APPLICATIONS OF APPROXIMATION THEORY TO DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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The authors utilize fundamental approximation theory concepts to establish an existence theorem for an initial value problem with a deviating argument. The techniques used to establish this theorem result in an algorithm to compute polynomial approximations to the solution of the initial value problem.

1. **Introduction.** In recent years several authors [2, 5, 8, 9, 10, 12] have considered best approximating in some sense the solutions to various types of nonlinear differential and integro-differential equations. These papers have generally been devoted to examining questions involving the existence of best approximations and with proving that appropriate sequences of approximations converge to a solution of the differential or integro-differential equation. The best approximation problems of the above references are ordinarily nonlinear and consequently computational questions are not readily resolved.

More recently Allinger [1], Henry [1, 6] and Wiggins [6] have developed alternative approximation theory approaches that are computationally adaptive.

In this regard [6] deals with nonlinear initial value problems without deviating arguments, whereas [1] primarily considers linear initial value problems with deviating arguments. The approximation problems of [6] and [1] are basically different. The goal of the present paper is to extend the results of [6] to arbitrary order and to the more difficult deviating argument case. The theory developed in the present paper will not contain the linear theory of [1]; in a subsequent paper [7] the authors will extend the results in [1] and relate those extensions to the fundamental results of the present paper.

2. **The initial value problem.** Consider the scalar initial value problem

$$(1) \quad x^{(n)}(t) - \widehat{f}(t, x(t), \cdots, x^{(n-1)}(t), x(h(t)), \cdots, x^{(n-1)}(h(t))) = 0 \,,$$
$$x^{(i)}(0) = \alpha_i, \ i = 0, 1, \cdots, n - 1, \ t \in J = [-\gamma, \tau] \,,$$

where $\gamma \geq 0$, $\tau \geq 0$, and $\gamma + \tau > 0$. This initial value problem is

considered in [6] for $h(t) = t$ and $n = 2$. A function x is a solution to initial value problem (IVP) (1) on J if the $x^{(i)}(t)$, $i = 0, \cdots, n$ are defined for $t \in J \cup h(J)$ and if x satisfies IVP (1) for $t \in J$. In (1) it is assumed that $\hat{f}: J \times R^{2n} \to R$ is continuous, and that $h \in C(J)$. Other hypotheses will be imposed later. We note that (1) generalizes not only the initial value problem considered in [6], but also the initial value problem with deviating argument of [1].

From a theoretical viewpoint, IVP (1) can be examined more efficiently in an equivalent vector setting. The scalar form will be utilized in the computations.

The standard transformation

$$(2) \qquad X(t) = [x_1(t), \cdots, x_n(t)]^T = [x(t), \cdots, x^{(n-1)}(t)]^T$$

converts IVP (1) into a companion vector IVP

$$(1A) \qquad \begin{aligned} \dot{X}(t) &= \bar{A}X(t) + F(t, X(t), X(h(t))) , \\ X(0) &= \varLambda, \quad t \in J , \end{aligned}$$

where $\varLambda = (\alpha_0, \cdots, \alpha_{n-1})^T$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \cdot & & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} ,$$

and

$$(3) \qquad F(t, X(t), X(h(t))) = [0, \cdots, 0, f_h[x](t)]^T ,$$

where for notational convenience

$$(4) \qquad f_h[x](t) = \hat{f}(t, x(t), \cdots, x^{(n-1)}(t), x(h(t)), \cdots, x^{(n-1)}(h(t))) .$$

The conditions on $\hat{f}$ imply that $F: J \times R^{2n} \to R^n$ may be assumed to be continuous. Let $\varPi_k$ denote the set of scalar polynomials of degree at most k. Define

$$\begin{aligned} (5) \qquad \mathscr{P}_k = \{ p \in \varPi_k &: p(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + b_n t^n \\ &+ \cdots + b_k t^k, t \in J, \text{ where } c_i = \alpha_i/i! \text{ and} \\ &b_i \in R, \quad (i = n, n+1, \cdots, k) \} . \end{aligned}$$

Thus $\mathscr{P}_k$ is the set of all scalar polynomials of degree at most k that satisfy the initial conditions of (1). Let $\hat{\varPi}_k$ denote the set of functions $P: J \to R^n$ such that each component of P is an element of $\varPi_k$. Corresponding to (5), define

$$(6) \qquad \hat{\mathscr{P}}_k = \{P \in \hat{\varPi}_k \colon P = [p, \dot{p}, \cdots, p^{(n-1)}]^T, \quad p \in \mathscr{P}_k\} .$$

Hence for each $P \in \hat{\mathscr{P}}_k$, $P(0) = \varLambda$. Let $\bar{Z}(t)$ be the fundamental matrix solution to the matrix initial value problem

$$(7) \qquad \dot{\bar{X}}(t) = \bar{A}\bar{X}(t), \ \bar{X}(0) = I, \ t \in J .$$

Hereafter, for $X = [x_1, \cdots, x_n] \in R^n$,

$$|X| = \max_{1 \leq i \leq n} (|x_i|) ;$$

if X is a continuous mapping from J to R^n, then

$$|X|_J = \max_J \max_{1 \leq i \leq n} (|x_i(t)|) .$$

Because of the nature of the matrix $\bar{A}$, the solution $\bar{Z}\varLambda$ to the vector initial value problem

$$(8) \qquad \dot{X}(t) = \bar{A}X(t), \quad X(0) = \varLambda, \quad t \in J ,$$

is an element of $\hat{\mathscr{P}}_k$. Due to the continuity of F, there exists a positive constant M depending on N such that $|F(t, X, Y)| \leq M$ for all $(t, X, Y) \in J \times R^{2n}$ with $\max(|X|, |Y|) \leq N$. Let $\bar{Z}(t) = (z_{ij}(t))$, $t \in J$. Define

$$\alpha = |\bar{Z}|_J = \max_J \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |z_{ij}(t)| \right) .$$

Then $J_1 = [-\gamma_1, \tau_1] \subseteqq J$, γ_1 and τ_1 nonnegative with $\gamma_1 + \tau_1 > 0$, and an appropriate deviating argument g are determined as follows:

H_1. Let $N \geq 1 + |\bar{Z}\varLambda|_J$. Choose J_1 to insure that for all $t \in J_1$, $2|t|M\alpha \leq 1$.

H_2. Assume an interval $J_2 = [-\gamma_2, \tau_2]$, γ_2 and τ_2 nonnegative with $\gamma_2 + \tau_2 > 0$, satisfying $J_2 \subseteqq J_1$ and $h(J_2) \subseteqq J_1$, exists. Define $g \in C(J_1)$ by $g(t) = h(t)$, $t \in J_2$, $g(J_1) \subseteqq J_1$.

REMARK. If $h(0) = 0$, a condition frequently required of the deviating argument (see [11]), then the existence of a J_2 and g satisfying H_2 is assured.

Assuming hypotheses H_1 and H_2, an alternative to initial value problem (1A) is

$$(1B) \qquad \begin{aligned} &\dot{X}(t) = \bar{A}X(t) + F(t, X(t), X(g(t))) , \\ &X(0) = \varLambda, \quad t \in J_1 . \end{aligned}$$

3. **An existence theorem.** We are now in a position to define an approximating scheme that results in a constructive existence theorem for IVP(1B) and hence for (1A) and (1).

Let

$$(9) \qquad S_k = \{P \in \hat{\mathscr{P}}_k \colon |P - \bar{Z}\varLambda| \leqq 2|t| M\alpha , \quad t \in J_1\} .$$

Then S_k is a compact, convex subset of $\hat{\mathscr{P}}_k$. For $x \in C^{n-1}(J_1)$, let

$$(10) \qquad \hat{F}[X](t) = F(t, X(t), X(g(t))) ,$$

where again $X(t)$ is given by (2). If $P \in S_k$, consider the best approximation problem

$$(11) \qquad \inf_{V \in \hat{\mathscr{P}}_k} |\dot{V} - \bar{A}V - \hat{F}[P]|_{J_1} .$$

Because of the nature of $\bar{A}$ and $\hat{\mathscr{P}}_k$, this minimization problem reduces to the scalar best approximation problem

$$(12) \qquad \inf_{v \in \varPi_{k-n}} ||v - f[p]||_{J_1} ,$$

where p is the first component of P and where $f[p] = f_g[p]$. Since $f \colon C^{n-1}(J_1) \to C(J_1)$, it is well known from classical approximation theory that there exists a $\bar{v} \in \varPi_{n-k}$ such that

$$(13) \qquad \inf_{v \in \varPi_{k-n}} ||v - f[p]||_{J_1} = ||\bar{v} - f[p]||_{J_1} .$$

Integrate $\bar{v}$ n times to obtain the kth degree polynomial $v_b \in \mathscr{P}_k$. If $V_b = (v_b, \cdots, v_b^{(n-1)})$, then

$$(14) \qquad \inf_{V \in \hat{\mathscr{P}}_k} |\dot{V} - \bar{A}V - \hat{F}[P]|_{J_1} = |\dot{V}_b - \bar{A}V_b - \hat{F}[P]|_{J_1} .$$

Define an operator T_k on S_k as follows: if $P \in S_k$, then

$$(15) \qquad T_k[P](t) = V_b(t) , \quad t \in J_1 .$$

Thus if $P \in S_k$, $T_k[P]$ is the best approximation from $\hat{\mathscr{P}}_k$ to $\hat{F}[P]$ in the sense of (14). Note that since (13) has a unique solution v_b, V_b is also unique.

THEOREM 1. *Assume that $\hat{f}$ and h satisfy the continuity requirements described below* (1), *and that H_1 and H_2 are valid. Then T_k is a continuous mapping from S_k into S_k.*

Proof. Suppose that $T_k[P] = V_b$. Let

$$(16) \qquad \dot{V}_b(t) - \bar{A}V_b(t) - \hat{F}[P](t) = E(t) .$$

Since $P \in S_k$, H_1 and H_2 imply that $|\widehat{F}[P]|_{J_1} \leqq M$, and consequently, since $\bar{Z}(t)\varLambda \in \widehat{\mathscr{P}}_k$ and V_b is best in the sense of (14),

$$
\begin{aligned}
| \dot{V}_b &- \bar{A}V_b - \widehat{F}[P]|_{J_1} \\
&\leqq |\dot{\bar{Z}}\varLambda - \bar{A}\bar{Z}\varLambda - \widehat{F}[P]|_{J_1} \\
&= |\widehat{F}[P]|_{J_1} \leqq M \, .
\end{aligned}
$$

This inequality and (16) imply that

$$
(17) \qquad\qquad |E|_{J_1} \leqq M \, .
$$

Now (16) holds if and only if

$$
(18) \qquad
\begin{aligned}
V_b(t) &- \bar{Z}(t)\varLambda \\
&= \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)\widehat{F}[P](s)ds \\
&\quad + \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)E(s)ds \, .
\end{aligned}
$$

A variation of parameters argument readily verifies equality (18). Since $\bar{A}$ is a constant matrix, $\bar{Z}(t)\bar{Z}^{-1}(s) = \bar{Z}(t-s)$; therefore (17) and (18) imply that

$$
\begin{aligned}
|V_b - \bar{Z}\varLambda| &\leqq 2\,|t|\,M|\bar{Z}|_{J_1} \\
&= 2|t|M\alpha \, , \quad t \in J_1 \, .
\end{aligned}
$$

Thus $V_b \in S_k$ and hence T_k maps S_k into S_k.

Establishing that T_k is continuous is somewhat more difficult and requires the use of fundamental concepts of approximation theory. Suppose that $P \in S_k$ and that $\bar{P}$ is any other element of S_k; if $T_k[P] = Q$ and $T_k[\bar{P}] = \bar{Q}$, then as in (13),

$$
\inf_{v \in \varPi_{k-n}} ||v - f[p]||_{J_1} = ||q^{(n)} - f[p]||_{J_1}
$$

and

$$
\inf_{v \in \varPi_{k-n}} ||v - f[\bar{p}]||_{J_1} = ||\bar{q}^{(n)} - f[\bar{p}]|| \, ,
$$

where p, $\bar{p}$, q, and $\bar{q}$ are the first components of P, $\bar{P}$, Q, and $\bar{Q}$, respectively. The classical Freud theorem [3, p. 82] of approximation theory implies that

$$
(19) \qquad ||q^{(n)} - \bar{q}^{(n)}||_{J_1} \leqq \lambda_p ||f[p] - f[\bar{p}]||_{J_1} \, .
$$

Let $\varepsilon > 0$ be given. Since $\hat{f}$ is uniformly continuous on compact subsets of $J \times R^{2n}$, there exists a $\delta > 0$ such that if $|P - \bar{P}|_{J_1} < \delta$,

(20) $\|f[p] - f[\bar{p}]\|_{J_1} < \dfrac{\varepsilon}{\lambda_p} \dfrac{1}{\max\limits_{0 \leq i \leq n} [\max (\gamma_1, \tau_1)]^{n-i}}$,

$$i = 0, 1, \cdots, n \ .$$

Inequalities (19) and (20) imply that

$$|q^{(i)}(t) - \bar{q}^{(i)}(t)| < \varepsilon \qquad i = 0, 1, \cdots, n - 1, t \in J \ .$$

Thus

(21) $|Q(t) - \bar{Q}(t)| < \varepsilon$.

Since $T_k[P] = Q$ and $T_k[\bar{P}] = \bar{Q}$, the above analysis establishes that given $P \in S_k$ and $\varepsilon > 0$, there exists $\delta_o > 0$ such that if $\bar{P} \in S_k$ and $|P - \bar{P}|_{J_1} < \delta$, then

$$|T_k[P] - T_k[\bar{P}]|_{J_1} < \varepsilon \ .$$

Hence T_k is a continuous mapping from S_k into S_k and the proof is complete.

COROLLARY 1.   *Assume the hypotheses of Theorem 1 are valid. Then the mapping T_k has a fixed point in S_k.*

Proof. Since S_k is a compact, convex subset of $\hat{\mathscr{P}}_k$, the result follows from the Schauder fixed point theorem [16, p. 25].

We summarize the results of Theorem 1 and Corollary 1. There exists a $P \in \hat{\mathscr{P}}_k$ such that if we seek the best approximation V_b to $\hat{F}[P]$ in the sense of (14), this best approximation is precisely $V_b = P$. In the equivalent scalar form, if we seek the best approximation $\bar{v}$ from Π_{k-n} to $f[p]$ in the sense of (13), then $\bar{v} = p^{(n)}$.

The proof of Theorem 1 does not depend on k, and consequently the hypotheses of Theorem 1 imply that there exists a sequence

(22) $\{P_k(t)\}_{k=n}^{\infty}$, $t \in J_1$,

of fixed points, one for each operator T_k, $k \geq n$.

Define E_k, $k \geq n$, by

(23) $\dot{P}_k(t) - \bar{A}P_k(t) - \hat{F}[P_k](t) = E_k(t)$,

$t \in J_1$, where P_k is an element of the sequence (22). Since $P_k \in S_k$, $k \geq n$,

(24) $|P_k(t) - \bar{Z}(t)\Lambda| \leq 2|t| M\alpha$, $t \in J_1$.

Equation (23) and inequalities (17) and (24) imply that the sequences

(25) $\{|P_k|_{J_1}\}_{k=n}^{\infty}$ and $\{|\dot{P}_k|_{J_1}\}_{k=n}^{\infty}$

are bounded. Thus the sequence (22) is an equicontinuous and uniformly bounded family on J_1. Let W be a cluster point of this sequence. Equality (23) is valid if and only if

$$(26) \qquad P_k(t) - \bar{Z}(t)\Lambda = \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)\hat{F}[P_k](s)ds$$
$$+ \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)E_k(s)ds .$$

Thus if

$$(27) \qquad \lim_{k \to \infty} |E_k|_{J_1} = 0 ,$$

then (26) implies that W is a solution to IVP (1B). Thus W is a solution to IVP (1A) on J_2, and the first entry w of W satisfies IVP (1) on J_2. These observations constitute part of the conclusions of the next theorem.

THEOREM 2. *Suppose the hypotheses of Theorem 1 are satisfied. Then (27) is valid. If W is a cluster point of (22) on J_1, then W is a solution of IVP (1B) on J_1 and W is a solution of IVP (1A) on J_2.*

Proof. Based on the remarks preceeding the statement of Theorem 2, only (27) needs to be established. Let

$$(28) \qquad P_k = (p_k, \cdots, p_k^{(n-1)}) .$$

Then since P_k is a fixed point of T_k,

$$\inf_{v \in \Pi_{k-n}} ||v - f[p_k]||_{J_1} = ||p_k^{(n)} - f[p_k]||_{J_1} .$$

If

$$e_{k-n}(f[p_k]) = ||p_k^{(n)} - f[p_k]||_{J_1} ,$$

then

$$(29) \qquad e_{k-n}(f[p_k]) = |E_k|_{J_1} .$$

Jackson's theorem [13, p. 22] implies that

$$(30) \qquad e_{k-n}(f[p_k]) \leq 6\omega_k\left(\frac{L}{k-n}\right) ,$$

where L is half the length of the interval J_1, and ω_k is the modulus of continuity of $f[p_k]$ on J_1. Let $\varepsilon > 0$ be given. Since the sequences (25) are bounded, the sequences

$$\{p_k^{(i)}\}_{k=n}^\infty , \qquad i = 0, \cdots, n-1 ,$$

are uniformly bounded and equicontinuous families on J_1. Thus the uniform continuity of $\hat{f}$ on compact subsets of $J_1 \times R^{2n}$ implies that the sequence $\{f[p_k]\}_{k=n}^{\infty}$ is an equicontinuous family on J_1. Consequently given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|t - s| \leq \delta$ implies that $|f[p_k](t) - f[p_k](s)| \leq \varepsilon$, independent of k. Therefore $\omega_k(\delta) \leq \varepsilon$, independent of k. If K is large enough to insure that for all $k \geq K$, $L/(k - n) \leq \delta$, then the monotonicity of ω_k [13, p. 14] implies for all $k \geq K$,

$$\omega_k\left(\frac{L}{k - n}\right) \leq \omega_k(\delta) \leq \varepsilon .$$

Thus for all $k \geq K$, (29) and (30) imply that $|E_k|_{J_1} \leq 6\varepsilon$. Consequently $\lim_{k \to \infty} |E_k|_{J_1} = 0$, completing the proof.

4. **Computations.** We now describe an algorithm based on the theory of section three. For k sufficiently large, initially choose $P_{k0}(t) = \bar{Z}(t)A$. At the $l + 1$ step solve via the second algorithm of Remes [3, p. 97] the best approximation problem

$$(31) \qquad \inf_{V \in \mathscr{S}_k} |\dot{V} - \bar{A}V - \hat{F}[P_{k,l}]|_{J_1} = \inf_{v \in \Pi_{k-n}} ||v - f[p_{k,l}]||_{J_1} .$$

Let the solution of this minimizing problem be $P_{k,l+1}$ and continue. We again note that $p_{k,l+1}^{(n)}$ is then the minimizing polynomial to the scalar problem in (31). This procedure yields a unique sequence

$$(32) \qquad\qquad \{P_{k,l}\}_{l=0}^{\infty}$$

with a corresponding sequence of first elements of (32),

$$(33) \qquad\qquad \{p_{k,l}\}_{l=0}^{\infty} .$$

The sequences (32) and (33) have cluster points P_k and p_k, respectively, which are approximate solutions of degree k to (1A) and (1) on J_2.

The actual calculations are carried out in the scalar setting. Furthermore, the computations are effected for any interval $J_1 \subseteq J$ for which $h(J_1) \subseteq J_1$. If no such interval exists, the construction of an appropriate g would be necessary (see H_2). If convergence does not occur, then J_1 is reduced (see H_1).

We note that a cluster point of (32) need not be a fixed point of T_k in the sense of Corollary 1. However, an additional checkable condition insures that a cluster point of (32) is a fixed point of T_k. In particular, entries of sequence (33) satisfy

$$\inf_{v \in \Pi_{k-n}} ||v - f[p_{k,l}]||_{J_1} = ||p_{k,l+1}^{(n)} - f[p_{k,l}]||_{J_1} .$$

In calculating $p_{k,l+1}$, an alternating set (see [3, p. 75]) consisting of $k - n + 2$ arises. If $X_l = \{t_{1,l}, \cdots, t_{k-n+2,l}\}$ is an alternating set, and if

$$e_{k,l+1}(t) = p_{k,l+1}^{(n)}(t) - f[p_{k,l}](t) ,$$

then

$$e_{k,l+1}(t_{i,l}) = -e_{k,l+1}(t_{i+1,l}) ,$$
$$i = 1, \cdots, n - k + 1 .$$

Let $X = \{t_1, \cdots, t_{n-k+2}\}$ be a cluster point of $\{X_l\}_{l=0}^{\infty}$, and let

(34) $$e_k(t) = p_k^{(n)}(t) - f[p_k](t)$$

where p_k is the first entry of the cluster point P_k of (32) (and hence a cluster point of (33)). Without loss of generality we can assume that the subsequences of $\{X_l\}$ and of (33) that converge to X and p_k, respectively, involve the same subscripts.

H_3. Suppose for $t_i, t_{i+1} \in X$ that $e_k(t_i) = -e_k(t_{i+1})$, $i = 1, \cdots, k - n + 1$, where e_k is defined in (34).

We note that this condition is weaker than assuming the alternation theorem.

Assuming H_3 is valid, P_k is a fixed point of T_k, and consequently the theory of §3 applies. If (33) actually converges, H_3 is satisfied.

The following examples employ the algorithm described above. A Fortran IV program using double precision arithmetic is utilized in the calculations. The calculations for Examples 1 and 2 were executed on the Xerox Sigma 7 computer, located at Montana State University. The calculations for Example 3 were effected on the IBM 370, at The College of Charleston.

EXAMPLE 1.

$$\ddot{x}(t) - 2(2 - t)\dot{x}(t)x^2(t) = 0 , \quad J = [-.7, .7]$$
$$x(0) = 1/2 , \quad \dot{x}(0) = 1/4$$

The algorithm is employed over all of J, since $g(t) = t$. For $k = 5$ approximation is from

$$\mathscr{P}_5 = \{p(t): p(t) = .5 + .25t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5\} .$$

Let $B_l = (.5, .25, b_{2l}, b_{3l}, b_{4l}, b_{5l})$ be the coefficients of the l-th iterate $p_{k,l}$ of sequence (33). Then

$$B_0 = (.5, .25, 0, 0, 0, 0)$$
$$B_1 = (.5, .25, .125000, .020833, -.005208, -.001562)$$
$$\cdot \qquad \cdot \qquad \cdot$$
$$B_{15} = (.5, .25, .119879, .057090, .042496, .023350) .$$

If the algorithm is terminated with B_{15} (a stopping criteria is employed here), then the cluster point of (33) is

$$p_5(t) = .5 + .25t + .119879t^2 + .570909t^3 + .424964t^4 + .233505t^5 .$$

The actual solution is $x(t) = 1/(2 - t)$, and $||x - p_5||_J = .001779$.

EXAMPLE 2.

$$\ddot{x}(t) + 4x(t/2)\dot{x}(t/2) = 0, \ J = [-1, 1] ,$$
$$x(0) = 0 , \quad \dot{x}(0) = 2 .$$

We observe that $h(J) \subseteqq J$. Approximation is again from $\mathscr{P}_5$, and

$$B_0 = (0, 2, 0, 0, 0, 0) ,$$
$$\cdot \qquad \cdot \qquad \cdot$$
$$B_9 = (0, 2, 0, -1.290025, 0, .204262) .$$

Then B_9 and B_{10} are equal through six decimal places. Thus

$$p_5(t) = 2t - 1.290025t^3 + .204262t^5 .$$

The actual solution is $x(t) = \sin 2t$ and $||x - p_5||_J = .006865$.

EXAMPLE 3. $h(t) = 1/2(e^t - 1)$

$$\ddot{x}(t) + 4[2x(h(t)) + 1]^{-2} = 0 , \quad t \in [-.175, 1]$$
$$x(0) = 0 , \quad \dot{x}(0) = 2$$

In this example $h(J) = [-.08, .859] \subseteqq J$. For approximation from $\mathscr{P}_7$, the alternating set is

$$X = \{-.1750, -.1211, .0509, .3313, .6519, .9054, 1.0000\} .$$

The actual solution is $x(t) = ln(2t + 1)$, and

$$p_7(t) = 2t - 1.9688t^2 + 2.8151t^3 - 4.7085t^4$$
$$+ .58950t^5 - 4.0266t^6 + 1.0974t^7 .$$

Hypothesis H_3 is satisfied by e_7, and $||e_7|| = .08519$. Finally, $||x - p_7|| = .00496$.

5. Conclusions. This paper utilizes approximation theory techniques to obtain uniform approximate solutions to variants of *IVP* (1). The primary result of the paper is an existence theorem for *IVP* (1). An algorithm based on this existence theorem is used to compute approximate solutions to *IVP* (1). This paper extends results of the authors [6] to higher order initial value problems

with deviating arguments. The present paper is closely related to [1], but results in an approximation problem and a companion algorithm that differ from those given in [1] (for additional literature involving other deviating argument problems, see the references of [1]). In a subsequent paper [7] the authors intend to present a theory that parallels the theory of the present paper but contains the linear theory of [1]. The results of the present paper do substantially improve the nonlinear results contained in [1].

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