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RINGS WITH QUIVERS THAT ARE TREES

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Associated with each artinian ring R are two diagrams called the left and right quivers of R. We generalize a well-known theorem on hereditary serial rings by proving that if these quivers have no closed paths then R is a factor ring of a certain ring of matrices over a division ring. It follows that the categories of finitely generated left and right R-modules are Morita dual to one another. Applying our theorem and theorems of Gabriel and Dlab and Ringel, we show how to write explicit matrix representations of all hereditary algebras of finite module type.

A *quiver* is, in the terminology of Gabriel [8], [9], a finite set of points (vertices) connected by arrows. Given an artinian ring R and a basic set of primitive idempotents $e_1, \cdots, e_n$ of R (see, for example, [1, §27]), one forms $\mathscr{Q}({}_RR)$ the *left quiver of* R (see [11]): The vertices of $\mathscr{Q}({}_RR)$ are $v_1, \cdots, v_n$, one for each basic idempotent, with n_{ij} arrows from v_i to v_j iff Re_j/Je_j appears exactly n_{ij} times in a direct decomposition of the semisimple left R-module Je_i/J^2e_i. The right quiver $\mathscr{Q}(R_R)$ is formed similarly, with vertices $v'_1, \cdots, v'_n$ and n'_{ij} arrows from v'_i to v'_j iff e_jR/e_jJ appears exactly n'_{ij} times in a direct decomposition of e_iJ/e_iJ^2. Note that $n'_{ij} \neq 0$ iff $n_{ji} \neq 0$. Also, R is indecomposable iff $\mathscr{Q}({}_RR)$ is connected, i.e., there is a nonoriented path from v_i to v_j for every $i, j = 1, \cdots, n$.

A quiver $\mathscr{Q}$ is called a *tree* in case it is connected and contains no cycles, i.e., in case it has a unique nonoriented path from v_i to v_j, for every i, j. Let $\mathscr{Q}$ be such a quiver. Then the vertices of $\mathscr{Q}$ are partially ordered by $\leqq$, where $v_i \leqq v_j$ iff there is an oriented path from v_j to v_i (or $i = j$), and we can relabel the vertices so that $v_i \nleqq v_j$ implies $i \leqq j$. Having done this, we see that for any ring D, the set of matrices

$$T = \{[\![d_{ij}]\!] \mid d_{ij} \in D, d_{ij} = 0 \quad \text{if} \quad v_i \nleqq v_j\}$$

is a subring of the ring of upper triangular matrices over D. Moreever, if D is a division ring, then $\mathscr{Q}({}_TT) = \mathscr{Q}$, $\mathscr{Q}(T_T)$ is the dual quiver of $\mathscr{Q}$, and T is the unique basic *tic tac toe ring* (in the sense of Mitchell [12, §10.8]) over D with left quiver $\mathscr{Q}$.

Murase [14] showed that an indecomposable artinian ring whose quivers are of the form

$$v_1 \longleftarrow v_2 \longleftarrow v_3 \cdots v_{n-1} \longleftarrow v_n$$

is a factor ring of a block upper triangular matrix ring (i.e., of one whose basic ring is an upper triangular matrix ring) over a division ring. (Goldie [10] proved a similar result.) A ring with such a quiver is a serial ring, and an indecomposable hereditary artinian ring is serial iff it has quivers of this form. We extend this result, showing that any artinian ring whose quivers are trees is a factor ring of a tic tac toe ring over a division ring. As an application we also prove that such rings are self-dual in the sense that there is a Morita duality between their categories of finitely generated left and right modules.

Before proceeding to the proofs we note that, by the work of Gabriel [8],]9], and Dlab and Ringel [4], an indecomposable hereditary algebra over an algebraically closed field is of finite module type iff its quivers are Dynkin diagrams of type A_n, D_n, E_6, E_7, or E_8. These diagrams are all trees, so the theorem we are about to prove allows one to apply Gabriel's argument [8] (see also [2], [11]) to show that any artinian ring with quivers of type A_n, D_n, E_6, E_7, or E_8 is a ring of finite module type.

LEMMA 1. *Let R be an artinian ring with $e_1, \cdots, e_n$ a basic set of primitive idempotents. If Re_i/Je_i is isomorphic to a direct summand of $J^ke_j/J^{k+1}e_j$, then in $\mathscr{Q}({}_RR)$ there is an oriented path from v_j to v_i of length k. Moreover, if in addition R is hereditary, then the converse is also true.*

Proof. We induct on k. The cases $k=0$ and $k=1$ follow immediately from the definition of a quiver. Now let Re_i/Je_i be isomorphic to a direct summand of $J^ke_j/J^{k+1}e_j$. Let

$$\bigoplus_{r=1}^{t} Re_{j_r} \xrightarrow{f} J^{k-1}e_j \longrightarrow 0$$

be a projective cover. Then f induces an epimorphism

$$\bigoplus_{r=1}^{t} (Je_{j_r}/J^2e_{j_r}) \xrightarrow{\bar{f}} J^ke_j/J^{k+1}e_j \longrightarrow 0 \ .$$

Since $\bigoplus_{r=1}^{t} (Je_{j_r}/J^2e_{j_r})$ and $J^ke_j/J^{k+1}e_j$ are semisimple R/J-modules, $\bar{f}$ splits. Thus there exists an r with Re_i/Je_i isomorphic to a direct summand of $Je_{j_r}/J^2e_{j_r}$. By induction, there is an oriented path of length $(k-1)$ from v_j to v_{j_r} and one of length 1 from v_{j_r} to v_i, which combine to give the desired path of length k.

For the moreover part, suppose we have an oriented path

$$v_i = v_{i_k} \longleftarrow v_{i_{k-1}} \cdots v_{i_1} \longleftarrow v_{i_0} = v_j \ .$$

Assume that Re_{i_m}/Je_{i_m} is a direct summand of $J^me_j/J^{m+1}e_j$ $(m < k)$.

Then since R is hereditary, $J^m e_j \cong Re_{i_m} \oplus M$, some M. Thus

$$J^{m+1}e_j/J^{m+2}e_j \cong Je_{i_m}/J^2e_{i_m} \oplus JM/J^2M ,$$

and we are done since $Re_{i_{m+1}}/Je_{i_{m+1}}$ is a direct summand of $Je_{i_m}/J^2e_{i_m}$.

Now we are ready to prove the promised result.

THEOREM 2. *If the left and right quivers of an artinian ring R are trees, then there is an indecomposable tic tac toe ring T over a division ring D such that R is isomorphic to a factor ring of T. Moreover, $\mathscr{Q}({}_RR) = \mathscr{Q}({}_TT)$; and R is hereditary iff $R \cong T$.*

Proof. It is easy to see that a ring is Morita equivalent to an upper triangular tic tac toe ring over a division ring D iff it is isomorphic to a (block-upper-triangular) tic tac toe ring over D. Thus we may assume that R is basic.

Suppose that $\mathscr{Q} = \mathscr{Q}({}_RR)$ and $\mathscr{Q}(R_R)$ are trees, and correspondingly, relabel the vertices of $\mathscr{Q}$ and the idempotents of R as in the earlier discussion. In particular then, v_1 is minimal with respect to the partial order $\leqq$, and hence no arrows leave v_1.

Note that for each basic idempotent e_i, e_iRe_i is a division ring since $e_iJe_i = 0$ by Lemma 1. For each pair of idempotents e_p and e_q with an arrow from v_q to v_p in $\mathscr{Q}$ (and hence one arrow from v'_p to v'_q in $\mathscr{Q}(R_R)$), we have a left e_pRe_p- right e_qRe_q-bimodule e_pJe_q with $\dim({}_{e_pRe_p}e_pJe_q) = 1 = \dim(e_pJe_{q\,e_qRe_q})$. So we may choose $e_{pq} \in e_pJe_q$ with $e_{pq} \neq 0$ and define a division ring isomorphism $\sigma_{pq}\colon e_pRe_p \to e_qRe_q$ via $xe_{pq} = e_{pq}\sigma_{pq}(x)$ for $x \in e_pRe_p$. Since $\mathscr{Q}$ is connected, we have $e_pRe_p \cong e_rRe_r$ for all primitive idempotents e_p and e_r. Define $e_{ii} = e_i$ and for each $v_i \leqq v_j$ with oriented path

$$v_i = v_{i_0} \longleftarrow v_{i_1} \cdots v_{i_{k-1}} \longleftarrow v_{i_k} = v_j ,$$

define

$$e_{ij} = e_i e_{i_0 i_1} \cdots e_{i_{k-1} i_k} e_j .$$

For $v_p \leftarrow v_q$ in $\mathscr{Q}$, define $\gamma_{pq} = \sigma_{pq}$ and $\gamma_{qp} = \sigma_{pq}^{-1}$. Now let $v_1 = v_{i_0}, v_{i_1}, \cdots, v_{i_k} = v_j$ be the vertices of a nonoriented path from v_1 to v_j for $j \neq 1$. Define

$$\sigma_{1j} = \gamma_{i_{k-1}i_k} \circ \cdots \circ \gamma_{i_1 i_2} \circ \sigma_{i_0 i_1} \qquad \text{for} \quad j = 2, \cdots, n .$$

Define $\sigma_{11} = 1_{e_1Re_1}$. Let

$$D = \left\{ \sum_{j=1}^{n} \sigma_{1j}(x) \,\middle|\, x \in e_1Re_1 \right\} .$$

Then $D \cong e_jRe_j$ and D is a division subring of R.

Let $v_i \leqq v_j$ via an oriented path of length k. Then $e_iRe_j = e_iJ^ke_j$

by Lemma 1. It is then straightforward to verify that $De_{ij} = e_iJ^ke_j$, using the equalities $De_p = e_pRe_p$ and $e_pRe_{pq} = e_pRe_pe_{pq} = e_pJe_q$ for $v_p \leftarrow v_q$. Hence we have shown that

$$(*) \qquad R = \sum_{v_i \leqq v_j} De_{ij} .$$

Next we claim that $de_{ij} = e_{ij}d$ for any $d \in D$ and $v_i \leqq v_j$ in $\mathscr{Q}$. Suppose we have $v_p \leftarrow v_q$. Let $v_1 = v_{i_0}, v_{i_1}, \cdots, v_{i_m} = v_q$ be the non-oriented path from v_1 to v_q. If $v_p = v_{i_{m-1}}$, then $\sigma_{1q} = \sigma_{pq} \circ \sigma_{1p}$, and

$$e_{pq}\Big(\sum_{r=1}^{n} \sigma_{1r}(x)\Big) = e_{pq}\sigma_{1q}(x) = e_{pq}\sigma_{pq}(\sigma_{1p}(x)) = \sigma_{1p}(x)e_{pq}$$
$$= \Big(\sum_{r=1}^{n} \sigma_{1r}(x)\Big)e_{pq} .$$

If $v_p \neq v_{i_{m-1}}$, then $\sigma_{1p} = \sigma_{pq}^{-1} \circ \sigma_{1q}$, and

$$e_{pq}\Big(\sum_{r=1}^{n} \sigma_{1r}(x)\Big) = e_{pq}\sigma_{1q}(x) = \sigma_{pq}^{-1}(\sigma_{1p}(x))e_{pq} = \sigma_{1p}(x)e_{pq}$$
$$= \Big(\sum_{r=1}^{n} \sigma_{1r}(x)\Big)e_{pq} .$$

Now the claim follows by induction on the length of the path from v_j to v_i.

Let T be the tic tac toe ring

$$T = \{[\![d_{ij}]\!] \mid d_{ij} \in D, d_{ij} = 0 \quad \text{if} \quad v_i \nleqq v_j\} .$$

Define

$$\Phi: T \longrightarrow R \quad \text{via} \quad \Phi: [\![d_{ij}]\!] \longmapsto \sum_{v_i \leqq v_j} d_{ij}e_{ij} .$$

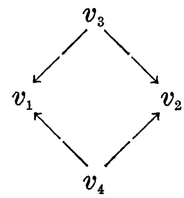
Since the elements of D commute with each e_{ij}, and since

$$e_{km}e_{pq} = \begin{cases} e_{kq} & \text{if} \quad m = p \\ 0 & \text{if} \quad m \neq p , \end{cases}$$

Φ is a ring homomorphism. Also Φ is onto by $(*)$.

Clearly $\mathscr{Q}({}_TT) = \mathscr{Q}({}_RR)$. If R is hereditary, then for $v_i \leqq v_j$ with oriented path of length k, $De_{ij} = e_iJ^ke_j \neq 0$ by Lemma 1. So $e_{ij} \neq 0$ and Φ is an isomorphism. If T is a tic tac toe ring whose quivers are trees, then T is hereditary by [12, Theorem IX. 10.9].

One could apparently use an argument similar to the one in [4, Proposition 10.2] to show that the rings of Theorem 2 are factor rings of so-called tensor rings (see [5]). The same argument, however, shows that rings with quivers



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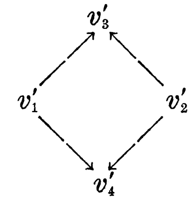

are also tensor rings. But these need not be tic tac toe rings. Indeed, let φ be an automorphism of a division ring D which does not fix the center of D. Then the ring R_φ of matrices

$$\begin{pmatrix} a & 0 & x & m \\ & b & y & z \\ & & c & 0 \\ & & & d \end{pmatrix}$$

with all entries in D except $m \in {}_DM_D$ where ${}_DM = {}_DD$ and multiplication in M_D is given by $m \cdot d = m\varphi(d)$, is a tensor ring that is not a tic tac toe ring. In contrast, by Theorem 2 or originally by Murase in [14], the ring S_φ of matrices

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$$

with $a, b \in D$ and $m \in M$ is isomorphic to the ring of upper triangular 2×2 matrices over D. (A word of caution: Associativity is lost if one tries this trick for 3×3 upper triangular matrices.) The ring R_φ fails to behave similarly, for the center of R_φ is all scalar matrices cI with $\varphi(c) = c \in \text{center}\,(D)$.

Note also that the above example indicates that Theorem 2 does not extend to include rings whose quivers are not trees.

A gap in the Morita duality theory that begs to be filled is the nearly total lack of knowledge of which artinian rings (in addition to artin algebras and QF rings) are self-dual. The characterization of artinian rings whose quivers are trees given in Theorem 2 enables us to show that such rings are self-dual. We employ the following lemma whose proof is dual to that of [7, Lemma 4]. In what follows, $E(M)$ is the injective envelope of M and $\text{Soc}_k(M)$ is the kth term in the lower Loewy series of M.

LEMMA 3. *Let R be any ring. Then the following statements about a left R-module M are equivalent*:

(a) *M is distributive.*

(b) *For each simple left R-module T, the set of submodules $\{\ker \gamma \mid \gamma \in \text{Hom}_R(M, E(T))\}$ is linearly ordered.*

(c) *For each simple left R-module T the right* End $({}_RE(T))$-*module* $\mathrm{Hom}_R(M, E(T))$ *is uniserial.*

PROPOSITION 4. *If R is an artinian ring whose quivers are trees, then there is a Morita duality between the categories of finitely generated left and finitely generated right R-modules.*

Proof. Assume that R is indecomposable and basic with identity element a sum of orthogonal primitive idempotents $1_R = e_1 + \cdots + e_n$. Let $E_i = E(Re_i/Je_i)$ for $i = 1, \cdots, n$, let $E = E_1 \oplus \cdots \oplus E_n$, and let $S = \mathrm{End}({}_RE)$. Then $(\ )^* = \mathrm{Hom}_R(-, {}_RE_S)$ defines a duality between the categories of finitely generated left R-modules and finitely generated right S-modules [13] and [7, Lemma 5]. Write $1_S = f_1 + \cdots + f_n$ where the f_i are the orthogonal primitive idempotents in S such that $Ef_i = E_i$. Let $N = J(S)$. We will show that the quivers of S are the same as the quivers of R.

From the results in [1, §24], we see that for $i = 1, \cdots, n$,

$$(Re_i/Je_i)^* \cong (\mathrm{Soc}\, E_i)^* \cong f_iS/f_iN$$
$$(\mathrm{Soc}_2(E_i)/\mathrm{Soc}\,(E_i))^* \cong f_iN/f_iN^2 .$$

So by [7, Lemma 5], f_iN/f_iN^2 is square free, and by [6, Theorem 2.4],

$$\begin{aligned} & e_iR/e_iJ \ \text{ embeds in } \ e_jJ/e_jJ^2 \\ \text{iff } \ & Re_i/Je_i \ \text{ embeds in } \ \mathrm{Soc}_2(E_j)/\mathrm{Soc}\,(E_j) \\ \text{iff } \ & f_iS/f_iN \cong (Re_i/Je_i)^* \ \text{ embeds in } \ (\mathrm{Soc}_2(E_j)/\mathrm{Soc}\,(E_j))^* \\ & \cong f_jN/f_jN^2 . \end{aligned}$$

Thus the right quiver of S is the same as the right quiver of R.

Now to see that the left quivers of R and S are the same we need only show that $\dim({}_{f_iSf_i}f_iSf_j) = 0$ or 1 for all i, j. But (writing maps on the right), $f_iSf_j \cong \mathrm{Hom}_R(E_i, E_j) \cong \mathrm{Hom}_{e_jRe_j}(e_jE_i, e_jE_j)$ by [6, Lemma 2.1]. Note that since the quivers of R are trees, Re_i and e_iR are distributive R-modules for each $i = 1, \cdots, n$ [3]. So by Lemma 3 and [7, Lemmas 4 and 5], ${}_{e_jRe_j}e_jE_{i\,f_iSf_i} \cong \mathrm{Hom}_R(Re_j, E_i)$ is left and right uniserial, so since f_iSf_i is also a division ring, e_jE_i is both left and right one-dimensional or zero. Now since ${}_{e_jRe_j}e_jE_j$ is also one-dimensional, it follows that ${}_{f_iSf_i}f_iSf_j$ is zero or one-dimensional. Note also that $f_iSf_j \neq 0$ iff $e_jE_i \neq 0$ iff $e_iRe_j \neq 0$. Thus R and S are isomorphic factor rings of tic tac toe rings with the same quivers over isomorphic division rings $e_iRe_i \cong f_iSf_i$.

Regarding algebras of finite module type, we conclude with

REMARK 5. Let R be an indecomposable hereditary artin algebra of finite module type which does not satisfy the hypotheses of Theorem

2. Then according to Dlab and Ringel [4], [5] the quivers of R or of its opposite ring are Dynkin diagrams of one of the types

$$\begin{array}{ll} B_n\colon v_n - \cdots - v_2 \longleftarrow v_1 & v'_n - \cdots - v'_2 \rightrightarrows v'_1 \\ C_n\colon v_1 \longleftarrow v_2 - \cdots - v_n & v'_1 \rightrightarrows v'_2 - \cdots - v'_n \\ F_4\colon v_1 - v_2 \longleftarrow v_3 - v_4 & v'_1 - v'_2 \rightrightarrows v'_3 - v'_4 \\ G_2\colon v_1 \longleftarrow v_2 & v'_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} v'_2\,. \end{array}$$

Using an argument similar to that in the proof of Theorem 2, one can show that if R is an artinian ring with quivers of one of the above types, then R is a factor ring of a generalized tic tac toe ring; that is, R is isomorphic to a factor of a matrix ring with some of the entries from a division subring C of a division ring D and the other nonzero entries from D. (For example, if R is hereditary with quivers

$$\mathscr{Q}({}_R R) = v_3 \longrightarrow v_1 \longleftarrow v_2 \longleftarrow v_4$$
$$\mathscr{Q}(R_R) = v'_3 \longleftarrow v'_1 \rightrightarrows v'_2 \longrightarrow v'_4$$

then R is isomorphic to a ring T of matrices

$$\begin{bmatrix} D & D & D & D \\ & C & 0 & C \\ & & D & 0 \\ & & & C \end{bmatrix}$$

with $\dim(D_C) = 2$.) To show this, assume that R is basic and that $\mathscr{Q}({}_R R)$ is a tree. Arrange the right and left quivers of R so that the multiple arrows point to the right. Let v'_α be the vertex of $\mathscr{Q}(R_R)$ at the tails of the multiple arrows, and let v'_β be the vertex of $\mathscr{Q}(R_R)$ at the heads of the multiple arrows. Let $\mathscr{Q}_\alpha$ be the subquiver of $\mathscr{Q}({}_R R)$ containing v_α and the arrows and vertices to the left of v_α, and let $\mathscr{Q}_\beta$ be the subquiver of $\mathscr{Q}({}_R R)$ containing v_β and the arrows and vertices to the right of v_β. Notice that $\dim({}_{e_\alpha R e_\alpha} e_\alpha J e_\beta) = 1$ since $\mathscr{Q}({}_R R)$ is a tree. For $v_p \leftarrow v_q$ in $\mathscr{Q}({}_R R)$, let e_{pq} generate ${}_{e_p R e_p} e_p J e_q$, and define e_{ij} as before for $v_i \leqq v_j$. Define $\sigma_{\alpha j}$ for $v_j \in \mathscr{Q}_\alpha$ and $\sigma_{\beta j}$ for $v_j \in \mathscr{Q}_\beta$ as in the proof of Theorem 2. Let

$$C' = \left\{ \sum_{v_j \in \mathscr{Q}_\beta} \sigma_{\beta j}(x) \,\middle|\, x \in e_\beta R e_\beta \right\}.$$

Let

$$D = \left\{ \sum_{v_j \in \mathscr{Q}_\alpha} \sigma_{\alpha j}(x) \,\middle|\, x \in e_\alpha R e_\alpha \right\}.$$

Define $\theta\colon C' \to D$ via $e_{\alpha\beta}e_{\beta}c' = \theta(c')e_{\alpha}e_{\alpha\beta}$ for $c' \in C'$. Then $C = \operatorname{im}\theta \cong C'$. Now let

$$T = \{[\![d_{ij}]\!] \mid d_{ij} \in D,\ d_{ij} = 0 \quad \text{if} \quad v_i \not\leqq v_j\ , \\ \text{and} \quad d_{ij} \in C \quad \text{if} \quad v_i \in \mathscr{C}_\beta\}\ .$$

Then T is a ring, since if $v_k \in \mathscr{C}_\alpha$ and $v_i \in \mathscr{C}_\beta$, then $v_i \not\leqq v_k$ (and hence $d_{ik} = 0$). So in any nonzero product $d_{ij}d_{jk}$, we must have $v_i \leqq v_j$ and $v_j \leqq v_k$, giving $v_i \leqq v_k$, and thus either $v_i \in \mathscr{C}_\alpha$, or both v_i, $v_j \in \mathscr{C}_\beta$ and $d_{ij}d_{jk} \in C$. Now define

$$\Phi\colon\ T \to R \quad \text{by}$$
$$\Phi\colon\ [\![d_{ij}]\!] \longmapsto \sum_{v_i \leqq v_j,\, v_i \in \mathscr{C}_\beta} \theta^{-1}(d_{ij})e_{ij} + \sum_{v_i \leqq v_j,\, v_i \in \mathscr{C}_\alpha} d_{ij}e_{ij}\ .$$

The map Φ is clearly additive and onto. To show that Φ preserves the multiplication, we need only add to the proof of Theorem 2 that for $d \in D$ and $c \in C$,

$$de_{\alpha\beta}\theta^{-1}(c)e_{\beta k} = dce_{\alpha\beta}e_{\beta k} = dce_{\alpha k}\ ,$$

which is immediate by the definition of θ.

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