HERMITIAN QUADRATIC FORMS AND HERMITIAN MODULAR FORMS

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It is shown that if H is a positive definite Hermitian quadratic form in r variables which is even integral over the imaginary quadratic field of discriminant $-d$ and if det $H = 2^r d^{-r/2}$, then 4 divides r.

1. It has long been known that an even integral unimodular positive quadratic form over Z must have rank divisible by 8 ([6], [7]). An interesting consequence of this fact is that if $s > r$, then the C-linear space of Siegel modular forms of degree s and weight $r/2$ is $\{0\}$ unless 8 divides r ([12]).

The purpose of the present note is to prove an analogous theorem about Hermitian quadratic forms over imaginary quadratic number fields, and to draw the corresponding conclusions about certain C-linear spaces of Hermitian modular forms.

Let K be the imaginary quadratic number field of discriminant $-d$ with ring of integers $\mathscr{O} = Z + Z\omega$. Let $\mathfrak{A}_i$, $\mathfrak{B}_i$, $\mathfrak{C}_i$ be fractional ideals and $L = \mathfrak{A}_1 x_1 + \cdots + \mathfrak{A}_r x_r$ a finitely generated $\mathscr{O}$-module. Then the ideal class of $\mathfrak{A} = \mathfrak{A}_1 \cdots \mathfrak{A}_r$ is an invariant of L. Let H: $L \times L \to K$ be a Hermitian form of rank r. We define the determinant of H by $|H| = \mathrm{Norm}_{K/Q}(\mathfrak{A}) \, |H(x_i, x_j)|$. Then $|H|$ depends only on H and L [8, page 229].

Associated to H are the ideal scale $(H) = \{H(v, w) \mid v, w \in L\}$ and the ideal norm (H) generated by $\{H(v, v) \mid v \in L\}$. Since $H(v, w) = \overline{H(w, v)}$, norm $(H) \subset Q$. H is called *positive definite* if norm $(H) \subset Q^+$, and *even integral* if norm $(H) \subset 2Z$. Since $\mathrm{trace}_{K/Q}(H(v, w)) = H(v, w) + H(w, v) = H(v + w, v + w) - H(v, v) - H(w, w)$, $\mathrm{trace}_{K/Q}$ (scale (H)) $\subset$ norm $(H) \subset$ scale (H) and an even integral Hermitian form H need not have integral scale.

THEOREM 1. *If H is a positive, even integral Hermitian form of rank r, and $|H| = 2^r d^{-r/2}$, then 4 divides r.*

The proof of this theorem consists of the assignment to H of the Z-bilinear form B: $L \times L \to Q$ defined by $B(v, w) = (H(v, w) + H(w, v))/2$. If H is positive, even integral and of determinant $|H| = 2^r d^{-r/2}$, then B is positive even integral and unimodular. Since $\dim_Z L = 2r$, the assertion follows immediately from the corresponding classical fact.

Once the assignment $H \mapsto Q$ is specified, the desired properties can be obtained by direct algebraic considerations. However, in view of our intended application and because it provides a natural organization and thematic coherence for the various details of the proof, we prefer to cloak the algebra in the garb of the theory of theta functions. For convenience, we give the proof only in the case when $L = \mathscr{O} x_1 + \cdots + \mathscr{O} x_r$ is a free $\mathscr{O}$-module. Then $|H| = |H(x_i, x_j)|$ and without the risk of confusion we can let H also denote the Hermitian matrix $(H(x_i, x_j))$.

2. Denote by $\mathfrak{X}^s_{\mathscr{H}}$ the set of s-rowed Hermitian matrices considered as a real vector space. The subset $\mathfrak{Y}^s_{\mathscr{H}}$ of positive Hermitian matrices is a strictly convex open cone in $\mathfrak{H}^s_{\mathscr{H}}$. The subset $\mathfrak{Z}^s_{\mathscr{H}} = \mathfrak{X}^s_{\mathscr{H}} + i\mathfrak{Y}^s_{\mathscr{H}}$ of the complexification of $\mathfrak{X}^s_{\mathscr{H}}$ is biholomorphically equivalent to a bounded symmetric domain, and is called the *Hermitian half space of rank* s. The group of biholomorphic automorphisms of $\mathfrak{Z}^s_{\mathscr{H}}$ consists of the mappings

$$(1) \qquad\qquad Z \mapsto (AZ + B)(CZ + D)^{-1}$$

where the complex matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfies $\bar{M}' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The *Hermitian modular group* $\Gamma^s_{\mathscr{H}}(K)$ *of rank* s *associated with* K is the subgroup of mappings of the form (1) such that the entries of M belong to $\mathscr{O}$. A holomorphic function $\varphi \colon \mathfrak{Z}^s_{\mathscr{H}} \to C$ is a *Hermitian modular form for* K *of weight* w if $\varphi(AZ + B)(CZ + D)^{-1} = \varepsilon |CZ + D|^w \varphi(Z)$ for all mappings in $\Gamma^s_{\mathscr{H}}(K)$, where ε is a certain root of 1 (see [1], [2]).

There are theta functions naturally associated with $\mathfrak{Z}^s_{\mathscr{H}}$. Let H denote a positive Hermitian quadratic form in r variables and let $\mathfrak{L}$ stand for a lattice in $C^{r \times s}$ considered as a real vector space. Then the theta function on $\mathfrak{Z}^s_{\mathscr{H}}$ associated with H and $\mathfrak{L}$ is (cp. [11]):

$$(2) \qquad \Theta_\mathfrak{L}(Z, H) = (\operatorname{vol} \mathfrak{L})^{1/2} \sum_{N \in \mathfrak{L}} \exp i\pi\sigma(Z\bar{N}'HN) ,$$

where vol $\mathfrak{L}$ denotes the volume of the lattice $\mathfrak{L}$ with respect to a Haar measure on $C^{rs} = R^{2rs}$, and σ stands for matrix trace. This theta function satisfies the following basic functional equation, whose proof consists of a standard application of the Poisson summation formula ([11], Theorem 2.2 with $z_{1/2} = 0$):

$$(3) \qquad\qquad \Theta_{\hat{\mathfrak{L}}}(-Z^{-1}, H^{-1}) = |-iZ|^r |H|^s \Theta_\mathfrak{L}(Z, H)$$

where $\hat{\mathfrak{L}}$ is the dual lattice defined by

$$(4) \qquad \hat{\mathfrak{L}} = \{\hat{N} \in C^{r \times s} \colon \sigma(\hat{N}N) \in Z \quad \text{for all} \quad N \in \mathfrak{L}\} .$$

We will determine necessary and sufficient conditions on H such that $\Theta_\mathfrak{L}(Z, H)$ be a Hermitian modular form for $\Gamma^s_\mathscr{K}(K)$ of weight r if the lattice $\mathfrak{L} = (\mathscr{O})^{r \times s}$.

It will suffice to consider the behavior of $\Theta_\mathfrak{L}(Z, H)$ with respect to the following set of generators of $\Gamma^s_\mathscr{K}(K)$:

(1) *Translations*: $Z \mapsto Z + B$, where B is a Hermitian matrix over $\mathscr{O}$;

(2) *Unimodular transformations*: $Z \mapsto \bar{U}'ZU$, where U is a unimodular matrix over $\mathscr{O}$, and

(3) *Partial involutions*: $Z \mapsto J_{s_1}Z$, where s_1 is an integer, $0 < s_1 \leq s$, and the transformation J_{s_1} corresponds to the Hermitian modular matrix

$$
(5) \qquad
\left(
\begin{array}{cc|cc}
0 & 0 & -\mathbf{1}_{s_1} & 0 \\
0 & \mathbf{1}_{s_2} & 0 & 0 \\
\hline
\mathbf{1}_{s_1} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1}_{s_2}
\end{array}
\right)
=
\left(
\begin{array}{cc}
A & B \\
C & D
\end{array}
\right) ;
$$

$s_2 = s - s_1$, and $\mathbf{1}_k$ denotes the unit matrix of rank equal to k.

If $s_1 = s$, then the partial involution J_s is just the involution $Z \mapsto -Z^{-1}$.

First observe that a $\Gamma^s_\mathscr{K}(K)$-modular form is invariant under the translation $Z \mapsto Z + B$ where B is a Hermitian matrix over $\mathscr{O}$. It follows that H must be even integral over $\mathscr{O}$.

The function $\Theta_\mathfrak{L}(Z, H)$, with H even integral over $\mathscr{O}$, is invariant under the unimodular mappings $Z \mapsto \bar{U}'ZU$, where U is a unimodular matrix over $\mathscr{O}$ ([11], Theorem 2.3); hence, it remains to consider the constraints imposed by the partial involutions in conjunction with the functional equation (3).

First we will consider involution $Z \mapsto -Z^{-1}$, and determine the conditions it imposes on H. Then we will show that if H satisfies the conditions imposed by the translations and the involution, then it automatically satisfies the constrains imposed by the various partial involutions.

The lattice $\hat{\mathfrak{L}}$ can be easily expressed in terms of $\mathfrak{L}$. In fact, since the different of K is the principal ideal generated by $\sqrt{-d}$, one finds $\hat{\mathfrak{L}} = (2/\sqrt{-d})\mathfrak{L}$, and consequently $\Theta_{\hat{\mathfrak{L}}}$ can be expressed in terms of $\Theta_\mathfrak{L}$ by

$$
(6) \qquad \Theta_{\hat{\mathfrak{L}}}(Z, H) = \frac{1}{\mathrm{vol}\,\mathfrak{L}} \Theta_\mathfrak{L}\left(Z, \frac{4H}{d}\right) ;
$$

here we have used $(\mathrm{vol}\,\mathfrak{L})(\mathrm{vol}\,\hat{\mathfrak{L}}) = 1$. Then (3) becomes

$$(7) \qquad \Theta_{\mathfrak{L}}\!\left(-Z^{-1}, \frac{4H^{-1}}{d}\right) = |-iZ|^r \, |H|^s (\text{vol } \mathfrak{L}) \Theta_{\mathfrak{L}}(Z, H) \, .$$

If $\Theta_{\mathfrak{L}}(Z, H)$ is a $\Gamma^s_{\mathscr{K}}(K)$-modular form of weight r, then in particular

$$\Theta_{\mathfrak{L}}(-Z^{-1}, H) = |-iZ|^r \Theta_{\mathfrak{L}}(Z, H) \, .$$

Using (7) to rewrite the right side as a function of $-Z^{-1}$, and then replacing $-Z^{-1}$ by Z, obtain

$$\Theta_{\mathfrak{L}}\!\left(Z, \frac{4H^{-1}}{d}\right) = \text{vol } \mathfrak{L} \, |H|^s \Theta_{\mathfrak{L}}(Z, H) \, .$$

Now write $Z = X + iY$, with X and Y Hermitian, and compute the limit as the eigenvalues of Y approach infinity (and X remains in some compact set); only the constant term $(\text{vol } \mathfrak{L})^{1/2}$ survives in the theta function (because Y and H are positive) as one immediately sees from the defining equation (2). Therefore $|H|^s (\text{vol } \mathfrak{L}) = 1$. Since $\mathfrak{L} = \mathscr{O}^{r \times s}$, $\text{vol } \mathfrak{L} = (\sqrt{d}/2)^{rs}$ and hence

$$(8) \qquad\qquad\qquad |H| = 2^r d^{-r/2} \, .$$

Using this condition, the left side of (7) can be rewritten. Abbreviate $\tilde{H} = (\sqrt{-d}/2)H$; then $|\tilde{H}| = 1$ and

$$\Theta_{\mathfrak{L}}\!\left(-Z^{-1}, \frac{4\tilde{H}^{-1}}{d}\right) = \Theta_{\mathfrak{L}}\!\left(-Z^{-1}, 4\tilde{\tilde{H}}' \frac{H^{-1}}{d} \tilde{H}\right)$$

(since $\tilde{H}$ is integral and induces a mere basis change of $\mathfrak{L}$)

$$= \Theta_{\mathfrak{L}}(-Z^{-1}, H) \, .$$

Hence (7) reduces to the functional equation for a $\Gamma^s_{\mathscr{K}}(K)$-modular form of weight r under the involution $Z \mapsto -Z^{-1}$.

Now we will show that if H is even integral over $\mathscr{O}$ and $|H| = 2^r d^{-r/2}$, then $\Theta_{\mathfrak{L}}(Z, H)$ is a $\Gamma^s_{\mathscr{K}}(K)$-modular form of weight r. It is only necessary to verify that $\Theta_{\mathfrak{L}}(Z, H)$ satisfies the appropriate functional equation for the partial involutions since the other classes of generators of $\Gamma^s_{\mathscr{K}}(K)$ have already been considered.

Let $0 < s_1 \leqq s$, let J_{s_1} denote the corresponding partial involution with matrix given by (5), and write

$$Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{pmatrix}$$

with rank $Z_1 = s_1$; observe that Z_{21} is determined by Z_{12}. Then

$$(9) \qquad\qquad J_{s_1} Z = \begin{pmatrix} -Z_1^{-1} & Z_1^{-1} Z_{12} \\ Z_{21} Z_1^{-1} & Z_2 - Z_{21} Z_1^{-1} Z_{12} \end{pmatrix} \, .$$

In order to express $\Theta_\mathfrak{L}(J_{s_1}Z, H)$ in terms of $\Theta_\mathfrak{L}(Z, H)$, we will expand $\Theta_\mathfrak{L}(Z, H)$ relative to Z_2. Let $s_2 = s - s_1$ and decompose $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$, $\mathfrak{L}_i = \mathscr{O}^{r \times s_i}$. Write $\mathfrak{L} \ni N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ with $N_i \in \mathfrak{L}_i$ and introduce the theta function $\Theta_{\mathfrak{L}_1}(Z_1, Z_{12}, H)$ by

$$(10) \qquad \Theta_{\mathfrak{L}_1}(Z_1, Z_{12}, H) = (\text{vol } \mathfrak{L}_1)^{1/2} \sum_{N_1 \in \mathfrak{L}_1} \exp i\pi\sigma(N_1 Z_1 \bar{N}_1' H + N_1 Z_{12} H + Z_{21} \bar{N}_1' H).$$

Then, noting that $\text{vol } \mathfrak{L} = (\text{vol } \mathfrak{L}_1)(\text{vol } \mathfrak{L}_2)$, obtain

$$(11) \quad \Theta_\mathfrak{L}(Z, H) = (\text{vol } \mathfrak{L}_2)^{1/2} \sum_{N_2 \in \mathfrak{L}_2} \Theta_{\mathfrak{L}_1}(Z_1, Z_{12}\bar{N}_2', H) \exp i\pi\sigma(Z_2 \bar{N}_2' H N_2).$$

By (9),

$$(12) \quad \begin{aligned} &\Theta_\mathfrak{L}(J_{s_1}Z, H) \\ &= (\text{vol } \mathfrak{L}_2)^{1/2} \sum_{N_2 \in \mathfrak{L}_2} \Theta_{\mathfrak{L}_1}(-Z_1^{-1}, Z_1^{-1} Z_{12}\bar{N}_2', H) \\ &\quad \times \exp - i\pi\sigma(N_2 Z_{21} Z_1^{-1} Z_{12}\bar{N}_2') \exp i\pi\sigma(Z_2 \bar{N}_2' H N_2). \end{aligned}$$

The theta function $\Theta_{\mathfrak{L}_1}(Z_1, Z_{12}, H)$ satisfies a transformation formula analogous to (3) which will enable us to rewrite the right hand side of (12). In fact, ([10], Theorem 2.2) asserts

$$(13) \quad \begin{aligned} &\Theta_{\hat{\mathfrak{L}}_1}(-Z^{-1}, Z_1^{-1} Z_{12} H, H^{-1}) \\ &= |-iZ_1|^r |H|^s \exp i\pi\sigma(Z_{21} Z_1^{-1} Z_{12} H)\Theta_{\mathfrak{L}_1}(Z_1, Z_{12}, H). \end{aligned}$$

Since $\hat{\mathfrak{L}}_1 = (2/\sqrt{-d})\mathfrak{L}_1$,

$$\Theta_{\hat{\mathfrak{L}}_1}(Z_1, Z_{12}, H) = \frac{1}{\text{vol } \mathfrak{L}_1}\Theta_{\mathfrak{L}_1}\left(Z_1, \sqrt{-d}Z_{12}, \frac{4H}{d}\right)$$

(cp. (6)), whence (13) can be written

$$(14) \quad \begin{aligned} &\Theta_{\mathfrak{L}_1}\left(-Z_1^{-1}, -\frac{\sqrt{-d}}{2}Z_1^{-1} Z_{12}H, \frac{4H^{-1}}{d}\right) \\ &= |-iZ_1|^r \text{ vol } \mathfrak{L}_1|H|^{s_1} \exp i\pi\sigma(Z_{21} Z_1^{-1} Z_{12}H)\Theta_{\mathfrak{L}_1}(Z_1, Z_{12}, H). \end{aligned}$$

We have assumed that H is even integral over $\mathscr{O}$ and $|H| = 2^r d^{-r/2}$; therefore, with $\tilde{H} = (\sqrt{-d}/2)H$ as before, and replacing N_1 by $\tilde{H}N_1$, find

$$\Theta_{\mathfrak{L}_1}\left(Z_1, -\frac{\sqrt{-d}}{2}Z_1^{-1} Z_{12}H, \frac{4H^{-1}}{d}\right) = \Theta_{\mathfrak{L}_1}(Z_1, Z_1^{-1}Z_{12}, H).$$

Recalling that $\text{vol } \mathfrak{L}_1 = (2^r d^{-r/2})^{s_1}$, substitution in (14) yields

$$(15) \qquad \begin{aligned} &\Theta_{\mathfrak{L}_1}(-Z_1^{-1}, Z_1^{-1}Z_{12}, H) \\ &\quad = |-iZ_1|^r \exp i\pi\sigma(Z_{21}Z_1^{-1}Z_{12}H)\Theta_{\mathfrak{L}_1}(Z_1, Z_{12}, H) \,. \end{aligned}$$

Now write $Z_{12}\bar{N}_2'$ in place of Z_{12} and substitute (15) in (12) to obtain

$$(16) \qquad \begin{aligned} \Theta_{\mathfrak{L}}(J_{s_1}Z, H) &= |-iZ_1|^r(\mathrm{vol}\,\mathfrak{L}_2)^{1/2} \sum_{N_2 \in \mathfrak{L}_2} \Theta_{\mathfrak{L}_1}(Z_1, Z_{12}\bar{N}_2', H) \\ &\quad \times \exp i\pi\sigma(Z_2\bar{N}_2'HN_2) = |-iZ_1|^r\Theta_{\mathfrak{L}}(Z, H) \,. \end{aligned}$$

Since $|CZ + D| = \begin{vmatrix} Z_1 & Z_{12} \\ 0 & \mathbf{1}_{s_2} \end{vmatrix} = |Z_1|$ for the modular transformation J_{s_1}, (16) shows that $\Theta_{\mathfrak{L}}(Z, H)$ is a modular form of weight r with a multiplier system ε which equals 1 on the translations and unimodular transformations in $\Gamma^s_{\mathcal{H}}(K)$, and equals $(-i)^{rs_1}$ on the partial involution J_{s_1}.

The *Siegel half space of rank* s, denoted $\mathfrak{Z}^s_{\mathcal{S}}$, is the subspace of $\mathfrak{Z}^s_{\mathcal{H}}$ which consists of symmetric complex matrices. The *Siegel modular group* $\Gamma^s_{\mathcal{S}}$ of rank s is the subgroup of $\Gamma^s_{\mathcal{H}}(K)$ which consists of those mappings whose corresponding matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are real. It follows that the restriction of a $\Gamma^s_{\mathcal{H}}(K)$-modular form of weight r to $\mathfrak{Z}^s_{\mathcal{S}}$ is a $\Gamma^s_{\mathcal{S}}$-modular form of weight r ("Siegel" modular form).

In particular, if $\mathfrak{L} = (\mathcal{O})^{r\times s}$, H is even integral over $\mathcal{O}$, and $|H| = 2^r d^{-r/2}$, then $\Theta_{\mathfrak{L}}(Z, H)|_{\mathfrak{Z}^s_{\mathcal{S}}}$ is a Siegel modular form of weight r. We can identify this form.

If Q is the matrix of an even integral unimodular quadratic form in $2r$ variables over Z, and if $Z \in \mathfrak{Z}^s_{\mathcal{S}}$, then the theta function

$$(17) \qquad \theta(z, Q) = \sum_{n \in Z^{2r\times s}} \exp i\pi\sigma(zn'Qn)$$

is a Siegel modular form of weight r. In particular, it satisfies the relation

$$\theta(-z^{-1}, Q) = |-iz|^r\theta(z, Q) \,.$$

Set $\omega = -(d + \sqrt{-d})/2$; then $\{1, \omega\}$ is a Z-basis for $\mathcal{O}$, and the elements of $\mathfrak{L} = (\mathcal{O})^{r\times s}$ can be expressed in the form $N = N_1 + \omega N_2$ where $N_i \in Z^{r\times s}$. If Z is restricted to $\mathfrak{Z}^s_{\mathcal{S}}$, one obtains

$$\Theta_{\mathfrak{L}}(z, H) = (\mathrm{vol}\,\mathfrak{L})^{1/2}\theta(z, Q(H))$$

where

$$(18) \qquad Q(H) = \frac{1}{2}\begin{pmatrix} H + \bar{H} & \omega H + \bar{\omega}\bar{H} \\ \omega H + \bar{\omega}\bar{H} & \omega\bar{\omega}(H + \bar{H}) \end{pmatrix} \,.$$

Since the restriction of $\Theta_{\mathfrak{L}}$ to $\mathfrak{Z}^s_{\mathcal{S}}$ is a Siegel modular form, it automatically follows that $Q(H)$ is positive, even integral over Z,

and $|Q(H)| = 1$. But these conditions entail that 8 divides the rank of $Q(H)$, i.e., 4 divides r, which completes the proof of Theorem 1.

3. Theorem 1 gives necessary conditions for the existence of positive, even integral Hermitian forms H of rank r with determinant $|H| = 2^r d^{-r/2}$. In fact, these conditions are sufficient as the following theorem implies.

THEOREM 2. *There exists a positive Hermitian matrix H of rank 4, $|H| = 16/d^2$, which is even integral over $\mathscr{O}$.*

In modern terminology, we will show that there exists a Hermitian form $H: \mathscr{O}^4 \times \mathscr{O}^4 \to K$ such that norm $(H) \subset 2Z^+$ and $|H| = 16/d^2$.

Our construction is based on the following lemma of Jacobson [5]. See also Gerstein [3] for a similar construction.

LEMMA (Jacobson). *If $K_{\mathfrak{P}}$ is a local field, then any two regular n-ary Hermitian spaces over $K_{\mathfrak{P}}$ with the same determinant are isometric.*

(19)
$$
\begin{aligned}
\text{Let } L' &= \langle v_1 \rangle \perp \langle v_2 \rangle \perp \langle v_3 \rangle \perp \langle v_4 \rangle \\
&= \langle 2 \rangle \perp \langle 2 \rangle \perp \langle 2/d \rangle \perp \langle 2/d \rangle \; .
\end{aligned}
$$

Then $|L'| = 16/d^2$ and norm $(L') \subset (2/d)Z$. At all primes $\mathfrak{P}/d$ we will use Jacobson's lemma to change L' to L so that L will be even integral. Let $pZ = \mathfrak{P} \cap Z$. Then $p\mathscr{O} = \mathfrak{P}^2$ and $\mathfrak{P}$ is the only integral with $N(\mathfrak{P}) = p$.

For all $\mathfrak{P}|d$, let

(20)
$$
L_{\mathfrak{P}} = \begin{pmatrix} 0 & 2/\sqrt{-d} \\ -2/\sqrt{-d} & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 2/\sqrt{-d} \\ -2/\sqrt{-d} & 0 \end{pmatrix} .
$$

Then $L_{\mathfrak{P}}$ is a Hermitian form over $\mathscr{O}_{\mathfrak{P}}$, $|L_{\mathfrak{P}}| = 16/d^2$, and norm $(L_{\mathfrak{P}}) \subset 2Z_p$. Since $|L_{\mathfrak{P}}| = |L'|$, by Jacobson's lemma, the Hermitian spaces $L_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} K_{\mathfrak{P}}$ and $L' \otimes K_{\mathfrak{P}}$ are isometric, and there exists a lattice in $L' \otimes K_{\mathfrak{P}}$ which is isometric to $L_{\mathfrak{P}}$. Call that lattice $L_{\mathfrak{P}}$. We have a matrix $T_{\mathfrak{P}}$ with $\mathfrak{P}$-adic coefficients such that

$$
L_{\mathfrak{P}} = T_{\mathfrak{P}} v_1 + T_{\mathfrak{P}} v_2 + T_{\mathfrak{P}} v_3 + T_{\mathfrak{P}} v_4 \; .
$$

For some large integer n, $\mathfrak{P}^n L' \subset L_{\mathfrak{P}}$. Therefore if T is a matrix with coefficients in K which is close to $T_{\mathfrak{P}}$ in the $\mathfrak{P}$-adic topology, then $T(L') \otimes \mathscr{O}_{\mathfrak{P}} = L_{\mathfrak{P}}$. Since there are only a finite number of prime ideals $\mathfrak{P}|d$, we can use the Chinese Remainder Theorem to construct a matrix T, with coefficients in K, which is

sufficiently close to $T_\mathfrak{P}$ for all $\mathfrak{P} \mid d$. Let $M = T(L')$. Then $\forall \mathfrak{P} \mid d$, $M \otimes \mathcal{O}_\mathfrak{P} = L_\mathfrak{P}$.

Using the invariant factor theorem [8, page 214], we can find a basis x_1, x_2, x_3, and x_4 for $V = L' \otimes K = M \otimes K$ such that

$$(21) \qquad \begin{aligned} L' &= \mathfrak{A}_1 x_1 + \mathfrak{A}_2 x_2 + \mathfrak{A}_3 x_3 + \mathfrak{A}_4 x_4 \\ M &= \mathfrak{B}_1 x_1 + \mathfrak{B}_2 x_2 + \mathfrak{B}_3 x_3 + \mathfrak{B}_4 x_4 \, . \end{aligned}$$

Since L' is a free $\mathcal{O}$-module, $\mathfrak{A} = \mathfrak{A}_1 \mathfrak{A}_2 \mathfrak{A}_3 \mathfrak{A}_4$ is a principal ideal. Let $\mathfrak{C}_i$ be the ideal such that $\mathrm{Ord}_{\mathfrak{P}'}(\mathfrak{C}_i) = \mathrm{ord}_{\mathfrak{P}'}(\mathfrak{A}_i) \, \forall \mathfrak{P}' \nmid d$, and $\mathrm{Ord}_\mathfrak{P}(\mathfrak{C}_i) = \mathrm{Ord}_\mathfrak{P}(\mathfrak{B}_i) \forall \mathfrak{P} \mid d$. Let $\mathfrak{C} = \mathfrak{C}_1 \mathfrak{C}_2 \mathfrak{C}_3 \mathfrak{C}_4$ and

$$(22) \qquad L = \mathfrak{C}_1 x_1 + \mathfrak{C}_2 x_2 + \mathfrak{C}_3 x_3 + \mathfrak{C}_4 x_4 \, .$$

Then $L \otimes \mathcal{O}_\mathfrak{P} = M \otimes \mathcal{O}_\mathfrak{P} = L_\mathfrak{P} \, \forall \mathfrak{P} \mid d$, and $L \otimes \mathcal{O}_{\mathfrak{P}'} = L' \otimes \mathcal{O}_{\mathfrak{P}'} \mathscr{P}' \, \forall \mathfrak{P}' \mid d$. Therefore $\mid L \mid = 16/d^2$ and norm $(L) \subset 2\mathbf{Z}$.

Now $\mid L' \mid = N(\mathfrak{A}) \mid H(x_i, x_j) \mid$ and $\mid L \mid = N(\mathfrak{C}) \mid H(x_i, x_j) \mid$. Since $\mid L \mid = \mid L' \mid$, $N(\mathfrak{A}/\mathfrak{C}) = 1$. But $\mathfrak{A}$ and $\mathfrak{C}$ differ only at the ramified primes. Let $\mathfrak{A}/\mathfrak{C} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_m^{e_m}$ where the $\mathfrak{P}_i$ are the different ramified primes, and $p_i = N(\mathfrak{P}_i)$ are distinct prime numbers. Then $N(\mathfrak{A}/\mathfrak{C}) = 1 = p_1^{e_1} \cdots p_m^{e_m}$ and all $e_i = 0$. Therefore $\mathfrak{A} = \mathfrak{C}$. Since L' is a free $\mathcal{O}$-module, $\mathfrak{A} = \mathfrak{C}$ is principal, and L is free.

For $K = \mathbf{Q}(\sqrt{-1})$,

$$(23) \qquad I = \begin{pmatrix} 2 & -i & -i & 1 \\ i & 2 & 1 & i \\ i & 1 & 2 & 1 \\ 1 & -i & 1 & 2 \end{pmatrix}, \quad i = \sqrt{-1}$$

is even integral over $\mathbf{Z}[\sqrt{-1}]$ and $\mid I \mid = 1$, cp. Iyanaga [4].

One can show that the set of even integral Hermitian forms with determinant $16/d^2$ forms a genus. But its class number remains unknown; in view of [9], it seems unlikely that it is 1.

4. The results of the previous sections can be applied to the theory of Hermitian modular forms. Since, for $s > 4$. $(\Gamma^s_\mathscr{X}(K), 4)$ is spanned by theta functions associated with Hermitian forms of the type described in Theorem 1, Theorem 2 implies:

THEOREM 3. (a) *If $s > 4$, then $(\Gamma^s_\mathscr{X}(K), 4)$ is spanned by $\Theta_\mathfrak{P}(Z, L)$, where $\mathfrak{L} = \mathcal{O}^{4 \times s}$ and L runs through representatives of the $\mathcal{O}$-equivalence classes of positive Hermitian forms in 4 variables which are even integral over $\mathcal{O}$ and of determinant $16/d^2$;*

(b) *If $s > 4k$, then $\dim_C (\Gamma^s_\mathscr{X}(K), 4k) \geqq 1$;*

(c) *If $s > r$ and r is not divisible by 4, then $(\Gamma^s_\mathscr{X}(K), r) = \{0\}$.*

Freitag [2] has shown that $(\Gamma^2_{\mathscr{H}}(Q(\sqrt{-1})), 4)$ is one-dimensional, and has constructed a generator by means of theta functions with characteristic.   Define ([2], eq. (11))

$$\Theta(Z; \mathfrak{A}, \mathfrak{B}) = \sum \exp i\pi\Big(Z\Big\{\mathfrak{C} + \frac{1+i}{2}\mathfrak{A}\Big\} + 2\mathrm{Re}\Big(\frac{1+i}{2}\mathfrak{B}'\mathfrak{C}\Big)\Big) \, ,$$

where $Z\{\mathfrak{X}\} = \bar{\mathfrak{X}}'Z\mathfrak{X}$, $\mathfrak{A}, \mathfrak{B} \in \mathbf{Z}^2$, and $\mathfrak{C}$ runs through $\mathbf{Z}[\sqrt{-1}]^2$, and set

$$\varphi_4(Z) = \sum \Theta^4(Z; \mathfrak{A}, \mathfrak{B}) \, ,$$

where the sum runs over all "syzegetic triples" $(\mathfrak{A}, \mathfrak{B})$, cp. [2]. Then φ_4 is a Hermitian modular form of weight 4 for $Q(\sqrt{-1})$.

THEOREM 4.   *Let* $\mathfrak{L} = \mathbf{Z}[\sqrt{-1}]^{4\times 2}$ *and* $Z \in \mathfrak{H}^2_{\mathscr{H}}$. *There is a non-zero constant* c *such that*

$$\varphi_4(Z) = c\Theta_{\mathfrak{L}}(Z, I) \, .$$

Indeed, the proof of Theorem 1 shows that $\Theta_{\mathscr{L}}(Z, I)$ is a Hermitian modular form of weight 4 for $Q(\sqrt{-1})$, where I is Iyanaga's Hermitian quadratic form, given by (23).

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