

TRIGONOMETRIC APPROXIMATION THEORY IN COMPACT TOTALLY DISCONNECTED GROUPS

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We study some aspects of the problem of approximating functions on a compact totally disconnected group by trigonometric polynomials. In the classical case, approximation is connected with smoothness properties. By appropriately defining smoothness, one obtains this connection in the totally disconnected case also, and there are analogues of the classical results.

In particular it is shown that the Lipschitz class to which a function belongs can be identified by the best approximation characteristics of the function by trigonometric polynomials (Theorems 2 and 3), and that functions which are easily approximated by trigonometric polynomials have absolutely convergent Fourier series (Theorems 1 and 4).

Let G be a compact group. Let σ be an equivalence class of continuous irreducible unitary representations of G . The set of all such σ is called the dual object of G and is denoted by Σ . For each $\sigma \in \Sigma$ fix a $U^{(\sigma)} \in \sigma$ and let H_σ be the Hilbert space in which $U^{(\sigma)}$ acts. The dimension of H_σ is denoted by d_σ . This notation is consistent with the notation in the book of Hewitt and Ross [5]. Any unexplained notation may be found there. We now restrict our attention to infinite compact totally disconnected groups whose dual objects are countable or equivalently, which have a countable neighborhood base $G = G_0 \supset G_1 \supset \dots$ at the identity e consisting of open (hence closed) normal subgroups [7, p. 132]. Since these subgroups are open, they have positive Haar measure, and since each coset of a given subgroup has the same Haar measure, it follows that the index of G_{n+1} in G_n is $m(G_n)/m(G_{n+1})$, where m denotes the normalized Haar measure on G . We say that a bounded function is in the Lipschitz class of order $\alpha > 0$ (with respect to a neighborhood system $\{G_n\}$) if

$$\sup_{x \in G_k} \|f(x \cdot) - f(\cdot)\|_\infty \leq C m(G_k)^\alpha$$

where C is some constant independent of k .

This definition involved left translation. That right translation gives the same class of functions can be seen as follows. We have $|f(xy) - f(y)| = |f(y y^{-1}xy) - f(y)|$, and since G_k is a normal subgroup, $y^{-1}xy$ runs through G_k as x runs through G_k .

The Lipschitz class of order α will be denoted by $\text{Lip}_\alpha(G)$ and

is a closed subspace of the continuous functions $C(G)$ when endowed with the norm

$$\|f\|_{\text{Lip}_\alpha(G)} = \sup_k [\sup_{x \in G_k} \|f(x \cdot) - f(\cdot)\|_\infty m(G_k)^{-\alpha}] + \|f\|_\infty.$$

The space $\text{Lip}_\alpha(G)$ is nontrivial for all $\alpha > 0$. For example,

$$f(x) = \sum_{n=1}^{\infty} m(G_{n-1})^\alpha d_{\sigma_n}^{-1} \chi_{\sigma_n}(x)$$

where $\sigma_n \in A(\Sigma, G_n) \setminus A(\Sigma, G_{n-1})$ and $\chi_\sigma = \text{tr } U^{(\sigma)}$ is the character of the representation σ , belongs to $\text{Lip}_\alpha(G)$. This may be seen as follows. Let $x \in G_k$, $y \in G$, then

$$|f(xy) - f(y)| \leq \sum_{n=1}^{\infty} m(G_{n-1})^\alpha d_{\sigma_n}^{-1} |\chi_{\sigma_n}(xy) - \chi_{\sigma_n}(y)|.$$

However, $\chi_\sigma(xy) = \chi_\sigma(y)$ for $\sigma \in A(\Sigma, G_k)$, so that in the above sum, the terms for which $n \leq k$ vanish. Since $\|\chi_\sigma\|_\infty = d_\sigma$ we find that

$$|f(xy) - f(y)| \leq \sum_{n=k+1}^{\infty} m(G_{n-1})^\alpha d_{\sigma_n}^{-1} 2d_{\sigma_n}.$$

Moreover, since $m(G_{n+1})/m(G_n) < 1/2$ it follows that $|f(xy) - f(y)| \leq C(m(G_k))^\alpha$ where C depends on α but not on k .

It is customary to express Lipschitz conditions in terms of metrics. This is possible here by defining a metric on G by

$$d(x, y) = \begin{cases} m(G_n), & \text{if } xy^{-1}G_n \setminus G_{n+1} \\ 0, & \text{if } xy^{-1} = e. \end{cases}$$

Let F be a finite subset of Σ , and let f be a continuous function on G . Define

$$E_F(f) = \inf \|f - T\|_\infty$$

where the infimum is taken over all trigonometric polynomials T with spectrum in F . In this paper we restrict our attention to the case where $F = A(\Sigma, G_n)$ for some n . We will then simply write $E_n(f)$ instead of $E_{A(\Sigma, G_n)}(f)$. The infimum is actually attained since the space of trigonometric polynomials with spectrum in $A(\Sigma, G_n)$ is finite dimensional.

PROPOSITION 1. *Given n , suppose there is $K \in L^1(G)$ such that $\hat{K}(\sigma) = I_{d_\sigma}$ for all $\sigma \in A(\Sigma, G_n)$. Then*

$$\|f - K*f\|_\infty \leq (1 + \|K\|_1)E_n(f)$$

for any $f \in C(G)$.

Proof. Write $f = T + R$ where $\|f - T\|_\infty = \|R\|_\infty = E_n(f)$. Since $K*T = T$, we have

$$f - K*f = (T + R) - (K*T + K*R) = R - K*R.$$

Hence

$$\|f - K*f\|_\infty \leq \|R\|_\infty + \|K\|_1 \|R\|_\infty = (1 + \|K\|_1)E_n(f)$$

and the proposition is proven.

The next result is the analogue of Jackson's theorem [6] which has been studied in the abelian case by Bloom [2, 3].

THEOREM 1. *Let $f \in \text{Lip}_\alpha(G)$, then there is a constant C such that*

$$E_k(f) \leq Cm(G_k)^\alpha$$

for all $k = 0, 1, 2, \dots$.

Proof. Let $P_k(x) = m(G_k)^{-1}1_{G_k}$ where 1_A denotes the indicator function of the set A . It follows that $\hat{P}_k(\sigma) = I_{d_\sigma}$ for $\sigma \in A(\Sigma, G_k)$, and $\hat{P}_k(\sigma) = 0$ for all other $\sigma \in \Sigma$. In particular $\int_G P_k(x)dx = 1$. Since P_k*f has spectrum in $A(\Sigma, G_k)$,

$$\begin{aligned} E_k(f) &\leq \|P_k*f - f\|_\infty \\ &= \sup_{x \in G} \left| \int_G P_k(y)(f(y^{-1}x) - f(x))dy \right| \\ &\leq \sup_{x \in G} m(G_k)^{-1} \int_{G_k} |f(y^{-1}x) - f(x)|dy \\ &\leq m(G_k)^{-1} \int_{G_k} Cm(G_k)^\alpha dy = Cm(G_k)^\alpha \end{aligned}$$

which proves the theorem.

This rate of decay for $E_k(f)$ as k goes to infinity cannot be improved. Consider the function

$$f(x) = \sum_{n=1}^{\infty} m(G_{n-1})^\alpha d_{\sigma_n}^{-1} \chi_{\sigma_n}(x)$$

where $\sigma_n \in A(\Sigma, G_n) \setminus A(\Sigma, G_{n-1})$. It was noted earlier that $f \in \text{Lip}_\alpha(G)$. By using Proposition 1 and observing that $\chi_{\sigma_n}(e) = d_{\sigma_n}$ it follows that

$$\begin{aligned} 2E_k(f) &\geq \|P_k*f - f\|_\infty \\ &= \left\| \sum_{n=k+1}^{\infty} m(G_{n-1})^\alpha d_{\sigma_n}^{-1} \chi_{\sigma_n}(x) \right\|_\infty \\ &= \sum_{n=k+1}^{\infty} m(G_{n-1})^\alpha \geq Cm(G_k)^\alpha. \end{aligned}$$

The next theorem is the analogue of Bernstein's theorem [1] which characterizes the Lipschitz classes in terms of the behavior of the best approximations.

THEOREM 2. *Suppose there is a constant C such that*

$$E_k(f) \leq Cm(G_k)^\alpha$$

for all $k = 0, 1, 2, \dots$. Then $f \in \text{Lip}_\alpha(G)$.

Proof. There exists a sequence $\{T_k\}_{k=0}^\infty$ with spectrum of T_k in $A(\Sigma, G_k)$ which converges uniformly to f . Thus we can write

$$f = T_0 - \sum_{k=1}^{\infty} (T_k - T_{k-1})$$

the series converging uniformly and absolutely. Put $u_k = T_k - T_{k-1}$, and $g = T_0 - f$. As T_0 is a constant, it suffices to show that $g \in \text{Lip}_\alpha(G)$. Suppose $x \in G_k$ and $y \in G$, then

$$|g(xy) - g(y)| \leq \sum_{n=1}^{\infty} |u_n(xy) - u_n(y)|.$$

Since the spectrum of u_n is contained in $A(\Sigma, G_n)$ it follows that $u_n(xy) = u_n(y)$ for $n \leq k$. Hence

$$|g(xy) - g(y)| \leq \sum_{n=k+1}^{\infty} |u_n(xy) - u_n(y)|.$$

But $\|u_n\|_\infty \leq \|f - T_n\|_\infty + \|f - T_{n-1}\|_\infty \leq 2Cm(G_{n-1})^\alpha$ by hypothesis. It follows that

$$|g(xy) - g(y)| \leq 2C \sum_{n=k+1}^{\infty} m(G_{n-1})^\alpha \leq C'm(G_k)^\alpha.$$

For the remainder of this paper we will consider how the order of approximation affects a function's membership in the Fourier algebra $\mathfrak{K}(G)$.

LEMMA 1. *Let H be an open normal subgroup of a compact group G . Then*

$$\sum_{\sigma \in A(\Sigma, H)} d_\sigma^2 = m(H)^{-1}.$$

Proof. Since the dual object of the finite group G/H can (and will) be identified with $A(\Sigma, H)$ the Peter-Weyl theorem says

$$\sum_{\sigma \in A(\Sigma, H)} d_\sigma^2 = \dim l_2(G/H) = \text{card}(G/H) = m(H)^{-1}.$$

The next theorem is the totally disconnected version of another theorem of Bernstein [4].

THEOREM 3. *If*

$$\sum_{n=1}^{\infty} m(G_n)^{-1/2} E_n(f) < \infty$$

then $f \in \mathfrak{R}(G)$.

Proof. Let f satisfy the hypothesis, and denote by K_n the set $A(\Sigma, G_n) \setminus A(\Sigma, G_{n-1})$. Then

$$\begin{aligned} \|f\|_{\mathfrak{R}(G)} &= \sum_{\sigma \in \Sigma} d_{\sigma} \operatorname{tr} |\hat{f}(\sigma)| \\ &= \sum_{n=1}^{\infty} \sum_{\sigma \in K_n} d_{\sigma} |\operatorname{tr} \hat{f}(\sigma)| \\ &\leq \sum_{n=1}^{\infty} \left[\sum_{\sigma \in K_n} d_{\sigma}^2 \right]^{1/2} \left[\sum_{\sigma \in K_n} d_{\sigma} \operatorname{tr} (\hat{f}(\sigma) \hat{f}(\sigma)^{\sim}) \right]^{1/2}. \end{aligned}$$

By Proposition 1

$$\left[\sum_{\sigma \in K_n} d_{\sigma} \operatorname{tr} (\hat{f}(\sigma) \hat{f}(\sigma)^{\sim}) \right]^{1/2} \leq \|P_n * f - f\|_{\infty} \leq 2E_n(f).$$

By Lemma 1

$$\sum_{\sigma \in K_n} d_{\sigma}^2 \leq \sum_{\sigma \in A(\Sigma, G_n)} d_{\sigma}^2 = m(G_n)^{-1}.$$

Applying these last two inequalities to the estimate for $\|f\|_{\mathfrak{R}(G)}$ gives the result.

COROLLARY. $\operatorname{Lip}_{\alpha}(G) \subset \mathfrak{R}(G)$ for $\alpha > 1/2$.

Proof. By Theorem 1 $E_n(f) \leq Cm(G_n)^{\alpha}$ and $m(G_n) \leq 2^{-n}$. This is the totally disconnected version of the classical Bernstein theorem which has been treated for abelian groups by Walker [8, 9].

The summation condition in the previous theorem can be weakened by imposing an additional assumption on the variation of the function. This is analogous to the result of Zygmund [10]. We define the variation of a function f by

$$V(f) = \sup_k \sum_{\sigma \in G|G_k} \sup_{x, y \in \sigma} |f(x) - f(y)|$$

and say that f is of bounded variation if $V(f) < \infty$ and denote the set all such functions by $BV(G)$.

THEOREM 4. *Let f be of bounded variation and satisfy*

$$\sum_{n=1}^{\infty} E_n(f)^{1/2} < \infty .$$

Then $f \in \mathfrak{R}(G)$.

Proof. Put $N = m(G_n)^{-1}$, and let x_1, \dots, x_N be representatives from the cosets of G_n . Then for all $x \in G$ we have

$$\begin{aligned} & \sum_{i=1}^N |P_n * f(x_i x) - f(x_i x)|^2 \\ & \leq \|P_n * f - f\|_{\infty} \sum_{i=1}^N |P_n * f(x_i x) - f(x_i x)| . \end{aligned}$$

But

$$\begin{aligned} P_n * f(x_i x) - f(x_i x) &= \int_G P_n(y) (f(y^{-1} x_i x) - f(x_i x)) dy \\ &\leq \sup_{y \in G_n} |f(y^{-1} x_i x) - f(x_i x)| . \end{aligned}$$

Hence

$$\sum_{i=1}^N |P_n * f(x_i x) - f(x_i x)|^2 \leq E_n(f) V(f) .$$

Integrating both sides of this inequality over G gives

$$N \|P_n * f - f\|_2^2 \leq E_n(f) V(f) .$$

From the proof of Theorem 3 we find that

$$\begin{aligned} \|f\|_{\mathfrak{R}(G)} &\leq \sum_{n=1}^{\infty} (m(G_n)^{-1} - m(G_{n-1})^{-1})^{1/2} \|P_n * f - f\|_2 \\ &\leq \sum_{n=1}^{\infty} (m(G_n)^{-1} - m(G_{n-1})^{-1})^{1/2} m(G_n)^{1/2} (E_n(f) V(f))^{1/2} \end{aligned}$$

which proves the theorem.

COROLLARY. $\text{Lip}_{\alpha}(G) \cap BV(G) \subset \mathfrak{R}(G)$ for all $\alpha > 0$.

The converse of Theorem 3 is not true. There exist functions $f \in \mathfrak{R}(G)$ for which the series in the hypothesis of Theorem 3 is infinite. An example of such a function is

$$f(x) = \sum_{n=1}^{\infty} n^{-2} d_{\sigma_n}^{-1} \chi_{\sigma_n}(x)$$

where $\sigma_n \in A(\Sigma, G_n) \setminus A(\Sigma, G_{n-1})$. By the method in the remark following Theorem 1, $E_n(f) \geq Cn^{-1}$ so that the series in the hypothesis of Theorem 3 is infinite, but $\|f\|_{\mathfrak{R}(G)} = \sum n^{-2} < \infty$.

We next want to show that it is not possible to improve Theorem 3, at least when the sequence $\{m(G_n)\}$ satisfies certain conditions.

For this result we need some lemmas.

LEMMA 2. *Given $\{E_1, \dots, E_k\} \subset \mathcal{E}_2$, there exist complex numbers $\theta_1, \dots, \theta_k$ of modulus 1 such that*

$$\sum_{i=1}^k \|E_i\|_2 \leq 2^{1/2} \left\| \sum_{i=1}^k \theta_i E_i \right\|_1.$$

Proof. Let

$$\mathcal{G} = \prod_{\sigma \in \Sigma} \mathcal{U}(H_\sigma)$$

where $\mathcal{U}(H_\sigma)$ is the group of unitary operators on H_σ . Let

$$F_i(U) = \sum_{\sigma} d_{\sigma} \operatorname{tr}(E_i(\sigma) \pi_{\sigma} U)$$

where π_{σ} is the canonical projection of \mathcal{G} onto $\mathcal{U}(H_\sigma)$. This series converges in $L^2(\mathcal{G})$. By [1, p. 391]

$$\|E_i\|_2 = \left[\int_{\mathcal{G}} |F_i(U)|^2 dU \right]^{1/2} \leq 2^{1/2} \int_{\mathcal{G}} |F_i(U)| dU.$$

Adding these inequalities gives

$$\sum_{i=1}^k \|E_i\|_2 \leq 2^{1/2} \int_{\mathcal{G}} \sum_{i=1}^k |F_i(U)| dU.$$

Hence there exists a $U \in \mathcal{G}$ and complex numbers $\theta_1, \dots, \theta_k$ of modulus 1 so that

$$\begin{aligned} \sum_{i=1}^k \|E_i\|_2 &\leq 2^{1/2} \sum_{i=1}^k |F_i(U)| = 2^{1/2} \sum_{i=1}^k \theta_i F_i(U) \\ &= 2^{1/2} \sum_{\sigma} d_{\sigma} \operatorname{tr} \left[\pi_{\sigma} U \left[\sum_{i=1}^k \theta_i E_i \right] (\sigma) \right] \\ &\leq 2^{1/2} \|U\|_{\infty} \left\| \sum_{i=1}^k \theta_i E_i \right\|_1 \end{aligned}$$

where U is now regarded as an element of \mathcal{E}_{∞} . Since $\|U\|_{\infty} = 1$ the lemma is proven.

LEMMA 3. *Given $n = 0, 1, 2, \dots$, there exists a trigonometric polynomial P with the following properties.*

- (a) $\|P\|_{\infty} = 1$
- (b) $\|P\|_{\mathfrak{R}(\sigma)} \geq 2^{-1/2} m(G_n)^{1/2} / m(G_{n-1})$
- (c) $\hat{P}(\sigma) = 0$ for all $\sigma \notin A(\Sigma, G_n) \setminus A(\Sigma, G_{n-1})$.

Proof. Given k , let 1_k denote the indicator function of G_k . Then $\hat{1}_k(\sigma) = m(G_k) I_{d_{\sigma}}$ for $\sigma \in A(\Sigma, G_k)$ and $\hat{1}_k(\sigma) = 0$ for $\sigma \notin A(\Sigma, G_k)$. Put

$$\psi = (m(G_n)^{-1}\mathbf{1}_n - m(G_{n-1})^{-1}\mathbf{1}_{n-1}) / (m(G_n)^{-1} - m(G_{n-1})^{-1}).$$

Note that ψ satisfies (a) and (c). Let $j = m(G_{n-1})^{-1}$ and let ψ_1, \dots, ψ_j denote the translates of ψ to the cosets of G_{n-1} . From Lemma 2 there exist complex numbers $\theta_1, \dots, \theta_j$ of modulus 1 such that

$$\left\| \sum_{i=1}^j \theta_i \widehat{\psi}_i \right\|_1 \geq 2^{-1/2} \sum_{i=1}^j \|\widehat{\psi}_i\|_2.$$

Put $P = \sum_{i=1}^j \theta_i \widehat{\psi}_i$. Then for each i

$$\|\psi_i\|_2 = \|\psi\|_2 \geq \left[\int_{G_n} 1 dx \right]^{1/2} = m(G_n)^{1/2}.$$

Therefore

$$\|P\|_{\mathfrak{R}(G)} \geq 2^{-1/2} m(G_n)^{1/2} j$$

which establishes (b). That $\|P\|_\infty = 1$ follows from the fact that ψ_i have disjoint supports and that $\|\psi_i\|_\infty = \|\psi\|_\infty = 1$. Finally, (c) follows from the fact that ψ satisfies (c) and the ψ_i are translates of ψ .

LEMMA 4. *Given a sequence $\{b_n\}_{n=1}^\infty$ of real numbers converging to zero, there exists a positive sequence $\{a_n\}_{n=1}^\infty$ with the following properties.*

- (a) $a_n \geq b_n$ for all n
- (b) $\{a_n\}_{n=1}^\infty$ is decreasing and converges to zero
- (c) $(a_n - a_{n+1})/a_n$ is decreasing and converges to zero.

Proof. Without loss of generality assume $0 < b_n \leq 1$ for all n . Let $\beta_n = \log b_n$. Let $N_0 = 1$, let N_1 be so large that $\beta_k \leq -1$ for all $k \geq N_1$. For $n = 2, 3, \dots$ define N_n by taking it so large that $\beta_k \leq -n$ for $k \geq N_n$ and $N_n - N_{n-1} > N_{n-1} - N_{n-2}$. Define for $n = 1, 2, \dots, \alpha_{N_n} = 1 - n$. Define α_1 by $\alpha_1 > (1 - N_1)/(N_1 - N_2)$ and define α_k by linear interpolation for the other indices k . By construction $\{\alpha_n\}_{n=1}^\infty$ is convex and converges to $-\infty$. Let $a_n = \exp(\alpha_n)$, then $\{a_n\}_{n=1}^\infty$ has the required properties.

THEOREM 5. *Suppose that the sequence $\{m(G_n)/m(G_{n+1})\}_{n=0}^\infty$ is bounded. Let $\{b_n\}_{n=1}^\infty$ be a sequence of real numbers which converges to zero. Then there exists a function f such that*

$$\sum_{n=1}^{\infty} m(G_n)^{-1/2} b_n E_n(f) < \infty$$

but $f \notin \mathfrak{R}(G)$.

Proof. Let $\{a_n\}_{n=1}^\infty$ be the sequence from Lemma 4. Put

$$e_n = m(G_n)^{1/2}(a_n - a_{n+1})/a_n .$$

Then

$$(1) \quad \sum_{n=1}^{\infty} m(G_n)^{-1/2} a_n e_n = a_1 < \infty .$$

But

$$(2) \quad \sum_{n=1}^{\infty} m(G_n)^{-1/2} e_n = \infty$$

which can be seen as follows. For arbitrary N , there exists a K such that

$$\begin{aligned} \sum_{n=N}^{N+K} (a_n - a_{n+1})/a_n &> a_N^{-1} \sum_{n=N}^{N+K} (a_n - a_{n+1}) \\ &= 1 - a_{N+K+1}/a_N > 1/2 . \end{aligned}$$

Since both of the sequences $\{m(G_n)^{1/2}\}_{n=0}^{\infty}$ and $\{(a_n - a_{n+1})/a_n\}_{n=1}^{\infty}$ are decreasing and converge to zero, it follows that $\{e_n\}_{n=1}^{\infty}$ is also decreasing and converges to zero. Now put

$$f = \sum_{n=1}^{\infty} (e_{n-1} - e_n) P_n$$

where P_n is from Lemma 3. Then

$$\begin{aligned} E_n(f) &\leq \left\| \sum_{k=n+1}^{\infty} (e_{k-1} - e_k) P_k \right\|_{\infty} \\ &\leq \sum_{k=n+1}^{\infty} (e_{k-1} - e_k) = e_n . \end{aligned}$$

Combining this with (1) proves the first half of the conclusion of the theorem. It remains to show that $f \notin \mathfrak{R}(G)$.

Using the fact that $m(G_n)^{-1/2} \leq 2(m(G_n)^{-1/2} - m(G_{n-1})^{-1/2})$ and summation by parts gives the following:

$$\begin{aligned} &\sum_{n=1}^N m(G_n)^{-1/2} e_n \\ &\leq C \sum_{n=1}^N (m(G_n)^{-1/2} - m(G_{n-1})^{-1/2}) e_{n-1} \\ (3) \quad &= C \left[\sum_{n=1}^{N-1} (m(G_n)^{-1/2} - m(G_0)^{-1/2}) (e_{n-1} - e_n) \right. \\ &\quad \left. + (m(G_N)^{-1/2} - m(G_0)^{-1/2}) e_N \right] \\ &= C \sum_{n=1}^{N-1} m(G_n)^{-1/2} (e_{n-1} - e_n) - e_0 m(G_0)^{-1/2} + m(G_N)^{-1/2} e_N . \end{aligned}$$

Now suppose $f \in \mathfrak{R}(G)$, then by Lemma 3 and the boundedness of

$$m(G_{n-1})/m(G_n)$$

$$\begin{aligned} \|f\|_{\mathbb{R}(G)} &= \sum_{n=1}^{\infty} (e_{n-1} - e_n) \|P_n\|_{\mathbb{R}(G)} \\ &\geq \sum_{n=1}^{\infty} (e_{n-1} - e_n) m(G_n)^{-1/2} C' . \end{aligned}$$

Hence

$$\begin{aligned} m(G_N)^{-1/2} e_N &= m(G_N)^{-1/2} \sum_{n=N+1}^{\infty} (e_{n-1} - e_n) \\ &\leq \sum_{n=N+1}^{\infty} m(G_n)^{-1/2} (e_{n-1} - e_n) \leq \|f\|_{\mathbb{R}(G)} < \infty . \end{aligned}$$

Using these two estimates in (3) gives that

$$\sum_{n=1}^{\infty} m(G_n)^{-1/2} e_n < \infty$$

contrary to (2).

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