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FREE SEMIGROUPS OF 2×2 MATRICES

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Let $A = [1, m; 0, 1]$, $B = [1, 0; m, 1]$. The semigroup $S_m = sgp\langle A, B\rangle$ (including identity) generated by A, B is nonfree if two formally different words (with positive exponents) are equal; free otherwise. Theorem. S_m is free if $-\pi/4 \leqq \arg m \leqq \pi/4$, $|m| \geqq 1$.

Thus S_m can be free when $G_m = gp\langle A, B\rangle$ is nonfree.

THEOREM. Values of m for which S_m is nonfree are dense on the line segment joining $-2i$ to $2i$; there are nonfree values of m arbitrarily close to $m = 1$.

The group $G_m = gp\langle A, B\rangle$ generated by $A = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ is free if m is transcendental [6], if $m = 2$ [13] if $|m| \geqq 2$ [2], and if m satisfies none of the three inequalities $|m|^2 < 2$, $|m^2 - 2| < 2$, $|m^2 + 2| < 2$ [5]. Further results appear in [1, 3, 7, 8, 9, 10, 11, 12]. A diagonal similarity transformation carries A to $C = [1, 2; 0, 1]$ and B to $D = [1, 0; \lambda, 1]$, $\lambda = m^2/2$. Most of the known results are summarized in the diagram given in [8, p. 1392], which is drawn in the λ plane. A value of λ is "free" if $gp\langle C, D\rangle$ is free. The nonfree values of λ are dense in $|\lambda| < 1/2$ [5]. The semigroup $S_m = sgp\langle A, B\rangle$ (including identity) generated by A, B is nonfree if two formally different words W_1, W_2 (with positive exponents) are equal, or if $W_1 = I$; free otherwise. In conversation, S. Stein and D. Hickerson asked whether S_m can be free when G_m is nonfree. Theorems 2.4–2.6 give an affirmative answer to this question (take $m = 1$). For orientation, two trivial lemmas are worth stating.

1.1. LEMMA. *If S_m is nonfree, then G_m is nonfree.*

1.2. LEMMA. *If G_m is free then S_m is free.*

Let H_λ (K_λ) be the group (semigroup) generated by C and D. Then we have:

1.3. LEMMA. *H_λ (K_λ) is free if and only if G_m (S_m) is free.*

As noted in [8, p. 1391] we also have:

1.4. LEMMA. *H_λ is free if and only if $H_{-\lambda}$ is free.*

However it will be seen that it is possible for K_λ to be free while $K_{-\lambda}$ is not free.

1.5. PROBLEM. Let $|\lambda| < 1/2$. Is it true that K_λ is free whenever $K_{-\lambda}$ is free?

1.6. PROBLEM. If G_m is not free, is it generated by elements E and F such that $sgp\langle E, F\rangle$ is not free?

1.7. LEMMA. *Let* $\lambda = m^2/2$. *Then* $K_{-\lambda}$ *is free if and only if* $sgp\langle[1, m; 0, 1], [1, 0; -m, 1]\rangle$ *is free.*

Proof. Conjugate by $[2, 0; 0, m]$.

In §2 it is shown that if $\mathrm{Re}\,\lambda \geqq 1/2$, K_λ is free. This is a best possible result in the sense that (as shown in §3) $\lambda = 1/2$ is a limit of nonfree values.

In §4 it is shown that nonfree values of λ are dense on $[-2, 0]$. Probably they are also dense on $[0, 1/2]$; some results to support this conjecture are given. It is also shown that there exists a value of λ in $[-2, 0]$ for which K_λ is not free, but is torsion free.

Section 5 applies the methods of the preceding sections to the group H_λ. It is shown that, in some respects, the methods are more powerful than those previously used. The extensive machine calculations in [3] are simplified.

In §6 it is shown that S_m is almost always free if m is a root of unity.

2. **Free regions.** In this section $R(z)$ and $I(z)$ denote the real and imaginary parts of the complex number z in the extended complex plane. Also, if $U = [a, b; c, d]$, $\det U = 1$, then we denote by $U(z)$ the complex number $(az + b)(cz + d)^{-1}$. Clearly if V is another such matrix then $(UV)(z) = U(V(z))$. As usual a word in $sgp\langle A, B\rangle$ means either the identity or $A^{x_1}B^{x_2}\cdots$ or $B^{x_2}A^{x_3}\cdots$ where all exponents are positive.

2.1. LEMMA. (a) *If* $R(z) > 2$ *then* $|z^{-1} - 1/4| < 1/4$.
(b) *If* $|z - 1/4| > 1/4$ *and* $R(z) > 0$ *then* $0 < R(z^{-1}) < 2$.

Proof. (a) The map $T(z) = z^{-1}$ carries the line $R(z) = 2$ onto the circle $|w - 1/4| = 1/4$. Since $T(4) = 1/4$, T must carry the region $R(z) > 2$ onto the interior of the circle $|w - 1/4| = 1/4$.

(b) The map $T(z) = z^{-1}$ carries the circle $|z - 1/4| = 1/4$ onto the line $R(w) = 2$. Since $T(1) = 1$, T must map the exterior of the circle onto the region $R(w) < 2$. Clearly $R(z) > 0$ implies $R(T(z)) > 0$.

2.2. LEMMA. *Let* $|\lambda| \geqq 1/2$, $R(\lambda) \geqq 0$, $R(z) > 2$, $C = [1, 0; \lambda, 1]$. *Then* $0 < R(C^n(z)) < 2$ *for every positive integer* n.

Proof. Let $z' = z^{-1} + n\lambda$. Then $C^n(z) = 1/z'$. By 2.1a we have $|z^{-1} - 1/4| < 1/4$. Hence

$$\left|z' - \frac{1}{4}\right| = \left|n\lambda - \left(\frac{1}{4} - z^{-1}\right)\right| \geqq |n\lambda| - \left|\frac{1}{4} - z^{-1}\right|$$
$$> \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Now $R(z') \geqq R(z^{-1}) > 0$. Hence by 2.1b

$$0 < R(1/z') < 2 .$$

2.3. LEMMA. *Let*

$$R(u) = 1, \sum = \{w \mid R(wu) > 2\}, \Delta = \{w \mid 0 < R(wu) < 2\} .$$

Let $|\lambda| \geqq 1/2$, $R(\lambda) \geqq 0$, $A = [1, 2; 0, 1]$, $B = [1, 0; \lambda u, 1]$. *Let* n *and* m *be any positive integers. Then*:

(a) $w \in \Sigma$ *implies* $B^n(w) \in \Delta$

(b) $w \in \Delta$ *implies* $A^n(w) \in \Sigma$

(c) $A^n B^m(1) \in \Sigma$

(d) $B^n A^m(1) \in \Delta$.

Proof. Let $U = [u, 0; 0, 1]$, $C = [1, 0; \lambda, 1]$. Then $B = U^{-1}CU$.

(a) Let $w \in \Sigma$, $z = wu$. Now $B^n(w) = U^{-1}C^nU(w) = u^{-1}C^n(z)$. Hence

$$R(uB^n(w)) = R(C^n(z)) .$$

But by 2.2 we have $0 < R(C^n(z)) < 2$. Thus $B^n(w) \in \Delta$.

(b) Let $w \in \Delta$. Then $0 < R(wu) < 2$.

Now

$$R(uA^n(w)) = R(u(w + 2n)) = R(uw) + 2n > 2n \geqq 2 .$$

Thus $A^n(w) \in \Sigma$.

(c) We have $uA^nB^m(1) = (\lambda m + u^{-1})^{-1} + 2nu$. Now $R(2nu) = 2n \geqq 2$. Also $R(\lambda m + u^{-1}) = R(\lambda m) + R(u^{-1}) > 0$, since $R(\lambda m) \geqq 0$ and $R(u^{-1}) > 0$. Thus $R(uA^nB^m(1)) > 2$ and $A^nB^m(1) \in \Sigma$.

(d) $R(uA^m(1)) = R(u + 2mu) = 1 + 2m > 2$. Thus $A^m(1) \in \Sigma$. Hence by (a) we have $B^nA^m(1) \in \Delta$.

2.4. THEOREM. *Let* $R(\lambda) \geqq 0$, $|\lambda| \geqq 1/2$, $R(u) = 1$, $A = [1, 2; 0, 1]$, $B = [1, 0; \lambda u, 1]$. *Then the semigroup* $K_{\lambda u}$ *generated by* A *and* B *is free.*

Proof. Suppose W_1 and W_2 are different words in $K_{\lambda u}$ with $W_1 = W_2$. Let Σ and Δ be as in 2.3.

Case 1. One of the words, say W_1 is the identity I. Clearly $A^n = I$ or $B^n = I$ is impossible for any positive n. Also $A^nB^m = I$ or $B^mA^n = I$ is impossible since $A^n \neq B^{-m}$ for positive n and m. Thus W_2 has length $\geqq 3$. Since the relation $W_2 = I$ implies the relation $W_2^* = I$, where W_2^* is any cyclic permutation of W_2, we may assume that W_2 starts with A and ends with B. Let $W_2 = A^{x_n}B^{y_n} \cdots A^{x_1}B^{y_1}$, $x_i > 0$, $y_i > 0$. It follows from 2.3 that $W_2(1) \in \Sigma$. But $W_2(1) = 1 \in \Delta$, a contradiction.

Case 2. Neither word is the identity but one of them (say W_1) has length 1. Let $P = [0, 1; \lambda u/2, 0]$. Then the map $X \to PXP^{-1}$ is an automorphism of $K_{\lambda u}$ sending $A \to B$ and $B \to A$. Because of this we may assume that $W_1 = A^{x_1}$. Clearly $W_2 \neq B^{y_1}$ since $A^{x_1} \neq B^{y_1}$ and $W_2 \neq A^{y_1}$ since $A^{x_1} = A^{y_1}$ implies $x_1 = y_1$. Thus W_2 is of length $\geqq 2$. We may assume that W_2 starts and ends with B, for otherwise we could cancel and either return to Case 1 or obtain the desired condition. Let $W_2 = B^{s_n}A^{t_n} \cdots B^{s_1}A^{t_1}B^{s_0}$. It follows from 2.3 that $W_2(1) \in \Delta$. But $R(uW_1(1)) = R(u(1+2x_1)) = 1+2x_1 > 2$, hence $W_1(1) \in \Sigma$, a contradiction.

Case 3. Each word is of length $\geqq 2$. We may assume that W_1 and W_2 do not start with the same letter or end with the same letter, for otherwise we could cancel it. We consider two cases.

3.1. One word (say W_1) starts with B and ends with A. Then $W_1 = B^{x_n}A^{y_n} \cdots B^{x_1}A^{y_1}$ and $W_2 = A^{r_n}B^{s_n} \cdots A^{r_1}B^{s_1}$. From 2.3 we conclude that $W_1(1) \in \Delta$ and $W_2(1) \in \Sigma$, a contradiction.

3.2. One word (say W_1) starts with B and ends with B. Then $W_1 = B^{x_n}A^{y_n} \cdots B^{x_1}A^{y_1}B^{x_0}$ and $W_2 = A^{r_n}B^{s_n} \cdots A^{r_1}B^{s_1}A^{r_0}$. From 2.3 we conclude that $W_1(1) \in \Delta$, $W_2(1) \in \Sigma$, a contradiction.

2.5. THEOREM. *If $R(\lambda) < 0$ and $|I(\lambda)| \geqq 1/2$ then K_λ is free.*

Analytic proof. Clearly one of the tangent lines drawn from $\lambda = x + yi$ to the circle $|z| = 1/2$ intersects the circle in a point (c, d) with $c \geqq 0$. Set $\lambda' = c + di$. First assume $c \neq 0$. Let $b = (y - d)c^{-1}$, $u = 1 + bi$. The condition on the tangent line yields $(y - d)(x - c)^{-1}dc^{-1} = -1$. Hence

$$x = (d^2 + c^2 - dy)c^{-1} = [d^2 + c^2 - d(bc + d)]c^{-1} = c - bd .$$

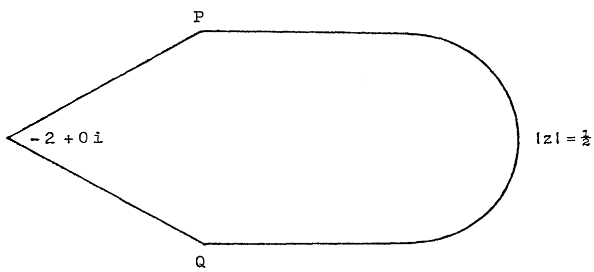
Thus $u\lambda' = c - bd + (bc + d)i = x + yi = \lambda$. By 2.4 we have $K_\lambda = K_{u\lambda'}$ is free. If $c = 0$ then $d = \pm 1/\sqrt{2}$, $y = d$. Let $u = 1 - xd^{-1}i$. Then $\lambda = u\lambda'$ and $K_\lambda = K_{u\lambda'}$ is free by 2.4.

Geometric proof. Let λ' lie on the semicircumference $|\lambda'| = 1/2$, $R(\lambda') \geqq 0$. If $R(u) = 1$, the locus $\lambda = u\lambda'$ is the line through λ' and perpendicular to the radius drawn from 0 to λ. As λ' varies, λ sweeps out all of the region $\{\lambda \mid R(\lambda) < 0, I(\lambda) \geqq 1/2\}$ (and more).

2.6. THEOREM. *Let*

$$P = \left(\frac{1}{2}(\sqrt{3} - 4), \frac{1}{2}\right), Q = \left(\frac{1}{2}(\sqrt{3} - 4), -\frac{1}{2}\right).$$

Then K_λ is free if λ is in the (closed) exterior of the bullet-shaped region illustrated.



Proof. By 2.4 we have $R(\lambda) \geqq 0$, $|\lambda| \geqq 1/2$ implies that K_λ is free and by 2.5 we have $R(\lambda) < 0$, $|I(\lambda)| \geqq 1/2$ implies K_λ is free. By [8, Theorem 3, p. 1390], the group H_λ (and hence the semigroup K_λ) is free if λ is not in the interior of the convex hull of $\{z \mid |z| = 1\} \cup \{2, -2\}$. But the tangent lines drawn from $(-2, 0)$ to $|z| = 1$ intersect $y = 1/2$ and $y = -1/2$ in P and Q respectively.

3. Some nonfree semigroups. In this and all remaining sections let A, B, C, D be as in §1.

It is known [3, 8] that there are some values of m for which $gp\langle A, B\rangle$ is not free; the value $m = 1$ has been known for long time. To obtain values of m for which $S_m = sgp\langle A, B\rangle$ is not free requires methods attuned to this special problem.

3.01. DEFINITION. A relation $w_1(A, B) = w_2(A, B)$ between 2 words in S_m is reduced if no cancellation is possible. The degree of a reduced relation is the greater of the lengths of the words w_1, w_2. (The degree of a reducible relation is defined by first reducing it to an equivalent reduced relation.)

Thus

$$AB^2A = B^3A^5B^4$$
$$ABAB^2AB^2 = AB^3A^5B^4$$

both have degree 3.

The following assertions have transparent proofs.

3.02. Lemma. *If $m \neq 0$, there is no relation of degree* 1 *or* 2 *in* S_m.

3.03. Lemma. *If a relation has degree* 3, *it can be written*

$$A^xB^yA^z = B^rA^sB^t ,$$

with x, y, z, r, s, t all positive.

The next theorem gives a complete account of the values of $m \neq 0$ for which S_m admits a relation of degree 3.

3.04. Theorem. *Let S_m admit a relation of degree* 3:

$$A^xB^yA^z = B^rA^sB^t .$$

Then

$$m^2 = x^{-1}(r^{-1} - y^{-1}) - t(rxy)^{-1} . \tag{3.05}$$

Furthermore if r, x, y, t are arbitrary positive integers such that $s = xyt^{-1}$ and $z = xrt^{-1}$ are integers, then for m^2 given by (3.05) the *stated relation of degree* 3 *holds.*

Note that both positive and negative values of m^2 arise, and that $-2 < m^2 < 1$. These bounds are exact. In fact, if $t = x = r = 1$, and $y \to \infty$ then $m^2 \to 1$. Also, if $x = y = 1$, $t = r \to \infty$, $\lim m^2 = -2$.

Proof of 3.04. Calculation shows that the relation

$$A^xB^yA^z = B^rA^sB^t$$

holds if and only if (3.06)-(3.09) all hold.

$$rs = yz , \tag{3.06}$$

$$st = xy , \tag{3.07}$$

$$s = x + z + m^2xyz , \tag{3.08}$$

$$y = r + t + m^2rst . \tag{3.09}$$

From (3.06)-(3.07) follows $rx = tz$. From (3.06)-(3.08) it follows that

$st = xt + rx + m^2strx$; this is (3.09) which is therefore redundant. It is now apparent that the solutions of (3.06)-(3.09) can be parametrized by taking r, x, y arbitrary positive integers, subject to $t \mid xy$, $t \mid rx$, setting $s = xy/t$, $z = rx/t$ and solving (3.08) for m^2. But (3.05) is a paraphrase of (3.08).

3.10. COROLLARY. *The values* $\lambda = 1/2$, $\lambda = -1$ *are limits of nonfree values.*

The relations of degree 4 are described in the next theorem.

3.11. THEOREM. *Any relation of degree* 4 *in* S_m *must have the form*

$$B^u A^x B^y A^z = A^q B^r A^s B^t \ , \tag{3.12}$$

with u, x, y, z, q, r, s, t *all positive.*

Proof. A priori, the relation $B^u A^x B^y A^z = A^q B^r$ would be conceivable. Detailed examination of this possibility shows, however, that such a relation is not possible unless $q = 0$. Similarly, the relation $B^u A^x = A^q B^r A^s B^t$ does not arise.

There are many values of m that satisfy (3.12), but do not satisfy (3.05).

Other nonfree values of m are given in § 5.

4. Semigroups with torsion. There are values of m such that S_m contains elements of finite order. It may be conjectured that every value of m with this property is a pure imaginary unmber. In fact, the pure imaginary numbers m with this property are denes on the line segment joining $-2i$ and $2i$.

4.1. THEOREM. *The nonfree values of* λ *are dense on* $[-2, 0]$.

Recall that $\lambda = m^2/2$.

Proof. Note $CD = [1 + 2\lambda, 2; \lambda, 1]$. This matrix has finite order if (and only if) its trace is $2 \cos k\pi/l$ for some integers k, l. But this is easily arranged: $\lambda = -2 \sin^2 k\pi/(2l)$.

4.2. THEOREM. *Let* $w = w(C, D)$ *have length* 2 *or* 3, *and have finite order. Then* λ *is real and negative.*

The proof is straightforward, so is omitted.

4.3. THEOREM. *Let $w = w(C, D)$ have length 4, and have finite order. Then λ is real and negative.*

Proof. Calculation shows that

$$\operatorname{tr} D^u C^x D^y C^z = 2 + 2\lambda(xy + yz + xu + zu) + 4xyzu\lambda^2 .$$

The condition that this is equal to $2\cos k\pi/l$ leads to a quadratic in λ. It must be proved that the discriminant of this quadratic is nonnegative. This fact is seen to follow from the arithmetic-geometric mean inequality applied to the four numbers xy, yz, xu, zu.

4.4. THEOREM. *Let n be a nonzero integer. Then S_m has torsion for the following values of m:*

$$(1)\quad m = i/n \qquad (2)\quad m = \sqrt{2}\, i/n \qquad (3)\quad m = \sqrt{3}\, i/n .$$

Proof. (1) Let $U = A^3 B^{nn} = [-2, 3m; mn^2, 1]$.
Then U has order 3.
(2) Let $U = AB^{nn} = [-1, m; mn^2, 1]$.
Then U has order 4.
(3) Let $U = A^{nn}B = [-2, mn^2; m, 1]$.
Then U has order 3.

4.5. THEOREM. *If m is real then S_m is torsion free.*

Proof. We may assume $m > 0$. If a nontrivial word W in S_m has finite order, the proper values of W are roots of unity and are reciprocals (since $\det W = 1$). Hence trace $W = z + \bar{z} < 2$, since z is a root of unity. An easy inductive argument shows, however, that every entry of W is nonnegative, and that each diagonal entry is at least 1. Thus trace $W \geqq 2$, a contradiction.

In [4, p. 747] it is shown that if m is rational and not the reciprocal of an integer then G_m (and hence S_m) is torsion free. In the same vein we have:

4.6. THEOREM. *If $m = pi/q$, p and q integers, $p \neq 0$, $q \neq 0$, $p \neq \pm 1$, $(p, q) = 1$, then G_m (and hence S_m) is torsion free.*

Proof. Assume G_m has a nontrivial element of finite order. Then it has an element U of prime order π. If $\pi = 2$, then $U = -I$; if $\pi > 2$, U has trace $\omega + \omega^{\pi-1}$ where ω is a primitive πth root of unity. It is easily seen by induction that U is of the form:

$$U = \begin{pmatrix} 1 + f_1(m^2) & mf_2(m^2) \\ mf_3(m^2) & 1 + f_4(m^2) \end{pmatrix}$$

where the f_i are polynomials with integer coefficients and f_1 and f_4 are without constant term. Thus U has trace $2 + f_1(m^2) + f_4(m^2) = 2 + h(m^2)$ where h is a polynomial with integer coefficients and without constant term.

Case 1. $\pi = 2$. Then $U = -I$, whence $1 + f_1(m^2) = -1$, that is $f_1(m^2) + 2 = 0$. This implies that $p^2 | 2$, a contradiction.

Case 2. $\pi = 3$. Then U has trace $\omega + \omega^2 = -1 = 2 + h(m^2)$, that is $h(m^2) + 3 = 0$. This implies that $p^2 | 3$, a contradiction.

Case 3. $\pi > 3$. Since U has trace $\omega + \omega^{\pi-1} = 2 + h(m^2)$, $\omega + \omega^{\pi-1}$ must be rational. But this contradicts the fact that the minimal polynomial of ω over the rationals is $1 + x + x^2 + \cdots + x^{\pi-1}$.

It is possible for S_m to be torsion free but not free. When $m = 2i/3$, S_m is torsion free by 4.6 but is not free (see 5.1e).

5. More nonfree values of m. We now examine certain relations of degree 4 in S_m. A computation shows that $A^x B^y A^z B^w = B^w A^z B^y A^x$ if and only if the following condition holds:

$$yz = wx + xy + wz + m^2 xyzw \ . \tag{5.1}$$

Thus for a given m we seek solutions of (5.1) in positive integers x, y, z, w.

5.2. THEOREM. *Let n be an integer. Then S_m is not free for the following values of m:*

(a) $m = 1/n, \quad |n| > 1,$
(b) $m = 2/n, \quad |n| > 2,$
(c) $m = 4/n, \quad |n| > 4,$
(d) $m = i/n, \quad |n| \geqq 1,$
(e) $m = 2i/n, \quad |n| \geqq 2,$
(f) $m = 4i/n, \quad |n| \geqq 4.$

Proof. Since S_m is free if and only if S_{-m} is free, we may assume that n is positive.

(a) If $n > 2$ then $x = 1, z = n, w = n^2 - 2n, y = (n + 1)w$ is a solution of (5.1). If $n = 2$ then $x = 1, y = 6, z = 2, w = 1$, is a solution of (5.1).

(b) We may assume n is odd.

Case 1. $n \equiv 1 \bmod 4$. Then $n = 1 + 4u$ and $u > 0$. If $u = 1$ then $n = 5$ and $x = 1, y = 50, z = 11, w = 5$ is a solution of (5.1). If

$u > 1$ then $x = u - 1$, $y = nu$, $z = n$, $w = 2 + 3u$ is a solution of (5.1).

Case 2. $n \equiv 3 \bmod 4$. Then $n = 3 + 4u$. If $u = 0$ then $n = 3$ and $x = 1$, $y = 3$, $z = 6$, $w = 1$ is a solution of (5.1). If $u \neq 0$ then $u > 0$ and $x = u$, $y = n^2$, $z = 2u(1 + u)$, $w = n$ is a solution of (5.1).

(c) We may assume n is odd. It follows that either $n^2 \equiv 1 \bmod 16$ or $n^2 \equiv 9 \bmod 16$.

Case 1. $n^2 \equiv 1 \bmod 16$. Then $x = (n^2-1)/16$, $y = 2n^2$, $z = x(1+2n^2)$, $w = 1$ is a solution of (5.1).

Case 2. $n^2 \equiv 9 \bmod 16$. Then $x = 1$, $w = (n^2 - 9)/16$, $y = n^2(1 + w)$, $z = 2w + 1$ is a solution of (5.1).

(d) $x = 1$, $y = 1 + n$, $z = n$, $w = n$ is a solution of (5.1).

(e) We may assume $n > 2$.

Case 1. $n \equiv 1 \bmod 3$. Then $x = (n-1)/3$, $y = n$, $z = n$, $w = n(n-x)$ is a solution of (5.1).

Case 2. $n \equiv 2 \bmod 3$. Then $x = (n-2)/3$, $y = n$, $z = n$, $w = n(1+x)$ is a solution of (5.1).

Case 3. $n \equiv 0 \bmod 3$. Then $x = n$, $y = n$, $z = 2n/3$, $w = n/3$ is a solution of (5.1).

(f) We may assume n is odd.

Case 1. $n^2 \equiv 1 \bmod 16$. Then $w = (n^2 - 1)/16$, $x = 8w$, $y = n^2w$, $z = 1$ is a solution of (5.1).

Case 2. $n^2 \equiv 9 \bmod 16$. Let $u = (n^2 - 9)/16$. Then $x = un^2$, $y = 2u + 1$, $z = u + 1$, $w = 1$ is a solution of (5.1) and the theorem is proved.

5.2. COROLLARY. [3, Theorem 3.1, p. 243]. *If b is any integer >2, the group $G_m = gp < [1, m; 0, 1], [1, 0; m, 1] >$ is not free whenever $m = 4/b$.*

Proof. Note that G_m is not free if $m = 4/3$ [8]; then apply 5.2(c).

(This proof supersedes an extensive computer calculation in [3].)
Finally we remark that we have not been able to prove that $S_{3/n}$ is not free ($|n| > 3$), although we presume that this is the case.

5.3. THEOREM. *In every neighborhood N of 1 there exists a real number r and a sequence r_n of reals such that S_{r_n} is not free and* $\lim_{n\to\infty} r_n = r$.

Proof. Choose an integer y such that $y > 3$, $y \in N$. Set $r = \sqrt{1-y^{-1}}$. Now if $x = 1$ and $w = 1$, (5.1) becomes:

$$m^2 = 1 - (yz)^{-1} - z^{-1} - y^{-1} . \tag{5.4}$$

Hence if m satisfies (5.4) then S_m is not free (for any z). For each integer $n > 3$ set $r_n = \sqrt{1 - (ny)^{-1} - n^{-1} - y^{-1}}$. Then S_{r_n} is not free and $\lim_{n\to\infty} r_n = r$.

6. Roots of unity. In [11, p. 69] it is conjectured that G_m is not free if m is a primitive qth root of 1. The situation for semigroups is quite different.

THEOREM 6.1. *If m is a primitive qth root of 1 and $q \neq 3$, 4 or 6 then S_m is free.*

Proof. Since any two primitive qth roots of 1 are conjugate, it suffices to prove the theorem for any particular primitive qth root of 1.

Case 1. Suppose $q \geqq 8$. Let $m = \cos(2\pi/q) + i\sin(2\pi/q)$. Then $\lambda = m^2/2 = (1/2)[\cos(4\pi/q) + i\sin 4\pi/q]$. Then $|\lambda| = 1/2$ and $R(\lambda) = (1/2)\cos(4\pi/q) \geqq 0$ (since $q \geqq 8$). Hence by 2.4 K_λ (and hence S_m) is free.

Case 2. $q < 8$. If $q = 1$ or 2 then $\lambda = m^2/2 = 1/2$ and again by 2.4, K_λ (and hence S_m) is free. Now suppose $q = 5$. Let $\omega = \cos(2\pi/5) + i\sin(2\pi/5)$. Let $m = \omega^3$. Then m is a primitive 5th root of 1. Let $\lambda = m^2/2 = \omega/2$. Then $|\lambda| = 1/2$, $R(\lambda) = (1/2)\cos(2\pi/5) \geqq 0$. Hence by 2.4, K_λ and (hence S_m) is free. Now assume $q = 7$. Let $\omega = \cos(2\pi/7) + i\sin(2\pi/7)$. Let $m = \omega^4$. Then m is a primitive 7th root of 1. Let $\lambda = m^2/2 = \omega/2$. Then $|\lambda| = 1/2$, and

$$R(\lambda) = (1/2)\cos(2\pi/7) \geqq 0 .$$

Hence K_λ is free and the proof is complete.

We note that if $q = 4$, $m = i$, so that S_m is not free by 5.2(d). If $q = 3$, $m = \cos(2\pi/3) + i\sin(2\pi/3)$, $\lambda = m^2/2 = (-1/4)(1 + \sqrt{3}i)$ while if $q = 6$, $m' = \cos(2\pi/6) + i\sin(2\pi/6)$, $\lambda' = m^2/2 = (-1/4)(1 - \sqrt{3}i)$. The two values of λ are conjugate; hence $K_\lambda \cong K_{\lambda'}$ and $S_m \cong S_{m'}$. Thus it suffices to treat the case $q = 3$. We have not been able to prove that S_m is not free when m is a primitive cube root of 1. However, we do have:

6.2. THEOREM. *Let* $\omega = \cos(2\pi/3) + i\sin(2\pi/3)$. *Then there exists a sequence* z_n *such that* $\lim_{n\to\infty} z_n = \omega$ *and* S_{z_n} *is not free.*

Proof. A computation shows that

$$A^x B^y A^u B^v A^z B^w = B^w A^z B^v A^u B^y A^x$$

if and only if $am^4 + bm^2 + c = 0$ where

$$\begin{aligned} a &= xyuvzw\ , \\ b &= xyuv + zwxy + zwuv + xvzw + uwxy - zvuy\ , \\ c &= xy + uv + zw + xv + uw + xw - zv - yu - zy\ . \end{aligned}$$

If we let $x = y = z = w = 1$, $u = v$ the above condition becomes

$$u^2m^4 + (u+1)^2m^2 + u^2 + 2 = 0\ . \tag{6.3}$$

Thus if m is solution of (6.3) (for any positive integer u), then S_m is not free. Let n be an integer, $n > 1$. It is easily seen that $4n^2(2+n^2) > (n+1)^4$. Let $r_n = \sqrt{4n^2(2+n^2) - (n+1)^4}$. Let $\varDelta_n = r_n i$. Then $\lim_{n\to\infty}[\varDelta_n/(2n^2)] = (\sqrt{3}/2)i$. Choose z_n so that $0 \leqq \arg z_n < \pi$ and $z_n^2 = [-(n+1)^2 - \varDelta_n]/(2n^2)$. Then $n^2z_n^4 + (n+1)^2z_n^2 + n^2 + 2 = 0$ and hence S_{z_n} is not free. Moreover $\lim_{n\to\infty} z_n^2 = -(1+\sqrt{3}i)/2 = \omega^2$. Hence $\lim_{n\to\infty} z_n = \omega$.

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Added in proof. Additional references have come to our attention.

The paper of Evans answers the conjecture of Newman [11] affirmatively. The papers of Merzljakov and Scharlemann improve on Bachmuth and Mochizuki's [1] results; Scharlemann answers a question in [1] negatively.

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