequence can be written down using (d).

4. As a last example we apply (3) to an explicit case. The results are indicative of the sorts of conclusions we will draw in the next section.

Let F be a stable space based on the Adem relation $Sq^3Sq^2 = 0$. Thus, letting n be a base stable dimension we have the following Postnikov system for F.

$$
\begin{array}{ccccc}
F^2 = K(Z_2, n+3) \xrightarrow{j_2} F^1 & & \xrightarrow{i^1} F & & \\
\downarrow & & \downarrow & & \\
K(Z_2, n+1) & \xrightarrow{i_2} & E_1 & \xrightarrow{k_2} & K(Z_2, n+4) \\
 & & \downarrow & & \\
 & & K(Z_2, n) & \xrightarrow{Sq^2} & K(Z_2, n+2)
\end{array}
$$

where $k_2 i_2 = Sq^3$ and, in the notation of 2.11 $i^1 j_2 = i_3$.

By (3) a, b, c, d above we conclude that the nonzero E^1-terms in our spectral sequence are

$$
\begin{aligned}
E^1_{1,0} &= H^{n+3}(K(Z_2, n+1), Z_2) = Z_2[Sq^2] \\
E^1_{1,1} &= E^1_{2,0} = Z_2[Sq^1] \\
E^1_{2,1} &= E^1_{3,0} = Z_2[Id] \, .
\end{aligned}
$$

All of these are immediate except the first which follows from (3) d by a simple evaluation of the various terms in the sequence. By 2.14 (C), $d^1_{2,1} = [Sq^2]^*\colon E^1_{2,1} \cong E^1_{1,0}$ thus $E^2_{2,1} = E^2_{1,0} = 0$.

Further, it is not difficult to verify that $d^1_{1,1} \neq 0$ thus $E^2_{1,1} = 0$.

Thus the nonzero E^2-terms are:

$$
\begin{aligned}
E^2_{2,0} &\cong Z_2 \\
E^2_{3,0} &\cong Z_2 \, .
\end{aligned}
$$

Again, checking that $\pi_1(B_\dagger(F)) = Z_2$ we have that

$$
\pi_i(B_\dagger(F)) \cong \begin{cases} Z_2 & i = 1, 3, 4 \\ 0 & \text{otherwise} \, . \end{cases}
$$

THEOREM 2.15. *In a Postnikov decomposition of $B_\dagger(F)$, $k_2 i_2 = 0$. Thus the simply connected covering space of $B_\dagger(F)$ is $K(Z_2, 3) \times K(Z_2, 4)$.*

Sketch of proof. $k_2 i_2$ is either Sq^2 or zero. Let $X = S(P^4)$, the suspension of real projective 4 space. If $k_2 i_2 = Sq^2$ then one verifies $[X, B_+(F)] \cong Z_2$. On the other hand, a direct calculation shows $H_0[X, F] \cong Z_2 \oplus Z_2$ which agrees with the assumption $k_2 i_2 = 0$.

3. $B_\dagger(F)$ **and** $B_\infty(F)$ **for stable two stage systems.** In this section we give explicit descriptions of $B_\dagger(F)$ and $B_\infty(F)$ for certain stable two stage systems. These descriptions are based on a presentation of $B_\dagger(F)$ as the total space of a fibration with fibre a product of $K(\pi, n)$'s and base a $K(\pi, 1)$. We first develop this presentation.

3.1. To fix notation for this section, assume we are given the following Postnikov decomposition of F

$$\begin{tikzcd} K(\pi_1, n_1) \arrow[r, "i"] & F \arrow[r] \arrow[d, "p"] & * \arrow[d] \\ & K(\pi_0, n_0) \arrow[r, "k"] & K(\pi_1, n_1+1)\,. \end{tikzcd}$$

Associated with this decomposition is the principal bundle pairing

$$\sigma\colon K(\pi_1, n_1) \times F \longrightarrow F\,.$$

Finally, for spaces X and Y, $C_0(X, Y)$ will denote the space of maps homotopic to a given constant map.

The following construction is basic to the remainder of this section.

3.2. Suppose we are given a pairing

$$\zeta\colon X \wedge K(\pi_0, n_0) \longrightarrow K(\pi_1, n_1+1)\,.$$

Let $\bar{\zeta}\colon X \times K(\pi_0, n_0) \to K(\pi_1, n_1+1)$ be the pairing induced by ζ.

Let $\bar{k}\colon X \times K(\pi_0, n_0) \to K(\pi_1, n_1+1)$ be the composition of projection onto the second factor and k.

Finally, let m denote multiplication in $K(\pi_1, n_1+1)$ and $\Delta\colon X \times K(\pi_0, n_0) \to (X \times K(\pi_0, n_0))^2$ denote the diagonal map.

We have the following pullback diagram of fibrations

$$\begin{tikzcd} E \arrow[r, dashed] \arrow[d, dashed, "\bar{t}"'] & * \arrow[d] \\ X \times K(\pi_0, n_0) \arrow[r, "m(\bar{\zeta}\times\bar{k})\Delta"] & K(\pi_1, n_1+1)\,. \end{tikzcd}$$

Letting t be $\bar{t}$ composed with projection onto the first factor, we have the following lemma.

LEMMA 3.3. *$E \xrightarrow{t} X$ is a fibration over X with fibre F and with cross-section.*

In general (E, t, X) is not a product. To see this we show by example that the geometric boundary homomorphism of the associated fibration $(E\langle F\rangle_0, \tilde{t}, X)$ may be nontrivial. We will use this computation for other purposes as well.

3.4. The pairing σ induces a pairing of mapping spaces.

$$\bar{\sigma}\colon C_0(F, K(\pi_1, n_1)) \times \langle F, F\rangle_0 \longrightarrow \langle F, F\rangle_0$$

which in turn induces

$$\tilde{\sigma}\colon C_0(F, K(\pi_1, n_1)) \longrightarrow \langle F, F\rangle_0$$

by the formula $\tilde{\sigma}(f) = \sigma(f, Id)$, where Id is the identity map on F.

On the other hand, ζ induces

$$\Omega\zeta\colon \Omega X \wedge K(\pi_0, n_0) \longrightarrow \Omega K(\pi_1, n_1 + 1)$$

which in turn induces

$$\widetilde{\Omega\zeta}\colon \Omega X \longrightarrow C_0(K(\pi_0, n_0), K(\pi_1, n_1)) .$$

We have the following

THEOREM 3.5. *A geometric boundary homomorphism* $d\colon \Omega X \to \langle F, F\rangle_0$, *for the fibration* $(E\langle F\rangle_0, \tilde{p}, X)$ *is given by the composition.*

3.6. $\Omega X \xrightarrow{\widetilde{\Omega\zeta}} C_0(K(\pi_0, n_0), K(\pi_1, n_1)) \xrightarrow{p^*} C_0(F, K(\pi_1, n_1)) \xrightarrow{\tilde{\sigma}} \langle F, F\rangle_0$.

Proof. Referring back to the definition of d (1.11) the proof amounts to a verification that the given composition is a suitable choice for d.

EXAMPLE 3.7. Let $X = C_0(K(\pi_0, n_0), K(\pi_1, n_1 + 1))$. One has the evaluation pairing

$$\zeta\colon C_0(K(\pi_0, n_0), K(\pi_1, n_1 + 1)) \wedge K(\pi_0, n_0) \longrightarrow K(\pi_1, n_1 + 1)$$

hence one may apply the construction 3.2.

LEMMA 3.3- *In the setting of* 3.7,

$$d_*\colon \pi_i(\Omega C_0(K(\pi_0, n_0), K(\pi_1, n_1 + 1))) \longrightarrow \pi_i(\langle F, F\rangle_0)$$

is an isomorphism for $i > 0$.

Firstly, $\widetilde{\Omega\zeta}$ is, in fact, the identity map on the spaces in question.

For $i > 0$, p^* is an isomorphism by a simple obstruction theoretic argument as in 2.14.2b and $\tilde{\sigma}^*$ is an isomorphism by similar consideration taking into account the properties of the pairing σ.

In what follows we denote the space $C_0(K(\pi_0, n_0), K(\pi_1, n_1 + 1))$ by the letter "K". Again, 3.2 gives a fibration over K with fibre F and cross-section.

THEOREM 3.9. *Let* $\kappa\colon K \to B_\dagger(F)$ *classify this fibration. Then*:

(a) *K is a product of spaces of type $K(\pi, i)$ for $0<i\leqq n_1-n_0+1$. Moreover $\pi_i(K) = H^{n_1+1}(K(\pi_0, n_0 + i), \pi_1)$.*

(b) *$\kappa_*: \pi_i(K) \longrightarrow \pi_i(B_\dagger(F))$ is an isomorphism for $i \neq 1$.*

Proof. (a) is a well known result ([13]).

(b) follows at once from 3.8, 1.10 and 1.12.

3.9 may be interpreted as saying that the universal covering space of $B_\dagger(F)$ is a product of $K(\pi, n)$'s. Thus, in order to determine the structure of $B_\dagger(F)$, we must know its fundamental group and the action of this group on the higher homotopy groups.

As we have indicated (2.8) $\pi_1(B_\dagger(F)) = Eq[F]$. Let

$$\hat{\tau}: Eq[f] \longrightarrow \text{aut}\,(\pi_{n_0}(F)) \times \text{aut}\,(\pi_{n_1}(F))$$

be defined by the formula $\tau[f] = f_* \times f_*$. Denote the image of τ as $\hat{\pi}$. $\hat{\tau}$ induces a map

$$\tau: B_\dagger(F) \longrightarrow K(\hat{\pi}, 1)\ .$$

The following theorem gives a fairly complete hold on the geometry of $B_\dagger(F)$.

THEOREM 3.10. *Up to homotopy, the sequence of maps*

$$K \xrightarrow{\kappa} B_\dagger(F) \xrightarrow{\tau} K(\hat{\pi}, 1)$$

is a fibration.

Moreover, the action of $\hat{\pi}$ on $\pi_i(K)$ is just the restriction of the usual action of $\text{aut}\,(\pi_{n_0}) \times \text{aut}\,(\pi_{n_1})$ *on $H^{n_1+1}(K(\pi_0, n_0 + i), \pi_1)$.*

Proof. Since the base of τ is a $K(\hat{\pi}, 1)$, to show that the sequence in question is a fibration it suffices to check that

(a) $\kappa_*: \pi_i(K) \cong \pi_i(B_\dagger(F)) i > 1$ (as was verified in 3.9).

(b) $0 \to \pi_1(K) \xrightarrow{\kappa_*} \pi_1(B_\dagger(F)) \xrightarrow{\tau_*} \pi_1(K(\hat{\pi}, 1)) \to 0$

is exact.

To verify (b) we first observe that τ_* is onto essentially by definition.

Next, self-equivalences in the image of κ_* are represented by compositions of the form

$$(3.11)\quad F \xrightarrow{\Delta} F \times F \xrightarrow{p \times 1} K(\pi_0, n_0) \times F \xrightarrow{h \times 1} K(\pi_1, n_1) \times F \xrightarrow{\sigma} F$$

where $h: K(\pi_0, n_0) \to K(\pi_1, n_1)$ is some map $[h] \in \pi_0(\Omega K) = \pi_1(K)$.

On checks that

1. Such compositions induce the identity map on homotopy. Hence $\tau_*\kappa_* = 0$.

2. Such compositions are homotopic to the identity if and only if h is homotopic to the constant map. Hence κ_* is a monomorphism.

To show that $\ker \tau_* = \operatorname{im} \kappa_*$, let $x \in \pi_1(B_\dagger(F))$ be such that $\tau_*(x) = 0$. Consider $x = [f] \in \bar{E}q[f]$ under the appropriate identification. By hypothesis,

$$p_*([f] - 1) = 0 \in [F, K(\pi_0, n_0)] .$$

Hence, there exists a map $g: F \to K(\pi_1, n_1)$ such that $i_*[g] = [f] - 1$. Again by hypothesis,

$$0 = g_*: \pi_n(F) \longrightarrow \pi_n(K(\pi_1, n_1)) .$$

Thus $i^*[g] = 0$. Since we are in the stable range $[g] = p^*[h]$ where $h: K(\pi_0, n_0) \to K(\pi_1, n_1)$. Finally considering $[h] \in \pi_1(K)$ (3.11) one verifies $\kappa_*[h] = x$.

That the action of $\hat{\pi}$ on $\pi_i(K)$ is as stated is a similar verification.

We now consider a situation where we can give an explicit model for $B_\dagger(F)$.

3.12. For a given prime p let k be a nontrivial stable operation of type (Z_p, Z_p, n_1, n_2). Then $\operatorname{aut}(\pi_1) = \operatorname{aut}(\pi_2) = Z_{p-1}$ and since k is nontrivial $\hat{\pi} = Z_{p-1}$ (see 2.8).

Consider the twisted Eilenberg-MacLane spaces $L_{Z_{p-1}}(\pi_i(K), i)$ [10]. Since $\pi_i(K)$ is p-primary the fundamental classes of the cohomology of the fibre live to E_∞ in the twisted Serre spectral sequence of $B_\dagger(F) \to K(Z_{p-1}, 1)$ [10, 11]. Hence there are geometric representations of these classes as commutative diagrams of the following form

$$\begin{array}{ccc} B_\dagger(F) & \longrightarrow & L_{Z_{p-1}}(\pi_i(K), i) \\ \big\downarrow & & \big\downarrow \\ K(Z_{p-1}, 1) & = & K(Z_{p-1}, 1) \end{array}$$

where the top map is an isomorphism in homotopy in dimension i and in dimension 1. We then have

THEOREM 3.13. *Let F be determined by a stable operation of type (Z_p, Z_p, n_1, n_2). Then*

$$B_\dagger(F) \cong \underset{K(Z_{p-1}, 1)}{\times} L_{Z_{p-1}}(\pi_i(K), i) ,$$

the fibre product over $K(Z_{p-1}, 1)$.

For $p = 2$ this reduces to an ordinary product.

We may use 3.13 to determine the structure of $B_\infty(F)$. For simplicity we only write down details for $p = 2$.

Recall that the fibration $B(F) \to B_\infty(F) \xrightarrow{r} B_\prime(F)$ is a model for

$\mathscr{E}_\dagger(B(F))$ (1.7). Hence $B_\infty(F)$ is the total space of this fibration. For the case $p = 2$ we have essentially constructed this total space in 3.5. In particular, let $k_1 \in H^{n_1+2}(K(Z_2, n_0 + 1), Z_2)$ be k delooped. Identifying $B_\infty(F) = K = C_0(K(Z_2, n_0), K(Z_2, n_1 + 1))$ and $B_\dagger(B(F)) = C_0(K(Z_2, n_0 + 1), K(Z_2, n_1 + 2))$, we have the following theorem.

THEOREM 3.14. *$B_\infty(F)$ is the pullback of the following diagram.*

$$\begin{tikzcd} B_\infty(F) \arrow[rr, dashed] \arrow[d, dashed] & & * \arrow[d] \\ B_\dagger(F)X \times K(Z_2, n_0 + 1) \arrow[rr, "m(\bar{\zeta}\times\bar{k}_1)\Delta"] & & K(Z_2, n_1 + 2) \end{tikzcd} \tag{3.15}$$

Proof. Since $\hat{\pi} = 0$ this is an immediate consequence of 3.10.

FINAL REMARKS 3.16. For $p \neq 2$ a similar construction can be given. The difference between the two cases is that the k-invariant for $p \neq 2$ is a twisted k-invariant. Therefore, in (3.15) one must replace $K(Z_2, n)$ by $L_{Z_{p-1}}(Z_p, n)$ and products by fibre products. The formula for the k-invariant in this case is similarly generalized.

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