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LOCAL AND GLOBAL CONVEXITY IN COMPLETE RIEMANNIAN MANIFOLDS

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A connected open set in Euclidean space is convex if it is locally supported at each boundary point; indeed, the same statement holds in any complete Riemannian manifold for which all geodesics are minimal. On the other hand, in an arbitrary complete n-dimensional Riemannian manifold M the question, under what circumstances global convexity properties are implied by local ones, involves the notion of cut locus. This question will be considered here.

Propositions 2 and 3 give sufficient conditions, in terms of the cut loci of boundary points, for a locally supported open subset of M to be weakly convex. Using a theorem of Karcher about hypersurfaces which do not intersect their own cut loci, we then obtain a condition for convexity (Proposition 4), as well as the following (Theorem 3): If H is an imbedded, compact, connected topological hypersurface of M which does not intersect its own cut locus (it follows then that $M\backslash H$ has two components, each with boundary H), and if H has a one-sided field of local support elements, then H is homeomorphic to S^{n-1} and the supported component of $M\backslash H$ is convex.

It is hoped that these observations may prove useful in investigating global convexity in certain classes of Riemannian manifolds M for which information on the behavior of cut loci is available.

The paper [6] by Karcher is our reference for facts concerning convexity and weak convexity of subsets of M, and cut loci of subsets of M. We also use the notion of local convexity defined and investigated by Cheeger and Gromoll [2].

Throughout, M will denote a complete Riemannian manifold of dimension n. A subset B of M is *strongly convex* if M contains exactly one minimal geodesic between any two points of B and that geodesic lies in B; *convex* if B contains exactly one minimal geodesic between any two points of B; and *weakly convex* if B contains at least one minimal geodesic between any two points of B. A weakly convex open set contains *every* minimal geodesic in M with endpoints in B; thus for open sets, convexity and strong convexity are equivalent. Any $p \in M$ has a strongly convex neighborhood, namely the open metric ball $B(p, \varepsilon)$ for ε sufficiently small [5]. Finally, a subset B of M is *locally convex* if each point of the closure $\bar{B}$ has a strongly convex neighborhood U such that $B \cap U$ is strongly convex. Clearly, a weakly convex set is locally convex.

If B is an open subset of M, then an open halfspace H_p of the tangent space M_p at $p \in \partial B$ is called a *support element for* B if H_p contains the initial tangent vectors of all minimal geodesics from p to points of B. H_p is a *local support element for* B if, for some open neighborhood U of p, H_p is a support element for $B \cap U$ [6].

The notation $[pq]$ (respectively, $[pq)$) will be used *only* when p and q are not cut points of each other, and will denote the unique (up to oriented reparametrization) minimal geodesic from p to q (resp., that geodesic with endpoint deleted). Whenever p and q lie on a minimal geodesic and one of p, q is not an endpoint, we may refer to the geodesic $[pq]$.

2. The theorem of Karcher. We shall need the generalized Jordan-Brouwer separation theorem for arbitrary compact topological hypersurfaces of E^n. A proof is included because we could not find a reference.

THEOREM 1 (*Generalized Jordan-Brouwer separation theorem*). *Let H be an imbedded, compact, connected topological $(n-1)$-manifold in E^n. Then $E^n \backslash H$ consists of two components, each with boundary H.*

Proof. By Alexander duality, $E^n \backslash H$ has two components A_1 and A_2 ([4], p. 179). Since these are open, $H \supset \partial A_1 \cup \partial A_2$; by invariance of domain, $H = \partial A_1 \cup \partial A_2$. Furthermore, since H is connected, there exists $q \in H \cap \partial A_1 \cap \partial A_2$. Observe that no closed subset H' of $H \backslash \{q\}$ separates E^n. Indeed, if V is an open ball about q in E^n such that $V \cap H' = \phi$, then V contains points of A_1 and A_2; therefore $A_1 \cup A_2 \cup V$ is a connected subset of $E^n \backslash H'$ whose closure contains $E^n \backslash H'$, and it follows that $E^n \backslash H'$ is connected.

If $p \in H$, then any two neighborhoods in H of p and q respectively contain neighborhoods whose complements in H are homeomorphic, as may be seen by joining p and q by an arc covered by finitely many coordinate neighborhoods. By a theorem of Borsuk ([3], p. 357), if E^n is separated by a compact subset C then E^n is separated by any homeomorph of C. Thus it follows from the preceding paragraph that no proper closed subset of H separates E^n. Therefore H is the boundary of each component of the complement of H ([3], p. 356). This completes the proof.

For any subset S of M, the *cut locus* $C(S)$ of S in M is defined by $C(S) := \bigcup_{p \in S} C(p)$ where $C(p)$ is the cut locus of p. The following theorem was proved by Karcher in [6]. (It is stated there for

$H = S^{n-1}$, but the proof holds in the present case also, with the only necessary change being the substitution of Theorem 1 for the original Jordan-Brouwer theorem.)

THEOREM 2 (*Karcher*). *Let H be an imbedded, compact, connected topological $(n-1)$-manifold in M satisfying $H \cap C(H) = \varnothing$. Then $M \backslash H$ consists of two open components A_1 and A_2, each with boundary H, where* (1) *A_1 is bounded, and* (2)*$C(\bar{A}_1) \subset A_2$.*

The component A_1 is uniquely determined by (1) and (2), and is referred to as the "inside" component of $M \backslash H$.

3. Local and global convexity. Concerning the question raised in the introduction, the following information may be found in the paper by Karcher:

PROPOSITION 1 [6]. *A connected open subset B of M is convex if and only if B possesses a local support element at every boundary point and does not intersect its own cut locus.*

If B is a locally convex open set, then as Cheeger and Gromoll have shown [2], $\bar{B}$ is an imbedded topological manifold-with-boundary; furthermore, B possesses a local support element at every boundary point. It is worth noting that an open set may possess a local support element at every boundary point and yet not be locally convex:

EXAMPLE 1. Let g be the standard Riemannian metric on $\boldsymbol{R}^2$, B be the open subset of $\boldsymbol{R}^2$ indicated in Figure 1, and H be the indicated arc in ∂B. We shall alter the metric g so that the inside loop beginning and ending at p is a geodesic in the new metric. Let U be a connected open set satisfying $H = \bar{U} \cap \partial B$ and carrying Fermi coordinates about H. Then there exists a Riemannian metric h on $\boldsymbol{R}^2$ such that (1) g and h agree on $\boldsymbol{R}^2 \backslash U$, and (2) H is the image of an h-geodesic. Indeed, h may be constructed from g and the flat metric $\tilde{h}$ on U determined by the Fermi coordinates; one uses a smooth Urysohn function vanishing on $\boldsymbol{R}^2 \backslash U$ and taking value 1 on

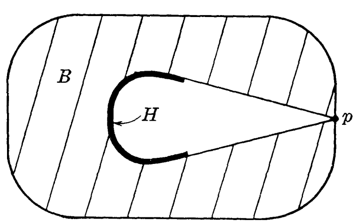

FIGURE 1

a neighborhood of every point of H which does not have a neighborhood on which $\tilde{h}$ agrees with g. The metric h is complete since it agrees with g except on a compact set. By (1) and (2), B has a local support element with respect to h at every boundary point, but B is not locally convex at p.

Local convexity does follow from being locally supported if ∂B is an imbedded manifold (it is not necessary to assume that $\bar{B}$ is a manifold-with-boundary):

LEMMA 1. *Let B be an open subset of M whose boundary is an imbedded topological $(n-1)$-manifold. If B possesses a local support element at every boundary point, then B is locally convex.*

Proof. Fix $p \in \partial B$, and choose ε sufficiently small that $B(p, \varepsilon')$ is convex for $\varepsilon' \leqq \varepsilon$. It follows from Proposition 1 that every connected component of $B \cap B(p, \varepsilon)$ is convex. Choose $\varepsilon'(0 < \varepsilon' \leqq \varepsilon)$ so that $\partial B \cap B(p, \varepsilon')$ lies in the component through p of $\partial B \cap B(p, \varepsilon)$; this is possible because ∂B is an imbedded manifold. Certainly there is a component C of $B \cap B(p, \varepsilon)$ such that $\partial C \cap B(p, \varepsilon') \neq \emptyset$. Since C is convex, ∂C is an imbedded $(n-1)$-manifold in M, and hence in ∂B. By invariance of domain, $\partial C \cap B(p, \varepsilon)$ is open in $\partial B \cap B(p, \varepsilon)$; obviously, it is also closed. Thus, by choice of ε', $p \in \partial C$. Now suppose C_1 is another component of $B \cap B(p, \varepsilon)$ whose boundary intersects $B(p, \varepsilon')$. Then $C \cap C_1 = \phi$, and $\bar{C}$ and $\bar{C}_1$ are imbedded manifolds-with-boundary having common boundary in a neighborhood of p, in contradiction to the local support hypothesis. Therefore $B \cap B(p, \varepsilon') = C \cap B(p, \varepsilon')$. Since both C and $B(p, \varepsilon')$ are strongly convex, so is $B \cap B(p, \varepsilon')$, as required.

PROPOSITION 2. *A connected open subset B of M is weakly convex if and only if B possesses a local support element at every boundary point and $B \backslash C(p)$ is connected for every $p \in \partial B$.*

Proof. Suppose that B is locally supported and $B \backslash C(p)$ is connected for every $p \in \partial B$. Fix $p \in \partial B$, and suppose further that the set $B(p) := \{q \in B \backslash C(p) : [pq] \subset \bar{B}\}$ is nonempty. For a fixed $q \in B(p)$, no point of $(pq]$ falls on ∂B, since the existence of a last such point would contradict the local support hypothesis. We may choose ε, by Proposition 1, so that every component of $B \cap B(p, \varepsilon)$ is convex; in particular, the component C containing an initial segment of $(pq]$ is convex. Then $(pq] \subset C \cup U$ where U is a neighborhood in B of $(pq]$. Consider a sequence of points $q_i \in B \backslash C(p)$ converging to q. For i sufficiently large, $[pq_i]$ contains a subarc $[r_i q_i] \subset U$ where $r_i \in C \cap U$. Since C is convex, then $[pr_i] \subset \bar{C}$ and hence $[pq_i] \subset \bar{B}$. It follows that $B(p)$ is open in $B \backslash C(p)$. Clearly, $B(p)$ is also closed in $B \backslash C(p)$, and so $B(p) = B \backslash C(p)$.

Now suppose γ is a minimal geodesic in M joining any $q, q' \in B$. If $\gamma \not\subset B$ then $\gamma \not\subset \bar{B}$. But then the interior of γ contains a point $p \in \partial B$, where $q, q' \in B \backslash C(p)$, $[pq] \subset \bar{B}$, and $[pq'] \not\subset \bar{B}$. Thus $B(p)$ is nonempty and properly contained in $B \backslash C(p)$, and we have just shown that this is impossible. Therefore $\gamma \subset B$ and B is weakly convex.

Conversely, if B is weakly convex, then for any $p \in \partial B$ and $q, r \in B \backslash C(p)$, B contains $[qp)$ and $(pr]$. Since B is locally convex, it is clear that q and r may be joined by a path in $B \backslash C(p)$.

For a locally convex set B, Proposition 2 yields a condition for global convexity which involves only the boundary of B:

PROPOSITION 3. *Let B be a connected, locally convex, open subset of M, and set $H = \partial B$. If $H \backslash C(p)$ is connected for all $p \in H$, then B is weakly convex.*

Proof. By Proposition 2, it suffices to observe that $B \backslash C(p)$ is connected for each $p \in H$. Suppose instead that $B \backslash C(p) = S_1 \cup S_2$, where the S_i are nonempty open separated subsets of M. Then $H \backslash C(p) = T_1 \cup T_2$ where $T_i = \partial S_i \backslash C(p)$. Assume $p \in T_1$. For any $q \in S_2$, since S_1 and S_2 are separated and $[pq] \cap C(p) = \emptyset$, $[pq]$ contains a point of T_2; thus T_2 is nonempty also. By assumption, there is a point r in $T_1 \cap \partial T_2$ or $\partial T_1 \cap T_2$. Since r has a neighborhood U in M not intersecting $C(p)$ and such that $B \cap U$ is connected, by local convexity, it follows that S_1 and S_2 may be joined by a path in $B \backslash C(p)$, which is impossible.

REMARK 1. The example of a weakly convex open ring on a cylinder illustrates Proposition 2 and shows that the converse of Proposition 3 is false.

PROPOSITION 4. *Let B be a connected, locally convex open subset of M, and set $H = \partial B$. If H is connected and compact and does not intersect its own cut locus, then $\bar{B}$ is bounded and strongly convex.*

Proof. (Assume $B \neq \emptyset$.) By Proposition 3, B and therefore $\bar{B}$ are weakly convex. Since H is an imbedded topological hypersurface of M, then by Theorem 2, $M \backslash H$ consists of two open components A_1 and A_2 with boundary H, where A_1 is bounded and $C(\bar{A}_1) \subset A_2$. Since B is open and connected and $\partial B = \partial A_i$, B coincides with A_1 or A_2.

Suppose $B = A_2$. Let γ be a geodesic ray from some $p \in \partial B$

having an initial segment in A_1; such a γ exists by local convexity of B. If there exists a cut point r of p along γ, then $r \in B$. Therefore there is a subarc $[pq]$ of γ such that $p, q \in H$ and $[pq]$ does not lie in $\bar{B}$. This contradicts weak convexity of $\bar{B}$. On the other hand, if γ contains no cut point of p, then since A_1 is bounded, again γ must enter B, in contradiction to weak convexity of $\bar{B}$. Therefore $B = A_1$. Since $C(\bar{B}) \cap \bar{B} = \varnothing$ and $\bar{B}$ is weakly convex, it is immediate that $\bar{B}$ is strongly convex.

THEOREM 3. *Let H be an imbedded, compact, connected topological $(n-1)$-manifold in M which does not intersect its own cut locus. By Theorem 2, $M \backslash H$ consists of two components, each with boundary H. If a component B of $M \backslash H$ has a local support element at every point of H, then H is homeomorphic to S^{n-1} and B is bounded and convex.*

Proof. By Lemma 1, B is locally convex. Therefore by Proposition 4, B is bounded and convex, and is the inside component of $M \backslash H$. Furthermore, the boundary of a nonempty, bounded, convex, open set is homeomorphic to S^{n-1} [6].

REMARK 2. Theorem 3 was proved by Karcher under the added assumption that B is the *inside* component of $M \backslash H$, in which case the theorem is a direct consequence of Theorem 2 and Proposition 1. We have shown that the outside component of $M \backslash H$ in Theorem 2 can never be locally supported.

COROLLARY 1. *Let H be a compact, connected Riemannian $(n-1)$-manifold, and $i: H \to M$ be an isometric imbedding such that $i(H)$ does not intersect its own cut locus. If sectional curvatures satisfy $K_H(\sigma) > K_M(i_*\sigma)$ for every 2-plane σ tangent to H, then H is homeomorphic to S^{n-1} and $i(H)$ is the boundary of a bounded convex open subset of M.*

Proof. By assumption, the second fundamental form of i is positive definite with respect to a continuous unit normal field. Therefore if N is a tubular neighborhood of $i(H)$ in M, a fixed component of $N \backslash i(H)$ is locally supported at every point of $i(H)$, as Bishop has shown [1]. Thus a component of $M \backslash i(H)$ is locally supported at every point of $i(H)$, and Theorem 3 applies.

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