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SCALAR DEPENDENT ALGEBRAS IN THE ALTERNATIVE SENSE

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Let R, a not necessarily associative algebra over a field F of characteristic $\neq 2$, be equipped with a map $g: R \times R \times R \to F$. We show that if R contains a nonzero idempotent and satisfies the identities (1) $(xy)z + (yx)z = g(x, y, z)[x(yz) + y(xz)]$ and (2) $(xy)z + (xz)y = g(x, y, z)[x(yz) + x(zy)]$ then R is an alternative algebra. The methods also apply to other pairs of identities.

1. Introduction. In [1, 2] several authors have studied scalar dependent algebras, i.e., not necessarily associative algebras R over a field F which are equipped with a map $g: R \times R \times R \to F$ such that $(xy)z = g(x, y, z)x(yz)$ for all $x, y, z \in R$. The main result there was that if a scalar dependent algebra contains an idempotent e, then it is associative. In [3] the study was extended to the case of algebras over a principal ideal domain. Here we shall look at the analogous situation in the alternative case.

Specifically, suppose that R is a not necessarily associative algebra over a field F of characteristic $\neq 2$ equipped with a map $g: R \times R \times R \to F$ and consider the identities:

$$(xy)z + (yx)z = g(x, y, z)[x(yz) + y(xz)] \tag{1}$$

$$(xy)z + (xz)y = g(x, y, z)[x(yz) + x(zy)] \tag{2}$$

$$(xy)z + (zy)x = g(x, y, z)[x(yz) + z(yx)] \tag{3}$$

$$x(yz) + z(yx) = g(x, y, z)[(xy)z + (zy)x] \tag{4}$$

$$x(yz) + x(zy) = g(x, y, z)[(xy)z + (xz)y] \tag{5}$$

$$x(yz) + y(xz) = g(x, y, z)[(xy)z + (yx)z] \,. \tag{6}$$

Note that if $g(x, y, z) \equiv 1$ then identities (1) and (6) each imply left alternativity, (2) and (5) each imply right alternativity and identities (3) and (4) each imply flexibility. (Recall that if (x, y, z) denotes $(xy)z - x(yz)$ then an algebra A is called left [right] alternative if $(x, x, y) = 0$ $[(y, x, x) = 0]$ for all x, y in A and is alternative if it is both left and right alternative. A is flexible if $(x, y, x) = 0$ for all x, y in A. A flexible left (right) alternative algebra is alternative.) The intent of this paper is to show that if R contains an idempotent e, then any pair of identities (1)-(6) which imply alternativity when $g(x, y, z) \equiv 1$, imply alternativity in all cases. Since the methods of proof are similar regardless of the choice of identities, to avoid repetition we present proofs only for the case of an algebra satisfying identities (1) and (2) and describe the results for the other cases at the end. Thus, unless otherwise specified, R will denote an algebra satisfying (1) and (2) over a field F of characteristic $\neq 2$.

It wil be useful to note that if $\alpha = g(x, y, z)$ then (1) and (2) easily reduce to

$$(1') \qquad (x, y, z) + (y, x, z) = (\alpha - 1)[x(yz) + y(xz)]$$

and

$$(2') \qquad (x, y, z) + (x, z, y) = (\alpha - 1)[x(yz) + x(zy)] .$$

2. Algebras with an identity element. In this section we assume that R contains an identity element 1.

LEMMA 1. *If R contains an identity element* 1 *and satisfies* (1) *then R is left alternative.*

Proof. Let $x, y, z \in R$ and let $\alpha = g(x, y, z)$, $\beta = g(x + 1, y, z)$, $\delta = g(x, y + 1, z)$, $\gamma = g(x + 1, y + 1, z)$. Then we have

$$\begin{aligned}(\alpha - 1)[x(yz) + y(xz)] &= (x, y, z) + (y, x, z)\\ &= (x + 1, y, z) + (y, x + 1, z)\\ &= (\beta - 1)[x(yz) + y(xz) + 2yz] .\end{aligned}$$

Thus,

$$(7) \qquad (\alpha - \beta)[x(yz) + y(xz)] = 2(\beta - 1)yz .$$

Similarly

$$(8) \qquad (\alpha - \delta)[x(yz) + y(xz)] = 2(\delta - 1)xz .$$

Suppose that $(x, y, z) + (y, x, z) \neq 0$ and that $\alpha = \beta$. Then by (7) $yz = 0$. If $\alpha = \delta$ then by (8) $xz = 0$ from which it follows that $(x, y, z) + (y, x, z) = 0$. Thus $\alpha \neq \delta$ and we get from (8)

$$(9) \qquad y(xz) = \delta'(xz) \qquad \text{for} \quad \delta' = \frac{2(\delta - 1)}{\alpha - \delta} \in F .$$

In a similar fashion $(\delta - 1)[x(yz) + y(xz) + 2xz] = (x, y + 1, z) + (y + 1, x, z) = (x + 1, y + 1, z) + (y + 1, x + 1, z) = (\gamma - 1)[x(yz) + y(xz) + 2xz + 2yz + 2z] = (\gamma - 1)[y(xz) + 2xz + 2z]$ so that

$$(10) \qquad (\delta - \gamma)[y(xz) + 2xz] = 2(\gamma - 1)z .$$

Since $\delta = \gamma$ implies that $z = 0$ we get $\delta \neq \gamma$ and $y(xz) + 2xz = \mu z$

for $\mu \in F$. Thus $(x+1, y+1, z) + (y+1, x+1, z) = (\gamma - 1)[y(xz) + 2xz + 2z] = tz$ for $t \in F$. By (9) $(x, y, z) + (y, x, z) = (\alpha - 1)\delta' xz$. Thus $xz = t'z$ for $t' \in F$ so that $y(xz) = t'yz = 0$. Since $y(xz) = x(yz) = 0$ we arrive at $(x, y, z) + (y, x, z) = 0$. Thus if x and y don't left alternate it follows that $\alpha \neq \beta$. Since $\alpha \neq \beta$, (7) leads to:

$$(x, y, z) + (y, x, z) = \mu' yz \quad \text{for} \quad \mu' \in F . \tag{11}$$

Applying the same procedure as above it follows that $g(x, y, z+1) \neq g(x+1, y, z+1)$ and that

$$(x, y, z+1) + (y, x, z+1) = \mu''(y(z+1)) \quad \text{for} \quad \mu'' \in F . \tag{12}$$

Combining (11) and (12) we have $(\mu' - \mu'')yz = \mu'' y$. If $\mu' = \mu''$ it follows that x and y left alternate. Suppose $\mu' \neq \mu''$, then $yz = cy$ for $c \in F$. Thus, by (11) $(x, y, z) + (y, x, z) = sy$ for $s \in F$. Analogously, $(x, y+1, z) + (y+1, x, z) = s'(y+1)$ for $s' \in F$. Comparing the last two equations we have $(s - s')y = s'1$. Thus, either $s' = 0$ or y is a scalar multiple of the identity element. In either case $(x, y, z) + (y, x, z) = 0$, so R is left alternative.

LEMMA 2. *If R contains an identity element* 1 *and satisfies* (2) *then R is right alternative.*

Proof. Let $x, y, z \in R$, $\alpha = g(x, y, z)$, and $\beta = g(x+1, y, z)$. Then $(\alpha - 1)[x(yz) + x(zy)] = (x, y, z) + (x, z, y) = (x+1, y, z) + (x+1, z, y) = (\beta - 1)[x(yz) + x(zy) + yz + zy]$. Thus we get

$$(\alpha - \beta)[x(yz) + x(zy)] = (\beta - 1)[yz + zy] . \tag{13}$$

Suppose $\alpha = \beta$. If $\beta = 1$ the result follows immediately whereas if $\beta \neq 1$ then $yz + zy = 0$. But in this case also $(x, y, z) + (x, z, y) = (\alpha - 1)[x(yz + zy)] = 0$ so the result holds. It follows that $\alpha \neq \beta$. In general then, if $(a, b, c) + (a, c, b) \neq 0$ then $g(a, b, c) \neq g(a+1, b, c)$. We are left with the case $\alpha \neq \beta$. Then by (13) and (2′) applied to the triple $x+1, y, z$ we have $(x, y, z) + (x, z, y) = l[zy + yz]$ for $l \in F$. Applying the same argument to the triple $x, y, z+1$, we get $(x, y, z+1) + (x, z+1, y) = l'[zy + yz + 2y]$ so that $(l - l')[zy + yz] = 2l'y$. If $l = l'$ we get right alternativity as in Lemma 1. Thus we may assume that $zy + yz = l''y$ for $l'' \in F$ so that $(x, y, z) + (x, z, y) = \gamma y$ for $\gamma \in F$. Similarly $(x, y+1, z) + (x, z, y+1) = \gamma'(y+1)$ for $\gamma' \in F$. Setting the last two equations equal we get $(\gamma - \gamma') = \gamma' 1$ and it follows that either $\gamma' = 0$ or y is a scalar multiple of the identity element. Thus, in any case $(x, y, z) + (x, z, y) = 0$.

Combining Lemmas 1 and 2 we have

THEOREM 1. *If R contains an identity element and satisfies* (1) *and* (2) *then R is an alternative algebra.*

3. Algebras containing an idempotent.

Henceforth, we drop the assumption that R contains an identity element but assume instead that it contains a nonzero idempotent.

LEMMA 3. *If R satisfies* (1) *and* (2) *then* $(e, e, R) = (R, e, e) = (e, R, e) = 0$.

Proof. Let $x \in R$. By (1) $ex = \alpha e(ex)$ for $\alpha = g(e, e, x)$. Thus $(e, e, x) = (\alpha - 1)e(ex)$. Similarly $(e, e, x + e) = (\beta - 1)[e(ex) + e]$ for $\beta = g(e, e, x + e)$. Thus $(\alpha - \beta)e(ex) = (\beta - 1)e$. Now $\alpha = \beta$ implies $\beta = 1$ so that $(e, e, x) = 0$. Assume $\alpha \neq \beta$. Then $e(ex)$ is a scalar multiple of e so that ex is a scalar multiple of e. Thus $(e, e, x) = ex - e(ex) = 0$ and we have $(e, e, R) = 0$. Similarly, by (2) we get $(R, e, e) = 0$.

For the last identity we first note that $(e, R, e)e = 0$. For if $x \in R$ then $(e, x, e)e = [(ex)e - e(xe)]e = (ex)e - [e(xe)]e = [e(x - xe)]e$. But by (1), $[e(x - xe)]e = -[(x - xe)e]e + g(e, x - xe, e)[e((x - xe)e) + (x - xe)e] = 0$ by the earlier remarks. Therefore $(e, R, e)e = 0$. Now by (1) $(e, x, e) = (\mu - 1)[e(xe) + xe]$ for $\mu = g(e, x, e)$ and $(e, x + e, e) = (\delta - 1)[e(xe) + xe + 2e]$ for $\delta = g(e, x + e, e)$. Therefore $(\mu - \delta)[e(xe) + xe] = 2(\delta - 1)e$. If $\mu = \delta$ then $\delta = 1$ so that $(e, x, e) = 0$. Otherwise $e(xe) + xe$ is a scalar multiple of e from which it follows that $(e, x, e) = te$ for some $t \in F$. But $0 = (e, x, e)e = te$ so that $t = 0$. Thus $(e, x, e) = 0$ so that $(e, R, e) = 0$.

LEMMA 4. *Let e be an idempotent of a ring R satisfying* (1). *Then to each $x, y \in R$ there are elements $a, b \in F$ such that $(x, y, e) + (y, x, e) = ae$ and $(x, e, y) + (e, x, y) = be$. Similarly, if R satisfies* (2) *there are elements $c, d \in F$ such that $(x, y, e) + (x, e, y) = ce$ and $(e, x, y) + (e, y, x) = de$.*

Proof. We prove the first identity only as the others are proved analogously. By (1) $(x, y, e) + (y, x, e) = (\alpha - 1)[x(ye) + y(xe)]$ and $(x, y + e, e) + (y + e, x, e) = (\beta - 1)[x(ye) + y(xe) + 2xe]$ for $\alpha = g(x, y, e)$ and $\beta = g(x, y + e, e)$. Thus, $(\alpha - \beta)[x(ye) + y(xe)] = 2(\beta - 1)xe$. If $\alpha \neq \beta$ then $x(ye) + y(xe) = l(xe)$ for $l \in F$. If $\alpha = \beta$ then either $\beta = 1$ or $xe = 0$. In any event we have $(x, y, e) + (y, x, e) = t(xe)$ for $t \in F$ or $xe = 0$. Similarly $(x + e, y, e) + (y, x + e, e) = t'(xe + e)$ for $t' \in F$ or $xe + e = 0$. A simple analysis of the four combinations yields $(x, y, e) + (y, x, e) = ae$ for some $a \in F$.

It is well known that Lemma 3 implies that, relative to an idempotent e, R has a Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ where $R_{ij} = \{x \in R \mid ex = ix, xe = jx\}$. Thus, we only have to prove the multiplicative properties in:

THEOREM 2. *Let R be an algebra satisfying* (1) *and* (2). *Then if e is an idempotent of R, R has a Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ relative to e and the Peirce subspaces multiply according to*:

(a) $R_{ij}R_{jk} \subseteqq R_{ik}$.

(b) $R_{ij}R_{ij} \subseteqq R_{ji}$.

(c) $R_{ij}R_{kl} = 0$ *if* $j \neq k$ *and* $(i, j) \neq (k, l)$.

(d) $r_{ij}^2 = 0$ *for any* $r_{ij} \in R_{ij}$, $i \neq j$.

Proof. Let $x, y \in R_{ii}$. By Lemma 4 $(x, y, e) + (x, e, y) = ce$ or $(xy)e - i(xy) = ce$. If $i = 1$ then by writing $xy = a_{11} + a_{10} + a_{01} + a_{00}$ and comparing component parts of both sides of the equation we get $xy \in R_{11} + R_{01}$. Thus $R_{11}^2 \subseteqq R_{11} + R_{01}$. From Lemma 4 again $(x, e, y) + (e, x, y) = be$ or $i(xy) - e(xy) = be$. From this we get $R_{11}^2 \subseteqq R_{11} + R_{10}$ and $R_{00}^2 \subseteqq R_{11} + R_{01} + R_{00}$. Hence $R_{11}^2 \subseteqq (R_{11} + R_{01}) \cap (R_{11} + R_{10}) = R_{11}$. In (2) let $x, y \in R_{00}$, $z = e$ to get $(xy)e = 0$. Thus $R_{00}^2 \subseteqq (R_{00} + R_{10}) \cap (R_{11} + R_{01} + R_{00}) = R_{00}$. Hence the R_{ii} are subalgebras.

To show that the subalgebras are othogonal, let $x \in R_{ii}$, $y \in R_{jj}$, $i \neq j$. Then from $(x, y, e) + (x, e, y) = ce$, we get $(xy)e + (i - 2j)xy = ce$ from which it follows that $xy = \delta e \in R_{11}$ for some $\delta \in F$. Thus $R_{ii}R_{jj} \subseteqq R_{11}$. Now let $x \in R_{11}$, $y \in R_{00}$. By (2) $(xy)e + (xe)y = \alpha[x(ye) + x(ey)] = 0$ so that $2xy = xy = 0$. By (1) $(yx)e = g(y, x, e)[y(xe) + x(ye)]$ and by (2) $(yx)e + (ye)x = g(y, x, e)[y(xe) + y(ex)]$. Since $yx \in R_{11}$ and $xy = 0$ we arrive at $yx = g(y, x, e)(yx) = 2g(y, x, e)(yx)$. Thus $yx = 0$ and we have $R_{11}R_{00} = R_{00}R_{11} = 0$.

Before proceeding note that $R_{ij}R_{jj} + R_{ii}R_{ij} \subseteqq R_{11} + R_{ij}$ for all i, j. For by Lemma 4, $(x, e, y) + (e, x, y) = be$. Then if $x \in R_{11}$, $y \in R_{10}$ or $x \in R_{10}$, $y \in R_{00}$ we obtain $xy - e(xy) = be \in R_{11}$ so that $xy \in R_{11} + R_{10}$. Similarly in $(x, e, y) + (x, y, e) = ce$ let $x \in R_{00}$, $y \in R_{01}$ or $x \in R_{01}$, $y \in R_{11}$ to get $(xy)e - xy = ce \in R_{11}$ so that $xy \in R_{11} + R_{01}$.

Next let $x \in R_{ii}$, $y \in R_{ij}$. Then $xy \in R_{11} + R_{ij}$. If $\alpha = g(x, e, y)$ then by (1), $(xe)y + (ex)y = \alpha[x(ey) + e(xy)]$ is $2ixy = \alpha[ixy + e(xy)]$. Also by (2), $(xe)y + (xy)e = \alpha[x(ey) + x(ye)]$ is $ixy + (yx)e = \alpha xy$. If $\alpha = 1$ then $ixy = e(xy)$ and $ixy + (xy)e = xy$ imply $(xy)_{11} = 0$ so that $xy \in R_{ij}$. If $\alpha \neq 1$ and $i = 1$ then by (1) $2xy = 2\alpha xy$ so that $xy = 0$, whereas if $i = 0$, (2) gives $xy = \alpha xy$ and again $xy = 0$. Therefore, in all cases $xy \in R_{ij}$ or $R_{ii}R_{ij} \subseteqq R_{ij}$.

By Lemma 4, $(y, x, e) + (y, e, x) = c'e$ reduces to

$$(yx)e + (j - 2i)yx = c'e$$

and we have $yx \in R_{11}$. Next let $\mu = g(e, y, x)$. By (1), $(ey)x + (ye)x = \mu[e(yx) + y(ex)]$ or $yx = \mu[yx + iyx]$. By (2), $(ey)x + (ex)y = \mu[e(yx) + e(xy)]$ or $iyx + ixy = \mu[yx + ixy]$. If $i = 1$, $yx = 2\mu yx$ and $yx + xy = \mu[yx + xy]$. If $\mu \neq 1$ then $yx = -xy$ so that $yx \in R_{11} \cap R_{ij} = 0$. If $\mu = 1$, $yx = 2yx$ implies $yx = 0$. If $i = 0$, then $\mu yx = 0$ implies $yx = 0$. Hence $yx = 0$ in all cases and $R_{ij}R_{ii} = 0$.

Now let $x \in R_{ii}$, $y \in R_{ji}$ and $\nu = g(y, e, x)$. Then $yx \in R_{11} + R_{ji}$. By (1) we obtain $yx = \nu(iyx + e(yx))$ and by (2) we have $i(yx) + (yx)e = 2\nu i(yx)$. Therefore, if $i = 0$ we have $(yx)e = 0$ so that $(yx)_{11} = 0$. If $i = 1$ we have $(1 - \nu)yx = \nu[e(yx)]$ and $2yx = 2\nu yx$. Therefore, if $\nu \neq 1$ then $yx = 0$ whereas, if $\nu = 1$, $e(yx) = 0$ so that $(yx)_{11} = 0$. Hence, in all cases $yx \in R_{ji}$ or $R_{ji}R_{ii} \subseteq R_{ji}$.

By Lemma 4, $(x, e, y) + (e, x, y) = be$ which reduces to $(3i - 1)xy - e(xy) = be$. This implies that $xy \in R_{11}$. Now let $\delta = g(x, y, e)$. Then (1) implies $xy + i(yx) = \delta i(xy + yx)$ so that if $i = 0$ then $xy = 0$. (2) implies $(1 + i)(xy) = \delta ixy$. Therefore, if $i = 1$ we obtain $xy + yx = \delta(xy + yx)$ and $2xy = \delta xy$. Therefore, if $\delta = 1$ then $xy = 0$ whereas, if $\delta \neq 1$ we have $xy = -yx \in R_{11} \cap R_{01} = 0$. Thus $R_{ii}R_{ji} = 0$.

Let $x \in R_{10}$, $y \in R_{01}$. Then $(x, e, y) + (e, x, y) = be$ reduces to $xy - e(xy) = be$ so that $xy \in R_{11} + R_{10}$. Also $(x, y, e) + (x, e, y) = ce$ reduces to $(xy)e - xy = ce$ so that $xy \in R_{11} + R_{01}$. Thus $xy \in (R_{11} + R_{01}) \cap (R_{11} + R_{10}) = R_{11}$ and $R_{10}R_{01} \subseteq R_{11}$. Now $(y, e, x) + (e, y, x) = b'e$ reduces to $-e(yx) = b'e$ so that $yx \in R_{11} + R_{01} + R_{00}$. Let $\alpha = g(x, y, e)$. Then $(xy)e + (yx)e = \alpha[x(ye) + y(xe)]$ and $(xy)e + (xe)y = \alpha[x(ye) + x(ey)]$ reduces to $xy + (yx)e = \alpha xy$ and $xy = \alpha xy$, respectively. Thus $(yx)e = 0$ and $yx \in R_{00}$. Hence $R_{01}R_{10} \subseteq R_{00}$.

We have established (a) and (c). For (b) let $x, y \in R_{10}$. Then $(x, y, e) + (x, e, y) = ce$ reduces to $(xy)e - xy = ce$ so that $xy \in R_{11} + R_{01}$. Let $\delta = g(x, e, y)$. Then $(xe)y + (xy)e = \delta[x(ey) + x(ye)]$ reduces to $xy = \delta(xy)$. Thus, if $xy \neq 0$ then $\delta = 1$. But then we get $(xe)y + (ex)y = x(ey) + e(xy)$ or $e(xy) = 0$. Thus $xy \in R_{01}$ so that $R_{10}^2 \subseteq R_{01}$.

If $x, y \in R_{01}$, then $(x, e, y) + (e, x, y) = be$ reduces to $xy - e(xy) = be$ or $xy \in R_{11} + R_{01}$. Let $\delta = g(x, e, y)$. Then $(xe)y + (ex)y = \delta[x(ey) + e(xy)]$ and $(xe)y + (xy)e = \delta[x(ey) + x(ye)]$ reduce to $xy = \delta xy$ and $xy + (xy)e = \delta xy$, respectively. Thus $(xy)e = 0$ and $xy \in R_{10}$. Hence $R_{01}^2 \subseteq R_{10}$.

Finally, for (d) let $x \in R_{ij}$ for $i \neq j$. Then $x^2 \in R_{ji}$. Let $y = x$ in $(x, y, e) + (y, x, e) = ce$ to obtain $2(i - j)x^2 = ae \in R_{11}$. Since $i \neq j$ and $x^2 \in R_{ji}$, it follows that $x^2 = 0$.

The alternative nucleus, $N_A(R)$, of an arbitrary ring R is defined by $N_A(R) = \{r \in R \mid (x, r, x) = 0$ and $(r, y, x) = (y, x, r) = (x, r, y)$ for all $x, y \in R\}$. It is shown in [4] that Theorem 2 is equivalent to the fact that $e \in N_A(R)$ for any idempotent e of R. It is immediate that if $i \neq j$ then $R_{ij} \subseteqq N_A(R)$ for if $r_{ij} \in R_{ij}$ then $(e + r_{ij})^2 = e + r_{ij}$. Thus $e + r_{ij}$ is idempotent so that $e + r_{ij} \in N_A(R)$ and so $r_{ij} \in N_A(R)$.

LEMMA 5. *R_{11} and R_{00} are alternative subalgebras of R.*

Proof. It is immediate from Theorem 1 that R_{11} is alternative since R_{11} is an algebra which contains an identity element e. Now let $x, y, z \in R_{00}$, $\alpha = g(x, y, z)$, and $\beta = g(x + e, y + e, z + e)$. Then by (1′) $(x, y, z) + (y, x, z) = (\alpha - 1)[x(yz) + y(x)z]$. Similarly

$$\begin{aligned}(x, y, z) + (y, x, z) &= (e + x, e + y, e + z) + (e + y, e + x, e + z)\\ &= (\beta-1)[(e + x)[(e + y)(e + z)] + (e + y)[(e + x)(e + z)]]\,.\end{aligned}$$

Comparing the last two identities we obtain $(\alpha - \beta)[x(yz) + y(xz)] = 2(\beta - 1)e$ since R_{00} is a subalgebra. If $\alpha = \beta$ then $\beta = 1$ so that R_{00} is left alternative. If $\alpha \neq \beta$ then $x(yz) + y(xz) = 2[(\beta - 1)/(\alpha - \beta)]e$. But $x(yz)+y(xz) \in R_{00}$ and $2[(\beta-1)/(\alpha-\beta)]e \in R_{11}$. Thus $x(yz)+y(xz)=0$ and $\beta = 1$. Hence R_{00} is left alternative. A similar argument using (2′) shows that R_{00} is right alternative.

THEOREM 3. *If R is an algebra satisfying* (1) *and* (2) *and if R contains a nonzero idempotent element, then R is alternative.*

Proof. Let $x, y \in R$. Then $x = \sum_{i,j=0}^{1} x_{ij}$ and $y = \sum_{i,j=0}^{1} y_{ij}$ so that $(x, x, y) = \sum_{i,j=0}^{1} (x, x, y_{ij})$. Now if $i \neq j$, then $y_{ij} \in N_A(R)$ so that, by the definition of $N_A(R)$, $(x, x, y_{ij}) = 0$. Thus, (x, x, y) reduces to $\sum_{l=0}^{1} (x, x, y_{ll}) = \sum_{i,j,k,r=0}^{1} (x_{ij}, x_{kr}, y_{ll})$. Let S denote the sum $\sum_{i,j,k,r,l=0}^{1} (x_{ij}, x_{kr}, y_{ll})$. The terms in S of the form (x_{ij}, x_{kk}, y_{ll}) are all zero by Theorem 2 and Lemma 5. The terms in S of the form (x_{ij}, x_{ij}, y_{ll}) for $i \neq j$ are all zero since $x_{ij} \in N_A(R)$. Finally the other terms in S come in pairs of the form $(x_{ij}, x_{kr}, y_{ll}) + (x_{kr}, x_{ij}, y_{ll})$. Since $i \neq j$ or $k \neq r$ the sum of each of these pairs is zero. Thus $(x, x, y) = 0$ and R is left alternative. Similarly R is right alternative.

The result of Theorem 1 holds true if the ring satisfies any pair of the identities (1)-(6) other than the pairs (1) and (6), (2) and (5), and (3) and (4) for which the result is obviously not true. The same holds true if the ring does not contain an identity but does contain a nonzero idempotent, except that here the case in which the identities (1) and (5) are satisfied is left open since we are unable to establish the property $R_{00}^2 \subseteqq R_{00}$ in this case. The proofs vary somewhat from those presented here but the basic attack is the same. Detailed proofs are available from the authors upon request.

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