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MINIMAL (G, τ)-EXTENSIONS

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In this paper, we are concerned with lifting minimality and topological transitivity through skew-extensions—the fibres being a compact group and the action intertwines with a group automorphism. It is shown that in the class of cocycles respecting the automorphism, these properties can be lifted when the automorphism is distal. This is obtained by a dynamical decomposition of an automorphism on a group, and subsequent analysis based on this decomposition. The lifting fails for hyperbolic automorphisms on a torus.

1. Introduction. Suppose (X, ψ) is a free abelian group extension of a minimal flow (Y, η). Then it was shown in [2] and, via different techniques, in [8] that under mild assumptions, almost all cocycle perturbations of (X, ψ) over (Y, η) are minimal. In this paper we study the corresponding problem in the more general situation when (X, ψ) is a free (G, τ)-extension of (Y, η) (see § 2 for definitions). The major dynamical results (Theorem 3.13 and Corollary 3.14) state that in cases which include finite or countably infinite dimensional tori, almost all cocycles lift topological transitivity, and, when (G, τ) is distal, they lift minimality.

In preparation for these results, detailed information on the dynamical properties of group automorphisms of compact abelian groups is necessary, and we carry out this analysis in § 2. To this end, we use a particular inverse limit decomposition of (G, τ) which identifies a distal tower and an ergodic extension. Certain aspects of this decomposition were previously studied by Seethoff and Brown, see [1]. Our results in § 2, which might be of independent interest, are the identification of the maximal equicontinuous factor in this case, and the fact that on an n-torus, distality is equivalent to some power being unipotent. Finally some indications of extensions to more general actions are given.

In order to show the main results, a notion of admissibility is required, and we discuss which (G, τ) are admissible in § 3. The remainder of § 3 is devoted to proving these results and noting some examples to illustrate the theory. One of the examples (Example 3.19) shows that even in the case of a periodic automorphism, the major result cannot be deduced from Ellis' original result.

The proof of Theorem 3.13 is a modification of Ellis' proof in [2] and we acknowledge our indebtedness to that paper.

2. The dynamics of group automorphisms. The basic situation which we will examine in this paper will be a discrete flow (X, ψ), with X compact Hausdorff, a compact Hausdorff abelian group G acting freely on X and an automorphism τ of G such that

$$\psi(gx) = \tau(g)\psi(x) \quad (x \in X,\ g \in G) .$$

Letting (Y, η) be the induced flow on X/G, and $\pi: X \to Y$ the canonical map, we say that (X, ψ) is a free (G, τ)-extension of (Y, η). Note that if $\tau = id$, we get the usual group extension. To simplify notation we will sometimes write $(X, \psi) \xrightarrow{(G,\tau)} (Y, \eta)$.

In addition to the dynamics of (Y, η) we will also be concerned with the dynamics of (G, τ). Thus to say that (X, ψ) is an equicontinuous-(G, τ)-extension means that (G, τ) is equicontinuous, and so on.

The following lemma is routine, and its proof will thus be omitted.

LEMMA 2.1. *Let (X, ψ) be a distal-(G, τ)-extension of (Y, η). Then (X, ψ) is a distal extension in the usual sense. In particular, if (Y, η) is distal, then (X, ψ) is distal, and, if (Y, η) is minimal, then (X, ψ) is pointwise almost periodic.*

We now turn to an examination of the dynamics of (G, τ). We denote the character group of G by $\Gamma(G)$ and normalized Haar measure on G by λ. We recall that (G, τ) is ergodic relative to λ if and only if $\alpha \in \Gamma(G)$, $\alpha \circ \tau^p = \alpha$ for some $p > 0$ implies $\alpha = 1$.

We define inductively a sequence of subgroups of G.

DEFINITION 2.2. Put $G_0 = G$, $\Gamma_0 = \{1\} \subset \Gamma(G)$. Having defined Γ_n, and $G_n = \operatorname{ann} \Gamma_n$, we put

$$\Gamma_{n+1} = \{\alpha \in \Gamma(G) \mid \alpha \circ \tau^p \cdot \alpha^{-1} \in \Gamma_n \text{ for some } p > 0\}$$
$$G_{n+1} = \operatorname{ann} \Gamma_{n+1} .$$

Each Γ_n is a τ-invariant subgroup of Γ_{n+1} and G_{n+1} is a τ-invariant subgroup of G_n. We put $G_\infty = \bigcap_{n=0}^\infty G_n$ and $\Gamma_\infty = \bigcup_{n=0}^\infty \Gamma_n$. Then $G_\infty = \operatorname{ann} \Gamma_\infty$, G_∞ is a τ-invariant subgroup of G and Γ_∞ is a τ-invariant subgroup of $\Gamma(G)$. We will denote by τ all the various restrictions and induced automorphisms associated with τ and these τ-invariant subgroups. We note that if (G_n, τ) is the restricted action on G_n then $(G_n)_1 = G_{n+1}$.

It is clear from this definition and the characterisation of ergodicity that (G, τ) is ergodic relative to Haar measure if and only if $G_1 = G$, and hence, if and only if $G_\infty = G$.

The sequence of subgroups above was first introduced by T. L. Seethoff, see [1]. He mainly considered the case $G_\infty = \{e\}$ and obtained some of the results of this section.

The dynamical portrait of this sequence is summed up by:

$$(G, \tau) \xrightarrow{(G_\infty, \tau)} (G/G_\infty, \tau) = \text{inv.}\lim (G/G_n, \tau) ,$$

where $(G/G_{n+1}, \tau) \xrightarrow{(G_n/G_{n+1}, \tau)} (G/G_n, \tau)$ for each $n \geqq 1$.

PROPOSITION 2.3.

1. *For each* $n \geqq 0$, $(G_n/G_{n+1}, \tau)$ *is equicontinuous.*
2. (G_∞, τ) *is ergodic relative to Haar measure on* G_∞.
3. $(G/G_n, \tau)$ *is distal,* $1 \leqq n \leqq \infty$.

Proof.

1. Since $(G_n)_1 = G_{n+1}$, it suffices to show that $(G/G_1, \tau)$ is equicontinuous. We identify Γ_1 with $\Gamma(G/G_1)$. Let $F = \{\alpha_1, \cdots, \alpha_n\}$ be a finite subset of Γ_1 and let $\langle F \rangle$ denote the smallest τ-invariant subgroup of Γ_1 containing F. Put $G_F = \text{ann}\,\langle F \rangle$. Then $(G/G_1, \tau) = \text{inv.}\lim_{\text{finite } F} (G/G_F, \tau)$. It now suffices to show that each $(G/G_F, \tau)$ is equicontinuous. Since $\alpha_1, \cdots, \alpha_n \in \Gamma_1$ there exist $p_1, \cdots, p_n > 0$ such that $\alpha_1 \circ \tau^{p_i} \cdot \alpha_i^{-1} = 1$. Put $p = \Pi p_i$. Then $\alpha \circ \tau^p \cdot \alpha^{-1} = 1$ for all $\alpha \in \langle F \rangle$. Therefore $(G/G_F, \tau)$ is periodic and hence is equicontinuous.

2. Since $\Gamma(G_\infty) \cong \Gamma(G)/\Gamma_\infty$, we have to show that if $\alpha \in \Gamma(G)$ satisfies $\alpha \circ \tau^p \cdot \alpha^{-1} \in \Gamma_\infty$ for some $p > 0$ then $\alpha \in \Gamma_\infty$. But if $\alpha \circ \tau^p \cdot \alpha^{-1} \in \Gamma_\infty$, then for some n, $\alpha \circ \tau^p \cdot \alpha^{-1} \in \Gamma_n$ and hence $\alpha \in \Gamma_{n+1} \subset \Gamma_\infty$ as required.

3. This follows from part 1 of this proposition, Lemma 2.1 and the comment immediately preceding this proposition regarding the structure of $(G/G_n, \tau)$.

We note that a consequence of part 2 of Proposition 2.3 is that $(\tau^p - I)G_\infty = G_\infty$ for all $p > 0$; here $(\tau^p - I)g$ denotes $\tau^p(g) \cdot g^{-1}$. It is also useful to note that $G_{n+1} = \bigcap_{p=1}^\infty (\tau^p - I)G_n$.

Regarding the induced automorphisms $(G_n/G_{n+1}, \tau)$ simple examples show that in general $\tau \neq I$. However if $\tau = I$ on G/G_1, then $\tau = I$ on G_n/G_{n+1} for each n as the following argument shows. First we note that $\tau = I$ on G_n/G_{n+1} is equivalent to saying $\Gamma_{n+1} = \{\alpha \in \Gamma(G) \mid \alpha \circ \tau \cdot \alpha^{-1} \in \Gamma_n\}$. We are assuming this is true for $n = 0$. Suppose that it is true for $n = k - 1$ and consider Γ_{k+1}. If $\beta \in \Gamma_{k+1}$ then there is a $p > 0$ such that $\beta \circ \tau^p \cdot \beta^{-1} = \gamma_0 \in \Gamma_k$. Put $\gamma = \beta \circ \tau \cdot \beta^{-1}$. Then $\gamma \cdot \gamma \circ \tau \cdots \gamma \circ \tau^{n-1} = \gamma_0$ and so $\gamma\tau^n \cdot \gamma^{-1} = \gamma_0 \circ \tau \cdot \gamma_0^{-1} \in \Gamma_{k-1}$. Thus $\gamma \in \Gamma_k$ and so we have shown $\Gamma_{k+1} = \{\alpha \in \Gamma(G) \mid \alpha \circ \tau \cdot \alpha^{-1} \in \Gamma_k\}$, as required.

Our next two propositions deal with the cases $G_1 = \{e\}$ and $G_\infty = \{e\}$.

PROPOSITION 2.4.

1. *If* (G, τ) *is equicontinuous, then for each* $\alpha \in \Gamma(G)$ *there is an* $n \geqq 1$, *depending on* α, *such that* $\alpha \circ \tau^n = \alpha$.

2. *If* H *is a closed* τ*-invariant subgroup of* G *such that* $(G/H, \tau)$ *is equicontinuous then* $G_1 \subset H$. *Hence* (G, τ) *is equicontinuous if and only if* $G_1 = \{e\}$.

3. *If* G *has a finitely generated character group, that is,* G *is isomorphic to the direct product of an m-torus and a finite group, then* (G, τ) *equicontinuous implies* $\tau^n = I$ *for some* n.

4. $(G/G_1, \tau)$ *is the maximal equicontinuous factor of* (G, τ).

Proof.

1. Let K denote the circle group, $\alpha \in \Gamma(G)$. Since τ is equicontinuous, $\overline{O_\tau(\alpha)} = \overline{\{\alpha \circ \tau^n \mid n \in Z\}}$ is compact in $\mathscr{C}(G, K)$ and hence is compact in $L^2(G, \lambda)$. Since $\alpha \neq \alpha \circ \tau^n$ implies $\alpha \perp \alpha \circ \tau^n$ and hence $\|\alpha - \alpha \circ \tau^n\|^2 = 2$ in L^2, it follows from compactness that there is an $n > 0$ for which $\alpha = \alpha \circ \tau^n$.

2. To show that $G_1 \subset H$, we have to show that if $\alpha \in \operatorname{ann} H$ then $\alpha \in \operatorname{ann} G_1 = \Gamma_1$. Now $\alpha \in \operatorname{ann} H$ implies $\alpha \in \Gamma(G/H)$ and hence, by equicontinuity of $(G/H, \tau)$, there is an n such that $\alpha \circ \tau^n = \alpha$, that is $\alpha \in \Gamma_1$. The last statement follows from this and part 1 of Proposition 2.3.

3. Let $\alpha_1, \cdots, \alpha_k$ be generators of $\Gamma(G)$. By equicontinuity of τ there exist $n_1, \cdots, n_k$ such that $\alpha_i \circ \tau^{n_i} = \alpha_i$, $1 \leqq i \leqq k$. Putting $n = \Pi n_i$ we get $\alpha \circ \tau^n = \alpha$ for all $\alpha \in \Gamma(G)$ and hence $\tau^n = I$.

4. Let $\eta: (G, \tau) \to (X, \phi)$ be a homomorphism with (X, ϕ) equicontinuous. We must show η factors through $(G/G_1, \tau)$. If we regard $C(X)$ as a subset of $C(G)$ via the map $f \to f \circ \eta$ then we must show that $f \in C(X)$ implies f is constant on cosets of G_1. Since (X, ϕ) is equicontinuous, $C(X)$ is generated by linear combinations of eigenfunctions (this is well known for minimal (X, ϕ) and is easily extended to the nonminimal case). Thus we must show that if $f \in C(X)$ satisfies $f \circ \tau = kf$, $|k| = 1$, then f is constant on cosets of G_1. Since λ is a supported measure, it suffices to show that as an $L^2(G, \lambda)$-function f is in the closed linear span of Γ_1. So write

$$f = \sum_{\alpha \in \Gamma(G)} k_\alpha \alpha \,, \quad \sum_{\alpha \in \Gamma(G)} |k_\alpha|^2 < \infty \,.$$

Since $f \circ \tau = kf$, $|k| = 1$, it follows that $|k_\alpha|$, as a function of α, is constant on orbits under τ. Thus $k_\alpha = 0$ whenever α has an infinite orbit, and so f is in the closed linear span of Γ_1 as required.

PROPOSITION 2.5. *The following statements are equivalent*:

1. (G, τ) *is distal.*

2. *(G, τ) has zero topological entropy.*
3. $G_\infty = \{e\}$.
4. *(G, τ) is pointwise almost periodic.*

Proof.

1. implies 2. It is known that a minimal distal flow on a compact Hausdorff space has zero topological entropy, and also that $h(X, \phi) = \sup_\alpha h(X_\alpha, \phi)$, if $X = \bigcup_\alpha X_\alpha$, where the X_α are ϕ-invariant closed sets [4]. This yields the result since a distal flow is a disjoint union of minimal distal flows.

2. implies 3. It is known that an ergodic automorphism on a nontrivial compact group has positive entropy. Since (G_∞, τ) is ergodic and has zero entropy if (G, τ) has zero entropy, it follows that $G_\infty = \{e\}$.

3. implies 1. This is immediate from part 3 of Proposition 2.3.

1. implies 4. This is a well known property of distal flows.

4. implies 3. First suppose G is metric. If (G, τ) is pointwise almost periodic (p.a.p.) then (G_∞, τ) is p.a.p. Since (G_∞, τ) is ergodic relative to a supported measure λ and G_∞ is metric then there is a point with dense orbit, that is, (G_∞, τ) is minimal. But $\{e\}$ is a closed invariant subset of (G_∞, τ) therefore $G_\infty = \{e\}$.

Now consider the general case of G not necessarily metric. Take any character on G_∞ and form the smallest τ-invariant subgroup of $\Gamma(G_\infty)$ containing it. This subgroup is countable and hence the factor group of G_∞ having it as its dual group is metric. But τ on this factor group is p.a.p. and ergodic and so by the metric proof above the factor group is trivial. But this means every character of G_∞ annihilates G_∞, in other words, $G_\infty = \{e\}$.

COROLLARY 2.6. *Let K be a closed τ-invariant subgroup of G such that $(G/K, \tau)$ is distal. Then $K \supset G_\infty$. Thus $(G/G_\infty, \tau)$ is the maximal distal group factor.*

Proof. Since $(G/K, \tau)$ is distal it follows that $(G/K)_\infty = \{e\}$. Let $K = \operatorname{ann} \Gamma_K$. Thus $\Gamma_K \cong \Gamma(G/K)$. If $\alpha \in \Gamma_K$, then $\alpha \in (\Gamma(G/K))_n$ for some n. Clearly $(\Gamma(G/K))_n \subset \Gamma_n$. Therefore $\alpha \in \Gamma_K$ implies $\alpha \in \Gamma_n$ for some n, and so $\Gamma_K \subset \bigcup_n \Gamma_n = \Gamma_\infty$. Thus $K \supset G_\infty$.

In view of this corollary it is natural to conjecture that $(G/G_\infty, \tau)$ is the maximal distal factor of (G, τ). We have however made no progress towards settling this.

We now recall the definition of unipotence, and study its relationship to distality and equicontinuity. We note that the notion of unipotence has proved to be important in the study of minimal affine transformations on connected groups.

DEFINITION 2.7. Let τ be an automorphism of a compact abelian group G. (G, τ) is called *unipotent* if $\bigcap_{j=0}^{\infty}(\tau - I)^j G = \{e\}$, and *strongly unipotent* if there is an n such that $\bigcap_{j=0}^{n}(\tau - I)^j G = \{e\}$.

PROPOSITION 2.8. *Let G be an n-torus.*

1. *If τ is unipotent, then $\bigcap_{j=0}^{n}(\tau - I)^j G = \{e\}$, that is, τ is strongly unipotent.*
2. *If τ is unipotent and equicontinuous, then $\tau = I$.*

Proof.

1. Put $G^{(m)} = (\tau - I)^m G$. Then $G^{(m)} \downarrow \{e\}$. We will prove by induction that $G^{(m)}$ is a torus whose dimension does not exceed $\max\{n - m, 0\}$. This is clearly true for $m = 0$. Assume true for $m = k$. If $k \geqq n$, then $G^{(k)}$ will be a 0-torus and automatically $G^{(k+1)}$ will be a 0-torus. If $k < n$, then $\dim G^{(k)} \leqq n - k$. Now $G^{(k+1)} = (\tau - I)G^{(k)}$ is a subtorus of $G^{(k)}$ and so either $G^{(k+1)} = G^{(k)}$, in which case $G^{k+m} = G^k$ for all $m \geqq 0$ and G^{k+1} will be a 0-torus, or $G^{k+1} \neq G^k$, in which case $\dim G^{k+1} \leqq \dim G^k - 1$ and so $\dim G^{k+1} \leqq \max\{n - m - 1, 0\}$ as required. It now follows that $G^{(n)} = \{e\}$.
2. By 1, τ is strongly unipotent, say $(\tau - 1)^m G = \{e\}$. By 3 of Proposition 2.4, $\tau^k = I$ for some $k > 0$. Now let τ be given by an $n \times n$ matrix A. Then A satisfies $(A - I)^m = 0$ and $A^k - I = 0$. The minimum polynomial of A is a factor of both $(t - 1)^m$ and $t^k - 1$, that is, is $t - 1$. Thus $A = I$ and hence $\tau = I$.

COROLLARY 2.9. *Let τ be an automorphism of a compact connected group G which is both unipotent and equicontinuous. Then $\tau = I$.*

Proof. Let $\Gamma(G)$ be the torsion free character group of G. Let $F = \{\alpha_1, \cdots, \alpha_n\}$ be any finite subset of $\Gamma(G)$, and let $\langle F \rangle$ be the smallest τ-invariant subgroup of $\Gamma(G)$ containing F. Putting $G_F = \operatorname{ann}\langle F \rangle$, we have $(G, \tau) = \text{inv.}\lim_{\text{finite } F}(G/G_F, \tau)$. To complete the proof we need only show $\tau = I$ on each G/G_F. By equicontinuity of τ, the $\alpha_i \in F$ have finite τ-orbits and so $\langle F \rangle$ is finitely generated. Therefore G/G_F is a finite-dimensional torus, and the result follows from Proposition 2.8.

We note that this result does not hold if G is not connected. For example, let $G = Z_4$, $\tau(x) = x^{-1}$. Then τ is equicontinuous, unipotent since $(\tau - I)^2 G = \{e\}$ and clearly $\tau \neq I$.

COROLLARY 2.10. *Let $A = a \cdot \tau$ be an affine transformation of a compact connected abelian group G. If τ is equicontinuous and $\tau \neq I$, then A is not minimal.*

Proof. It is a result of H. Hoare and W. Parry, [6], that a minimal affine transformation of a compact connected group has quasi-discrete spectrum, and hence, [5], τ is unipotent if A is minimal. The result now follows from Corollary 2.9.

Before proving our next result we need a lemma.

LEMMA 2.11. *Let G be an n-torus and τ an automorphism of G. Let $G_0 \supset G_1 \supset \cdots \supset G_\infty$ be the sequence of subgroups associated with τ by Definition 2.2. Then each G_k is an m-torus for some $m \leqq n$, and there is a k such that $G_k = G_{k+i}$ for all $i \geqq 0$.*

Proof. Noting that $G_{k+1} = (G_k)_1$, we need to show that if G is a torus, then G_1 is a torus. Since $\Gamma(G_1) \cong \Gamma/\Gamma_1$, and hence is finitely generated, we must show that Γ/Γ_1 is torsion free. Let $\bar{\alpha} = \alpha\Gamma_1 \in \Gamma/\Gamma_1$, $\alpha \in \Gamma(G)$, and suppose $\bar{\alpha}^k = 1$. Then $\alpha^k \in \Gamma_1$, and so there is a $p > 0$ such that $(\alpha^k)\circ\tau^p \cdot (\alpha^k)^{-1} = 1$, that is, $(\alpha\circ\tau^p\cdot\alpha^{-1})^k = 1$. But $\Gamma(G)$ is torsion free, therefore $\alpha\circ\tau^p\cdot\alpha^{-1} = 1$ and $\alpha \in \Gamma_1$. Thus $\bar{\alpha} = 1$.

Since G_{k+1} is a subtorus of G_k, we have either $\dim G_{k+1} < \dim G_k$ or $G_{k+1} = G_k$. Since $\dim G$ is finite the latter must occur for some k and then we get $G_k = G_{k+i}$ for all $i \geqq 0$.

We can now state the best result we have on the relation between distal and unipotent.

PROPOSITION 2.12. *Let τ be an automorphism of a compact abelian group G such that τ^p is unipotent for some $p > 0$. Then τ is distal. If, in addition, G is n-torus, then τ distal implies that τ^p is unipotent for some $p > 0$.*

Proof. Recall $G_1 = \bigcap_{k=1}^{\infty}(\tau^k - I)G \subset (\tau^p - I)G$. Inductively, $G_n \subset (\tau^p - I)^n G$. Thus $G_\infty \subset \bigcap_{n=0}^{\infty}(\tau^p - I)^n G = \{e\}$, and so, by Proposition 2.5, τ is distal.

Now let G be a torus. By the distality of τ and Lemma 2.11, there is a k such that $G_0 \supsetneqq G_1 \supsetneqq \cdots \supsetneqq G_k = \{e\}$. Dually we have $\{1\} = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma(G)$. Consider Γ_i: Γ_i is finitely generated with generators $\alpha_1^{(i)}, \cdots, \alpha_{m_i}^{(i)}$, say, and there are integers $p_1^{(i)}, \cdots, p_{m_i}^{(i)}$ such that $\alpha_j^{(i)}\circ\tau^{p_j^{(i)}}\cdot(\alpha_j^{(i)})^{-1} \in \Gamma_{i-1}$. Letting $p_i = \prod_j p_j^{(i)}$, then $\Gamma_i = \{\alpha | \alpha\circ\tau^{p_i}\cdot\alpha^{-1} \in \Gamma_{i-1}\}$. Next, putting $p = \prod_{i=1}^{k} p_i$ yields $\Gamma_i = \{\alpha|\alpha\circ\tau^p\cdot\alpha^{-1} \in \Gamma_{i-1}\}$, $1 \leqq i \leqq k$. Thus $\Gamma_i = (\tau^p - I)^{-1}\Gamma_{i-1}$ and hence $G_i = (\tau^p - I)G_{i-1}$. Thus $\bigcap_{j=0}^{k}(\tau^p - I)^j G = \{e\}$ and τ^p is strongly unipotent.

We note that Proposition 2.8 implies that a nontrivial equicontinuous automorphism of a torus is not unipotent. There are also distal, nonequicontinuous automorphisms of a torus which are not unipotent, for example, the automorphism

τ of the 3-torus given by the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

Finally we point out a relationship between the sequence of groups defined in this paper and a different sequence defined by P. Walters [11]. Walters defined his sequence as follows: if τ is an automorphism of a compact abelian group G, put $G'_0 = G$, $G'_n = \bigcap_{p=1}^{\infty} (\tau^n - I)^p G'_{n-1}$, $G'_\infty = \bigcap_n G'_n$. Then it can be shown that $G_\infty = G'_\infty$.

We end this section with a few comments on generalizations. If T is a locally compact separable metric abelian group which acts on G by automorphisms then one can define as before the notion of (X, T) being a free (G, T)-extension of (Y, T). However, the situation relating to the action (G, T) reduces in some sense to $T = Z^k$, $k \geqq 1$. To see this, if $S = \{t \in T \mid \pi^t = id\}$, then T/S acts effectively on G and so is isomorphic to a subgroup of the automorphism group of G. Since this group is totally disconnected, Iwasawa [7], it follows that T/S is isomorphic to $Z^k \times F$, where F is a totally disconnected compact group. Now Definition 2.2 utilizes finiteness of orbits of characters and since every character has finite orbit under F this property depends on the Z^k-part of the action.

Looking briefly at the Z^k situation, $k > 1$, we let (G, Z^k) be an action of Z^k generated by pairwise commuting automorphisms $\tau_1, \cdots, \tau_k$. We put $G_0 = G$, $\Gamma_0 = \{1\}$. Having defined Γ_n and $G_n = \operatorname{ann} \Gamma_n$ we put $\Gamma_{n+1} = \{\alpha \in \Gamma(G) \mid \alpha \circ \tau_i^{p_i} \cdot \alpha^{-1} \in \Gamma_n$ for some $p_i > 0$, each $1 \leqq i \leqq k\}$, $G_{n+1} = \operatorname{ann} \Gamma_{n+1}$, etc. Proposition 2.3 still holds and so we get a splitting of the action into an ergodic extension of a distal action and the distal action is an inverse limit of equicontinuous-τ-extensions. We shall make no use of these results and so omit the details.

3. Minimality in (G, τ)-extensions. We will assume that X, Y, and G are metric spaces. By a τ-cocycle for $(X, \psi) \xrightarrow{(G, \tau)} (Y, \eta)$ we mean a continuous map $\bar{\phi}: Y \times Z \to G$ such that $\bar{\phi}(\eta^n y, m) = \tau^m(\bar{\phi}(y, n)) \cdot \bar{\phi}(y, n + m)$. Setting $\phi(y) = \bar{\phi}(y, 1)$ we can define a new flow on X by $\psi_\phi(x) = \phi(\pi x)\psi(x)$. It is direct to verify that (X, ψ_ϕ) is still a (G, τ)-extension of (Y, η). We note that there is a one-to-one correspondence between τ-cocycles and $C(Y, G)$ as follows: if $\phi \in C(Y, G)$ then we put

$$\bar{\phi}(y, n) = \begin{cases} \prod_{i=0}^{n-1} \tau^i \phi(\eta^{n-1-i} y) & n \geqq 1 \\ e & n = 0 \\ \prod_{i=0}^{n} \tau^{-i} \phi(\eta^{-n-1+i} y) & n \leqq -1 . \end{cases}$$

Conversely, if $\bar{\phi}$ is a τ-cocycle we put $\phi(y) = \bar{\phi}(y, 1)$.

DEFINITION 3.1. We will say that (G, τ) satisfies (A) if for all $\varepsilon > 0$ there is an $N \geqq 1$ such that for all $n \geqq N$ and $g_0, \cdots, g_{n-1} \in G$ we have $\prod_{i=0}^{n-1} \tau^i(S_\varepsilon(g_i)) = G$. Hence $S_\varepsilon(g)$ denotes the sphere of radius ε about g relative to an invariant metric d on G.

We will say that G satisfies (B) if the following extension property holds: let $f \in C(Y, G)$, $\varepsilon > 0$. Then there exists $\delta > 0$ such that if F is a finite subset of Y and $u: F \to G$ satisfies $d(f(y), u(y)) < \delta$ for $y \in F$ then there exists $v \in C(Y, G)$ with $v|_F = u|_F$ and $d(f(y), v(y)) < \varepsilon$ for all $y \in Y$.

Finally we say that (G, τ) is admissible if (G, τ) satisfies (A) and G satisfies (B). Note that if (G, τ) is admissible, then (G, τ^{-1}) is admissible.

Our main results are for admissible (G, τ) and our first task is to show that this includes a reasonable class of groups and automorphisms. We first deal with condition (A). We will show that this holds for an arbitrary automorphism on a compact connected metric abelian (c.c.m.a.) group G. We do it by a series of lemmas.

LEMMA 3.2. *(G, τ) satisfies (A) if and only if for all $\varepsilon > 0$ there is an $N \geqq 1$ such that for all $n \geqq N$ we have $A_n^\varepsilon = \prod_{i=0}^{n-1} \tau^i(S_\varepsilon(e)) = G$.*

Proof. Clearly (A) implies the above condition on taking $g_0 = g_1 = \cdots = g_{n-1} = e$.

Suppose the above condition is satisfied. Let $g_0, \cdots, g_{n-1}$ be any n elements of G. Then

$$\begin{aligned}\prod_{i=0}^{n-1} \tau^i(S_\varepsilon(g_i)) &= \prod_{i=0}^{n-1} \tau^i(g_i S_\varepsilon(e)) \text{ by invariance of metric} \\ &= \prod_{i=0}^{n-1} \tau^i(g_i) \prod_{i=0}^{n-1} \tau^i(S_\varepsilon(e)) \\ &= \prod_{i=0}^{n-1} \tau^i(g_i) G = G\,.\end{aligned}$$

Thus (G, τ) satisfies (A).

LEMMA 3.3. *Suppose H is a closed τ-invariant subgroup of G and suppose (H, τ) and $(G/H, \tau)$ satisfy (A). Then (G, τ) satisfies (A).*

Proof. Let $p: (G, \tau) \to (G/H, \tau)$ be the natural map. We first show that for each $\delta > 0$ there is a $\delta_1 > 0$ such that for all $g \in G$ $S_{\delta_1}(p(g)) \subset p(S_\delta(g))$.

Let δ_1 be a Lebesgue number for the open cover $\{p(S_{\delta/2}(g)): g \in G\}$.

If $g \in G$ then there is a $g' \in G$ such that $S_{\delta_1}(p(g)) \subset p(S_{\delta/2}(g'))$. In other words, for some $h \in H$ we have $hg \in S_{\delta/2}(g')$. Thus $g' \in S_{\delta/2}(hg)$ and hence $S_{\delta/2}(g') \subset S_\delta(hg) = hS_\delta(g)$. Therefore $p(S_{\delta/2}(g')) \subset p(S_\delta(g))$. Thus we have the required $S_{\delta_1}(p(g)) \subset p(S_\delta(g))$.

Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that for all $g \in G$ we have $S_\delta(e)S_\delta(g) \subset S_\varepsilon(g)$. Choose δ_1 as above corresponding to δ. Apply (A) for $(G/H, \tau)$ using δ_1 as ε, and apply (A) for (H, τ) using δ as ε to obtain corresponding N_1 and N_0. Put $N = \max\{N_1, N_0\}$.

If $n \geqq N$ and $g_0, \cdots, g_{n-1} \in G$, then

$$p\Big(\prod_{i=0}^{n-1} \tau^i S_\delta(g_i)\Big) = \prod_{i=0}^{n-1} \tau^i(pS_\delta(g_i)) \supset \prod_{i=0}^{n-1} \tau^i S_{\delta_1}(p(g_i)) = G/H$$

and

$$\prod_{i=0}^{n-1} \tau^i S_\delta(\varepsilon) = H .$$

Thus

$$G = \Big(\prod_{i=0}^{n-1} \tau^i S_\delta(e)\Big)\Big(\prod_{i=0}^{n-1} \tau^i S_\delta(g_i)\Big) = \prod_{i=0}^{n-1} \tau^i(S_\delta(e)S_\delta(g_i)) \subset \prod_{i=0}^{n-1} \tau^i S_\varepsilon(g_i) .$$

Hence (G, τ) satisfies (A).

LEMMA 3.4. *Let $\{H_n\}$ be a decreasing sequence of closed τ-invariant subgroups of G with $\bigcap_n H_n = \{e\}$. Suppose that for each n, $(G/H_n, \tau)$ satisfies* (A). *Then (G, τ) satisfies* (A).

Proof. Let $\varepsilon > 0$. Choose $\delta > 0$ so that for all $g \in G$ we have $S_\delta(e)S_\delta(g) \subset S_\varepsilon(g)$. Since $H_n \downarrow \{e\}$, there is an N such that $\operatorname{diam}(H_N) < \delta$. Let $\pi_N: G \to G/H_N$ be the canonical map. Then, as in the proof of Lemma 3.3, there is a $\delta_1 > 0$ such that for all $g \in G$ we have $S_{\delta_1}(\pi_N(g)) \subset \pi_N(S_\delta(g))$. Now we use the fact that $(G/H_N, \tau)$ satisfies (A) and choose N_1 corresponding to δ_1.

Then for $n \geqq N_1$ and say $g_0, \cdots, g_{n-1} \in G$ we have

$$\pi_N\Big(\prod_{i=0}^{n-1} \tau^i(S_\delta(g_i))\Big) \supset \prod_{i=0}^{n-1} \tau^i S_{\delta_1}(\pi_N(g_i)) = G/H_N$$

and so

$$G = H_N\Big(\prod_{i=0}^{n-1} \tau^i(S_\delta(g_i))\Big) \subset S_\delta(e)\Big(\prod_{i=0}^{n-1} \tau^i S_\delta(g_i)\Big) \subset \prod_{i=0}^{n-1} \tau^i S_\varepsilon(g_i) .$$

Hence (G, τ) satisfies (A).

So far we have not shown that any (G, τ) actually satisfies (A). Our next proposition remedies this.

PROPOSITION 3.5. *Let G be a c.c.m.a. group and τ equicontinuous. Then (G, τ) satisfies* (A).

Proof. Let $\varepsilon > 0$. Since τ is equicontinuous there is a $\delta > 0$ such that $\tau^i(S_\varepsilon(e)) \supset S_\delta(\tau^i e) = S_\delta(e)$ for all i. Thus

$$A_n^\varepsilon = \prod_{i=0}^{n} \tau^i(S_\varepsilon(e)) \supset \prod_{i=0}^{n} S_\delta(e) .$$

Since G is connected there is an N such that $\prod_{i=1}^{N} S_\delta(e) = G$. Thus $A_n^\varepsilon = G$ for $n \geqq N$ and by Lemma 3.2 we have that (G, τ) satisfies (A).

LEMMA 3.6. *Let G be a c.c.m.a. group and τ an automorphism of G. Let $G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$ be the subgroups of Definition* 2.1. *Then each G_n is connected.*

Proof. Since $G_{n+1} = (G_n)_1$ it suffices to show that G_1 is connected and use induction. For this we have to show that Γ torsion free implies Γ/Γ_1 is torsion free and a proof of this is contained in the proof of Lemma 2.11.

PROPOSITION 3.7. *Let G be a c.c.m.a. group and τ a distal automorphism of G. Then (G, τ) satisfies* (A).

Proof. Let $G_0 \supset G_1 \supset \cdots \supset G_\infty$ be the sequence of subgroups associated with (G, τ) by Definition 2.2. Since (G, τ) is distal it follows by Proposition 2.5 that $G_\infty = \{e\}$. Thus, by Lemma 3.4, we need only show that each $(G/G_n, \tau)$ satisfies (A). First $(G/G_1, \tau)$ satisfies (A) since G/G_1 is connected and τ is equicontinuous. Now assume $(G/G_n, \tau)$ satisfies (A) and consider $(G/G_{n+1}, \tau)$. Then G_n/G_{n+1} is a τ-invariant subgroup of G/G_{n+1} and $(G_n/G_{n+1}, \tau)$ is equicontinuous, by Proposition 2.3. Now G_n/G_{n+1} is connected by Lemma 3.6 and so $(G_n/G_{n+1}, \tau)$ satisfies (A) by Proposition 3.5. Since $(G/G_{n+1} | G_n/G_{n+1}, \tau)$ is isomorphic to $(G/G_n, \tau)$, we can use Lemma 3.3 to assert that $(G/G_{n+1}, \tau)$ satisfies (A).

It now follows by induction that $(G/G_n, \tau)$ satisfies (A) for each n.

PROPOSITION 3.8. *Let (G, τ) be ergodic. Then (G, τ) satisfies* (A).

Proof. Let $A_n^\varepsilon = \prod_{i=0}^{n-1} \tau^i(S_\varepsilon(e))$. Since $A_n^\varepsilon \subset A_{n+1}^\varepsilon$, each A_n^ε is open and G is compact we need only show $\bigcup_{n=1}^\infty A_n^\varepsilon = G$ for each $\varepsilon > 0$ and then we can apply Lemma 3.2 to get the result.

Since G is metric and τ is ergodic we can find $x_0 \in S_\varepsilon(e)$ such that $\{\tau^i x_0 | i \geqq 0\}$ is dense in G. Let $y \in G$ and denote the invariant metric by d. Then there is an n such that $d(\tau^n x_0, y) = d(y(\tau^n x_0)^{-1}, e) < \varepsilon$. Therefore $y = (y(\tau^n x_0)^{-1})\tau^n x_0 \in S_\varepsilon(e)\tau^n S_\varepsilon(e) \subset A^\varepsilon_{n+1}$. Therefore $\bigcup_{n=1}^\infty A^\varepsilon_n = G$ and the result is proved.

We remark that in Proposition 3.8 we make no connected requirement on G.

THEOREM 3.9. *Let τ be an automorphism of a c.c.m.a. group. Then (G, τ) satisfies* (A).

Proof. Let $G_0 \supset G_1 \supset \cdots \supset G_\infty$ be the sequence of subgroups associated with (G, τ). Consider the closed τ-invariant subgroup G_∞. By Proposition 2.3, (G_∞, τ) is ergodic and hence by Proposition 3.8 satisfies (A). Since $(G/G_\infty, \tau)$ is distal and G/G_∞ is connected, then, by Proposition 3.7, $(G/G_\infty, \tau)$ satisfies (A). An application of Lemma 3.3 yields the result.

We now turn to consideration of condition (B) in the definition of admissibility. In [2] Proposition 2 it was shown by R. Ellis that if G is a connected Lie group whose left and right uniform structures coincide, then G satisfies (B). Moreover if G is compact his proof gives δ depending on ε and the metric and not on f. For our situation of a compact abelian G this gives:

THEOREM 3.10 (R. Ellis). *If G is a finite dimensional torus then G satisfies* (B). *Moreover in condition* (B) *δ can be chosen independently of $f \in C(Y, G)$.*

We now note a slight extension to a torus of countably infinite dimension.

PROPOSITION 3.11. *If G is a torus of countably infinite dimension then G satisfies* (B).

Proof. We shall sketch the details.

If $G = \prod_{i=1}^\infty K$, and d is the usual metric on K with $\operatorname{diam} K = 1$, we define $\bar{d}$ on G by

$$\bar{d}(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d(x_i, y_i)$$

where $x = (x_i)_{i=1}^\infty$, $y = (y_i)_{i=1}^\infty$. Let $\varepsilon > 0$. Using Theorem 3.10, choose δ for K and $\varepsilon/2$ and choose N so that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^N} < \frac{\varepsilon}{2}\,.$$

Put $\delta_0 = 1/2^N \cdot \delta$.

Let $f \in C(Y, G)$, F a finite subset of Y and let $u: F \to G$ satisfy $\bar{d}(f(x), u(x)) < \delta_0$ for $x \in Y$. Now a continuous function $h: Y \to G$ is determined by coordinate functions $h_i: Y \to K$, so in order to determine an extension v of u we will determine v_i for $i = 1, 2, \cdots$. A direct application of Theorem 3.10 to the first N coordinates and arbitrary extension in the remainder achieves this and shows that δ_0 satisfies (B).

Combining Theorems 3.9, 3.10 and Proposition 3.11, we have

THEOREM 3.12. *Let G be an n-torus, $1 \leqq n \leqq \infty$, and τ an arbitrary automorphism of G. Then (G, τ) is admissible.*

We now give our main result.

THEOREM 3.13. *Let $\pi: (X, \psi) \to (Y, \eta)$ be a (G, τ)-extension, Y infinite with no isolated points and (G, τ) admissible. If (Y, η) is point transitive, then for almost all $\phi \in C(Y, G)$, (X, ψ_ϕ) is point transitive.*

Proof. The proof follows the proof of Theorem 1 of [2].

Let y_0 be a point with dense orbit in Y, and let $x_0 \in X$ satisfy $\pi(x_0) = y_0$. We will show that the set of ϕ for which x_0 has a dense orbit under ψ_ϕ is a dense G_δ in $C(Y, G)$.

Let U be open in X. Since (G, τ^{-1}) is admissible, we may consider only $n \geqq 0$ and thus set

$$E(U) = \{\phi \in C(Y, G) \mid \psi_\phi^n(x_0) \in U \text{ for some } n \geqq 0\} .$$

If $\phi \in E(U)$ with $\psi_\phi^n(x_0) \in U$, then for ϕ_1 sufficiently close to ϕ we have $\prod_{i=0}^{n-1} \tau^i \phi_1(\eta^{n-1-i} y_0)$ is close to $\prod_{i=0}^{n-1} \tau^i \phi(\eta^{n-1-i} y_0)$, and hence $\psi_\phi^n(x_0) \in U$. Thus $E(U)$ is open.

We now show $E(U)$ is dense. Let $f \in C(Y, G)$ and $\varepsilon > 0$. Choose δ by the admissibility property (B) of (G, τ) and let $\{V_i \mid i = 1, \cdots, n\}$ be a covering of G by $\delta/2$-spheres. Choose N for $\delta/2$ from property (A) of (G, τ).

Now if $r > N$ and $\eta^r(y_0) \in \pi(U)$, then there is a $g \in G$ for which $g \cdot (\psi^r(x_0)) \in U$. Next pick V_{i_j} with $f(\eta^{r-1-j} y_0) \in V_{i_j}$ $(j = 0, \cdots, r-1)$. Then by (A), we can find $g_j \in V_{i_j}$ such that $g = \prod_{j=0}^{r-1} \tau^j(g_j)$. We define a function $\hat{\phi}: \{\eta^j(y_0)\ (0 \leqq j \leqq r-1)\} \to G$ by $\hat{\phi}(\eta^{n-1-j}(y_0)) = g_j$, and then extend $\hat{\phi}$ to $\phi \in C(Y, G)$ satisfying $d(f, \phi) < \varepsilon$. Now $\psi_\phi^r(x_0) = (\prod_{j=0}^{r-1} \tau^j \phi(\eta^{n-1-j}(y_0)))\ \psi^r(x_0) = g \cdot \psi^r(x_0) \in U$. Thus $\phi \in E(U)$ and so $E(U)$ is dense. Since X is a compact metric space, it has a countable base $\{U_n\}$ for the open sets. Putting $E = \bigcap_n E(U_n)$ we get the set

of ϕ for which x_0 has a dense orbit and this is a dense G_δ in $C(Y, G)$.
The theorem is proved.

The simple example of $X = Y \times G$, Y the orbit closure of an isolated point, G the circle, τ the identity, shows that the condition on isolated points is necessary. This is also true for Ellis' original result.

COROLLARY 3.14. *Let $\pi: (X, \psi) \to (Y, \eta)$ be a (G, τ)-extension, Y infinite with no isolated points and (G, τ) admissible and distal. If (Y, η) is minimal, then for almost all $\phi \in C(Y, G)$, (X, ψ_ϕ) is minimal.*

Proof. Since (Y, η) is minimal and π is a distal extension, it follows by Lemma 2.1 that (X, ψ_ϕ) is pointwise almost periodic for any $\phi \in C(Y, G)$. The result now follows from Theorem 3.13.

Recalling that (Y, η) is topologically weak-mixing if $(Y \times Y, \eta \times \eta)$ is point-transitive, the following result is shown by techniques similar to those in [9], Theorem 1.

COROLLARY 3.15. *Let $\pi: (X, \psi) \to (Y, \eta)$ be a (G, τ)-extension, Y infinite with no isolated points and (G, τ) admissible. If (Y, η) is topologically weak-mixing, then for almost all $\phi \in C(Y, G)$, (X, ψ_ϕ) is topologically weak-mixing. If, in addition, (Y, η) is minimal and (G, τ) is distal, then for almost all $\phi \in C(Y, G)$, (X, ψ_ϕ) is topologically weak-mixing and minimal.*

Recall that two point-transitive flows (Z, ρ) and (W, θ) are weakly disjoint if $(Z \times W, \rho \times \theta)$ is point-transitive, and two minimal flows (Z_1, ρ_1), (W_1, θ_1) are disjoint if $(Z_1 \times W_1, \rho_1 \times \theta_1)$ is minimal.

COROLLARY 3.16. *Let (Z, ρ) be a point-transitive flow and $\pi: (X, \psi) \to (Y, \eta)$ a (G, τ)-extension, Y infinite with no isolated points and (G, τ) admissible. If (Z, ρ) is weakly disjoint from (Y, η), then for almost all $\phi \in C(Y, G)$, (Z, ρ) is weakly disjoint from (X, ψ_ϕ). If, in addition, (Z, ρ), (Y, η) are minimal, (G, τ) is distal, and (Z, ρ) is disjoint from (Y, η), then for almost all $\phi \in C(Y, G)$, (Z, ρ) is disjoint from (X, ψ_ϕ).*

Proof. In either case we have that $(X \times Z, \psi \times \rho)$ is a $(G \times 1, \tau \times 1)$-extension of $(Y \times Z, \eta \times \rho)$. If (y_0, z_0) has a dense orbit, then by considering points (x_0, z_0) with $\pi(x_0) = y_0$, the proofs of 3.13 and 3.14 show that for almost all $\phi \in C(Y, G)$, $\phi \times 1: Y \times Z \to G \times 1$ yields the desired conclusion.

Note that if (Z, ρ) is weakly disjoint from (W, θ), their maximal equicontinuous factors $(Z/S, \rho)$, $(W/S, \rho)$ are minimal, and hence are disjoint, since they have no common nontrivial eigenvalue. Now suppose (X, ψ) is a (G, τ)-extension of (Y, η), (G, τ) admissible and (Z, ρ), (Y, η) minimal. Suppose further that $(Z/S, \rho)$ is disjoint from $(Y/S, \eta)$. By Corollary 6, [10], (Z, ρ) is weakly disjoint from (Y, η), and so by Corollary 3.16, (Z, ρ) is weakly disjoint from (X, ψ_ϕ) for almost all ϕ. Thus, $(Z/S, \rho)$ is disjoint from $(X/S, \psi_\phi)$ for almost all ϕ, independent of the action of τ on G.

We now look at some examples in connection with the results of this section.

EXAMPLE 3.17. Let $(Y, \eta) = (K, R_\alpha)$, where $R_\alpha(x) = \alpha x$ is an irrational rotation of the circle K. Let $G = K$ and $\tau(g) = g^{-1}$. Put $X = Y \times G = K^2$ and define $\psi(x_1, x_2) = (\alpha x_1, x_2^{-1})$. Then (X, ψ) is a (G, τ)-extension of (Y, η). If $\phi \in C(K, K)$, then $\psi_\phi(x_1, x_2) = (\alpha x_1, \phi(x_1)x_2^{-1})$. Corollary 3.14 applies here to say that for almost all $\phi \in C(K, K)$, (K^2, ψ_ϕ) is minimal. Suppose we choose a ϕ for which ψ_ϕ is minimal. Then by connectedness of K^2 it follows that $(\psi_\phi)^2$ is also minimal. But $\psi_\phi^2(x_1, x_2) = (\alpha^2 x_1, \phi(\alpha x_1)\phi(x_1)^{-1}x_2)$. In other words, it is a minimal group extension of (K, R_{α^2}). Now this means the equation $[\phi(\alpha x_1)/\phi(x_1)]^n = h(\alpha^2 x_1)/h(x_1)$ has no continuous solution $h\colon K \to K$ unless $n = 0$. In particular it has no solution for $n = 1$. Therefore the function $\phi(\alpha x_1)/\phi(x_1)$, which is a continuous coboundary for R_α is not a continuous coboundary for R_{α^2}. Since this is true for 'most' $\phi \in C(K, K)$ we get that 'most' continuous coboundaries for R_α are not continuous coboundaries for R_{α^2}.

EXAMPLE 3.18. The following example illustrates some qualitative differences between distal and equicontinuous automorphisms, and shows the strange situation that the equicontinuous automorphisms can have 'worse' dynamical behavior.

Let $(Y, \eta) = (K, R_\alpha)$, $G = K^2$ and let τ be given by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; τ is distal. Put $X = K^3$ and let ψ be the affine transformation

$$\psi(x_1, x_2, x_3) = (\alpha x_1, \tau(x_2, x_3)) = (\alpha x_1, x_2 x_3, x_3) .$$

By Corollary 3.14, for almost all $\phi \in C(K, K^2)$, $\phi = (\phi_1, \phi_2)$, the transformation

$$\psi_\phi(x_1, x_2, x_3) = (\alpha x_1, \phi_1(x_1)x_2 x_3, \phi_2(x_1)x_3)$$

is minimal. In fact, if α, β, γ are rationally independent, one can use the constant cocycle $\phi_1(x_1) = \beta$, $\phi_2(x_2) = \gamma$.

Now (K^2, τ) is a 2-step group extension of the identity $(K^2, \tau) \to (K, id) \to (e, id)$, and we can similarly decompose X into a pair of group extensions $X \to K \times K \to K$. Applying Ellis' result on group extensions twice we get that for almost all $\phi_2 \in C(K, K)$ there is a residual subset R_{ϕ_2} of $C(K^2, K)$ such that for all $\phi_1^2 \in R_{\phi_2}$ the transformation

$$\psi_{\phi_1^2 \phi_2}(x_1, x_2, x_3) \longrightarrow (\alpha x_1, \phi_1^2(x_1, x_3)x_2x_3, \phi_2(x_1)x_3)$$

is minimal.

We now consider $Y, \eta, G,$ and X as above and take τ to be given by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; τ is equicontinuous. Let ψ be the affine transformation

$$\psi(x_1, x_2, x_3) = (\alpha x_1, k_1x_3, k_2x_2) .$$

Since τ is equicontinuous and not equal to the identity, ψ is not minimal, by Corollary 2.10. If we perturb ψ by constant cocycles it will still be affine, and hence not minimal. Therefore ψ cannot be perturbed into minimality by constant cocycles. However for almost all $\phi = (\phi_1, \phi_2) \in C(K, K^2)$ we have minimality of

$$\psi_\phi(x_1, x_2, x_3) = (\alpha x_1, \phi_1(x_1)k_1x_2, \phi_2(x_1)k_2x_1) .$$

EXAMPLE 3.19. We now give the example, alluded to in the introduction, to show that our theory does not follow from Ellis' original result.

If (G, τ) is equicontinuous with $\tau^n = id$, then $\pi: (X, \psi^n) \to (Y, \eta^n)$ is a group extension. If G is an n-torus, then Ellis' result gives that for almost all $\phi \in C(Y, G)$, $(\psi^n)_\phi$ is minimal. However, even in this case, we cannot obtain Corollary 3.14, since there may not exist $\phi_1 \in C(Y, G)$ with $(\psi^n)_\phi = (\psi_{\phi_1})^n$. For example in the situation of Example 3.17

$$(\psi^2)_\phi = (\alpha^2x_1, \phi(x_1)x_2)$$
$$(\psi_{\phi_1})^2 = (\alpha^2x_1, \phi_1(\alpha x_1)\phi_1(x)^{-1}x_2) .$$

In other words we would have to perturb ψ^2 with continuous coboundaries of R_α and Ellis' result does not allow us to do this, since most cocycles are not continuous coboundaries for R_α.

EXAMPLE 3.20. Our theorem also enables us to extend certain examples of Furstenburg. To see this, let $(Y, \eta) = (K, R_\alpha)$, $G = K^{r-1}$ and let τ be given by an $(r-1) \times (r-1)$ lower triangular unimodular matrix of integer entries with 1's down the diagonal and 0's down the lower subdiagonal. Put $X = K^r$ and define ψ by

$$\psi(x_1, \cdots, x_r) = (\alpha x_1, \tau(x_2, \cdots, x_r)) .$$

Then for almost all $\phi = (\phi_2, \cdots, \phi_r) \in C(K, K^{r-1})$, we have ψ_ϕ is minimal and ψ is of the form

$$\psi_\phi(x_1, \cdots, x_r) = (\alpha x_1, \cdots, \phi_j(x_1)x_2^{n_2^j} \cdots x_{j-2}^{n_{j-2}^j}, x_j \cdots) .$$

In other words, ψ_ϕ is in the class of transformations of K^r considered by H. Furstenburg, [3] Theorem 2.1, where, using Furstenburg's notation, $g_j(x_1, \cdots, x_j) = \phi_{j+1}(x_1)x_2^{n_2} \cdots x_{j-1}^{n_{j-1}}$ $j \geqq 2$. However our g_j has degree 0 in x_j and so Furstenburg's minimality condition (Remark on p. 582 of [3]) of nonvanishing degree does not apply.

EXAMPLE 3.21. We are indebted to W. Parry and P. Walters for the following example, which shows that in the absence of the distal condition on (G, τ) there may be no minimal lifts (see Corollary 3.14). Let $(Y, \eta) = (K, R_\alpha)$, $G = K^n$ and let τ be a hyperbolic (hence ergodic) automorphism of K^n. Put $X = Y \times K^n$ and

$$\psi(x, g) = (\alpha x_1, \tau(g)) .$$

Then for $\phi \in C(K, K^n)$ we have

$$\psi_\phi(x, g) = (\alpha x, \phi(x)\tau(g)) .$$

We will show that if ϕ is null homotopic then ψ is (topologically) conjugate to ψ_ϕ. Let $S(x, g) = (x, p(x) \cdot g)$ and consider the equation $\psi_\phi S = S\psi$. This holds if p satisfies

$$p(\alpha x) = \phi(x)\tau(p(x)) .$$

Consider the equation in R^n

$$(*) \qquad p_0(\alpha x) = \phi_0(x) + \tau(p_0(x))$$

where p_0, ϕ_0 are maps K to R^n, ϕ_0 is a 'lift' of ϕ, that is

$$\begin{array}{ccc} K & \xrightarrow{\phi_0} & R^n \\ & \searrow^{\phi} & \downarrow \\ & & K^n \end{array}$$

is commutative, and τ is the linear map on R^n which induces τ on K^n. Since τ is hyperbolic $R^n = E \oplus C$ where C is the sum of the eigenspaces of τ corresponding to eigenvalues λ of τ with $|\lambda| < 1$, similarly E corresponds to eigenvalues with $|\lambda| > 1$. Let τ^c, τ^e denote the maps induced by τ on E and C. Then, with similar notation, (*) may be written

$$p_0^c(\alpha x) = \phi_0^c(x) + \tau^c(p_0^c(x))$$
$$p_0^e(\alpha x) = \phi_0^e(x) + \tau^e(p_0^e(x)) .$$

Let $\alpha^c = \sup_{k\in K} ||\phi_0^c(k)||$, $\alpha^e = \sup_{k\in K} ||\phi_0^e(k)||$. Then $||(\tau^c)^n\phi_0^c(\alpha^n x)|| \leqq ||\tau^c||^n\alpha^c$, $||(\tau^e)^{-n}\phi_0^e(\alpha^{-n}x)|| \leqq ||(\tau^e)^{-1}||^n\alpha^e$ and since τ^c and $(\tau^e)^{-1}$ are both contracting it follows that the series $\sum_{n=0}^\infty (\tau^c)^n(\phi_0^c(\alpha^n x))$ and $\sum_{n=1}^\infty (\tau^e)^{-n}(\phi_0^e(\alpha^{-n}x))$ both converge uniformly and hence define continuous functions $f^c: K \to C$, $f^e: K \to E$. Formal manipulation shows that $-f^c$ will serve as p^c and f^e will serve as p^e. We now translate to K^n by the exponential map to obtain a solution $p = \exp p_0$ of $p(\alpha x) = \phi(x)\tau(p(x))$. Thus whenever ϕ is nullhomotopic the perturbation ψ_ϕ is not minimal.

Now suppose we have two perturbations ψ_{ϕ_1} and ψ_{ϕ_2} with ϕ_1 homotopic to ϕ_2. Then a similar argument to the above, using the fact that $\phi_1\phi_2^{-1}$ is null homotopic, shows that ψ_{ϕ_1} is conjugate to ψ_{ϕ_2}. Hence we need only consider perturbations by some representative of each homotopy class. Thus we consider maps $\phi: K \to K^n$ of the form $\phi(x) = (x^{m_1}, x^{m_2}, \cdots, x^{m_n})$, $m_1, \cdots, m_n \in Z$. Then

$$\psi_\phi(x, x_1, \cdots, x_n) = \left(\alpha x, x^{m_1}\prod_{i=1}^n x_i^{a_{1i}}, \cdots, x^{m_n}\prod_{i=1}^n x_i^{a_{ni}}\right)$$

where τ is given by the matrix (a_{ij}). In other words ψ_ϕ in an affine transformation of K^{n+1} with automorphism part τ' given by (b_{ij}), where

$$b_{ij} = \begin{cases} 1 & i = j = 1 \\ m_{i-1} & j = 1, 2 \leqq i \leqq n+1 \\ a_{i-1,j-1} & 2 \leqq i, j \leqq n+1 \\ 0 & i = 1, 2 \leqq j \leqq n+1 . \end{cases}$$

Since the automorphism τ' is not unipotent it follows that ψ_ϕ is not minimal. Thus there are no minimal perturbations of ψ.

We do not know of any examples of minimal perturbations of ψ of this form if we drop the hyperbolic condition on τ, but retain that τ be ergodic or even have positive entropy.

We again close the section with some remarks on Z^k-action. For simplicity, we suppose $k = 2$. Thus we have commuting homeomorphisms ψ_1, ψ_2 on X, projecting to η_1, η_2 on Y, and automorphisms τ_1, τ_2 on G with $\psi_i(gx) = \tau_i(g)\psi_i(x)$, $i = 1, 2$. If ϕ is a $\{\tau_1, \tau_2\}$-cocycle, then we must have $\phi(y, (m, n)) = \phi(y, (m, 0) + (0, n)) = \phi(y, (0, n) + (m, 0))$ for all $y \in Y, (m, n) \in Z^2$. This relationship, together with the cocycle condition, enables us to identify the set of cocycles with a closed subset of $C(Y, G) \times C(Y, G)$, and hence we have a complete metric topology on the set of all $\{\tau_1, \tau_2\}$-cocycles.

We may replace condition (A) of admissibility, Definition 3.1, by: for all $\varepsilon > 0$ there exist $M, N \geqq 1$ such that for all $(m, n) \geqq (M, N)$ and $g_0, \cdots, g_{n+m-1} \in G$ we have

$$\left[\prod_{i=0}^{m-1} \tau_1^i S_\varepsilon(g_i)\right]\left[\prod_{j=0}^{n-1} \tau_2^i S_\varepsilon(g_{i+j})\right] = G .$$

Thus if either (G, τ_1) or (G, τ_2) satisfy (A), then $(G, \{\tau_1, \tau_2\})$ satisfies the above.

In attempting to generalize Theorem 3.13 a major problem is to identify the set of cocycles and "fit" a cocycle to a finite set of data. In general we have been unable to do this. One case we can deal with is when (Y, Z^2) is a free product of two flows $(Y_1, \bar{\eta}_1)$, $(Y_2, \bar{\eta}_2)$; in this case the set of cocycles coincides with $C(Y_1, G) \times C(Y_2, G)$. Thus the following extension of 3.13 and 3.14 holds.

THEOREM 3.22. *Let $\pi: (X, Z^2) \to (Y, Z^2)$ be a (G, Z^2)-extension with (Y, Z^2) a free product of (Y_1, η_1) and (Y_2, η_2), Y_1 and Y_2 infinite with no isolated points. Suppose (G, Z^2) is admissible and (Y_1, η_1), (Y_2, η_2) point-transitive. Then for almost all product cocycles $\{\phi_1, \phi_2\}$ we have $(X, Z^2_{(\phi_1,\phi_2)})$ is point-transitive. If, in addition, (Y_1, η_1) and (Y_2, η_2) are minimal and (G, Z^2) is distal, then for almost all product cocycles, $(X, Z^2_{(\phi_1,\phi_2)})$ is minimal.*

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