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SUFFICIENT CONDITIONS FOR THE SET OF HAUSDORFF COMPACTIFICATIONS TO BE A LATTICE

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Let $K(X)$ be the complete upper semilattice of compactifications of a completely regular Hausdorff space X. We show that if $\beta X\backslash X$ is C^*-embedded in βX and if either $\alpha X\backslash X$ is realcompact or is a P-space for some αX in $K(X)$, then $K(X)$ is a lattice.

1. Introduction. Throughout this paper, all topological spaces under consideration are supposed to be completely regular and Hausdorff, unless stated otherwise.

A compactification of a space X is a compact space αX which contains X as a dense subspace. We say $\alpha_1 X$ and $\alpha_2 X$ are equivalent compactifications of X if there is a homeomorphism h from $\alpha_1 X$ onto $\alpha_2 X$ such that h restricted to X in $\alpha_1 X$ is the identity map onto X in $\alpha_2 X$. We do not distinguish between equivalent compactifications. For compactifications $\alpha_1 X$ and $\alpha_2 X$, we say that $\alpha_1 X \geqq \alpha_2 X$ if and only if there is continuous function from $\alpha_1 X$ onto $\alpha_2 X$ such that h restricted to X is the identity. Thus, $\alpha_1 X$ is equivalent to $\alpha_2 X$ if and only if $\alpha_1 X \geqq \alpha_2 X$ and $\alpha_2 X \geqq \alpha_1 X$. Let $K(X)$ denote the set of all compactifications of X. Then $K(X)$ with the order $\geqq$ defined as above is a complete upper semilattice. Lubben [3] proved that X is locally compact if and only if $K(X)$ is a complete lattice. Next, Shirota [6] showed that if X is first countable then $K(X)$ is a lattice if and only if X is locally compact. Thus, Q (=rationals) provides us with the simplest example for which $K(Q)$ is not a lattice. Visliseni and Flaksmaier [9] showed that if there exists a sequence in $\beta X\backslash X$ which converges to a point in X, then $K(X)$ cannot be a lattice. In the same paper they also constructed a non-locally compact space X for which $K(X)$ is a lattice.

In this paper we determine two classes of spaces which properly contain the class of locally compact spaces and for which $K(X)$ is a lattice, whenever X is a member of either of them. Examples are constructed to show that none of these conditions are necessary.

2. Preliminaries. The terminology of [1] and [11] are used throughout. The following will be needed for subsequent development.

DEFINITION 2.1. Let $\alpha X \in K(X)$, $f_\alpha: \beta X \to \alpha X$ be continuous and $f_\alpha|_X = \mathrm{id}$. Then f_α is closed and hence we can consider αX as the quotient space of βX induced by f_α. Define

$$\mathscr{F}(\alpha X) = \{f_\alpha^{-1}(p) \mid p \in \alpha X \backslash X\} .$$

THEOREM 2.2 (Magill [4]). *Let $\alpha_1 X, \alpha_2 X \in K(X)$. Then $\alpha_1 X \leqq \alpha_2 X$ if and only if each set in $\mathscr{F}(\alpha_2 X)$ is a subset of a set in $\mathscr{F}(\alpha_1 X)$.*

DEFINITION 2.3. A space X is said to be of *countable type* if and only if every compact subset is contained in a compact set of countable character (i.e., one having a countable neighborhood system).

THEOREM 2.4 ([2], page 115). *A space X is of countable type if and only if $\beta X \backslash X$ is Lindelöf.*

THEOREM 2.5 ([1], page 115). *Lindelöf spaces are realcompact.*

DEFINITION 2.6. A space X is of *point countable type* if and only if every point is contained in a compact set of countable character.

THEOREM 2.7 ([8], page 341). *If X is a space of point countable type then $\beta X \backslash X$ is realcompact.*

THEOREM 2.8 ([9], page 1424). *If, in the subspace $\beta X \backslash X$ of the space βX, there exists a countable sequence of points converging to some point in X, then $K(X)$ is not a lattice.*

3. Major results.

LEMMA 3.1 ([10], page 28). $\beta X \backslash X$ is *C^*-embedded in βX if and only if $\mathscr{C}l_{\beta X}(\beta X \backslash X) = \beta(\beta X \backslash X)$.*

DEFINITION 3.2. For $\alpha X \in K(X)$, let $f_\alpha: \beta X \to \alpha X$ be the quotient map, define

$$\mathscr{M}_\alpha = \{p \in \beta X \backslash X \mid |f_\alpha^{-1}(f_\alpha(p))| > 1\} ,$$

and

$$\mathscr{E}_\alpha = \{F \subseteqq \mathscr{M}_\alpha \mid F = f_\alpha^{-1}(y), \text{ some } y \in \alpha X\} .$$

LEMMA 3.3. *If $\mathscr{C}l_{\beta X}(\mathscr{M}_\alpha) \subseteqq \beta X \backslash X$ for every $\alpha X \in K(X)$, then $K(X)$ is a lattice.*

Proof. Since $K(X)$ is a complete upper semi-lattice, it is sufficient to show any two elements of $K(X)$ have a lower bound. Let $\alpha_1X, \alpha_2X \in K(X)$. $A = \mathscr{C}l_{\beta X}(\mathscr{M}_{\alpha_1}) \cup \mathscr{C}l_{\beta X}(\mathscr{M}_{\alpha_2})$ is compact in $\beta X \backslash X$. Obtain αX by identifying A to a point, then αX is a compactification of X. Clearly, each set in $\mathscr{F}(\alpha_i X)$ is a subset of a set in $\mathscr{F}(\alpha X)$ for $i = 1, 2$. By Theorem 2.2, $\alpha X \leqq \alpha_1 X, \alpha_2 X$. Hence, $K(X)$ is a lattice.

LEMMA 3.4 ([1], page 62). *Let $f: X \to Y$ be continuous, A be dense in X. If $f|_A$ is a homeomorphism, then $f(X\backslash A) \subseteqq Y \backslash f(A)$.*

DEFINITION 3.5. Let Y be a quotient space of X with the quotient map P. Let $\{A_i\}_{i=1}^k$ be a collection of disjoint, nonempty subsets in X with $k \geqq 2$. We say $\{A_i\}_{i=1}^k$ is *a section partition induced by P* if and only if there exists $B \subseteqq Y$ such that $P(A_i) = B$ and $P^{-1}(b) \cap A_i$ is a singleton for $1 \leqq i \leqq k$, $b \in B$. P induces a partition on $A = \bigcup_{i=1}^k A_i$; namely, $A = \bigcup_{b \in B} A_b$, $A_{b_1} \cap A_{b_2} = \phi$ if $b_1 \neq b_2$, where $A_b = \bigcup_{i=1}^k (P^{-1}(b) \cap A_i)$. This partition induces the *section correspondence induced by P on A.*

LEMMA 3.6. *If $\beta X \backslash X$ is C^*-embedded in βX then for every $\alpha X \in K(X)$, $\mathscr{M}_\alpha$ contains no copy of N which is C-embedded in $\beta X \backslash X$.*

Proof. Let $\alpha X \in K(X)$ such that $\mathscr{M}_\alpha$ contains a copy of N which is C-embedded in $\beta X\backslash X$. F is compact for each $F \in \mathscr{E}_\alpha$, so it can contain only finitely many points of N. Form A by choosing one point from each nonempty $F \cap N$, then A is infinite. Let $h \in C(\beta X\backslash X)$ such that $h(A) = N \subseteqq \boldsymbol{R}$. $h|_A$ carries A homeomorphically onto a closed set in $\boldsymbol{R}$, so A is C-embedded in $\beta X\backslash X$ by 1.19 of [1]. Therefore, A is a copy of N, which is C-embedded in $\beta X\backslash X$. If $F = f_\alpha^{-1}(f_\alpha(a))$ for some $a \in A$ then since $a \in \mathscr{M}_\alpha$, we have $F \in \mathscr{E}_\alpha$. Let $\mathscr{A} = \{F \in \mathscr{E}_\alpha | F \cap A \neq \phi\}$. Form B by choosing one point from each $F \backslash A$, $F \in \mathscr{A}$. $\{A, B\}$ is a section partition induced by f_α. We want to show that B is closed in $\beta X\backslash X$. Let (b_λ) be an ultranet in B, and $b_\lambda \to b \in (\beta X\backslash X)\backslash B$. Let (a_λ) be the corresponding ultranet in A through the section correspondence induced by f_α on $f_\alpha(A)$. Since βX is compact, $a_\lambda \to a \in \beta X$. Clearly, (a_λ) is nontrivial, since (b_λ) is nontrivial. Also, $a \in X$, since A is closed and discrete in $\beta X\backslash X$. It is known that f_α is continuous, so $f_\alpha(a_\lambda) \to f_\alpha(a)$ and $f_\alpha(b_\lambda) \to f_\alpha(b)$. Since $f_\alpha(a_\lambda) = f_\alpha(b_\lambda)$ for all λ, and the limit points of these nets are unique, it follows that $f_\alpha(a) = f_\alpha(b)$. This is not possible since $f_\alpha(\beta X\backslash X) \subseteqq \alpha X \backslash f_\alpha(X)$ by Lemma 3.4. Thus B is closed in $\beta X\backslash X$. Since A is a C-embedded copy of N and B is a closed set disjoint from A, so A and B are completely separated in $\beta X \backslash X$ by $3B$ of [1]. As $\beta X \backslash X$ is C^*-embedded in βX, therefore A and B are completely separated in βX by 1.17 of [1]. It follows that $\mathscr{C}l_{\beta X}(A) \cap \mathscr{C}l_{\beta X}(B) = \phi$. Choose (a_λ) in A and (b_λ) in B as before, with $a\lambda \to a \in X$, $b_\lambda \to b \in X$. Then $f_\alpha(a) = f_\alpha(b)$. This is a contradiction, since $f_\alpha|_X$ is one-to-one. Hence $\mathscr{M}_\alpha$ contains no copy of $\boldsymbol{N}$, which is C-embedded in $\beta X \backslash X$ for all αX in $K(X)$.

THEOREM 3.7.[1] *If $\beta X \backslash X$ is C^*-embedded in βX, and if $\alpha X \backslash X$ is realcompact for some αX in $K(X)$ then $K(X)$ is a lattice.*

Proof. If $\alpha X \backslash X$ is realcompact for some αX, then $\beta X \backslash X$ is realcompact by 8.13 of [1].

Claim. $\mathscr{C}l_{\beta X}(\mathscr{M}_\alpha) \subseteqq \beta X \backslash X$ for every $\alpha X \in K(X)$. Suppose not, then there exists $\alpha X \in K(X)$ such that $\mathscr{M}_\alpha$ has a limit point $x_0 \in X$. Let $Y = \{x_0\} \cup (\beta X \backslash X)$ endowed with the relative topology as a subspace of βX. $\beta X \backslash X$ is realcompact and dense in Y, so $\beta X \backslash X$ is not C-embedded in Y. Let $f \in C(\beta X \backslash X)$ such that f cannot be extended to Y. Let $[-\infty, \infty]$ be the two-point compactification of $\boldsymbol{R}$. Clearly, f can be considered as a continuous function of $\beta X \backslash X$ into $[-\infty, \infty]$. f has an extension $\bar{f}$ from $\beta(\beta X \backslash X) = \mathscr{C}l_{\beta X}(\beta X \backslash X)$ into $[-\infty, \infty]$. Without loss of generality, we may assume $\bar{f}(x_0) = \infty$. Since $x_0 \in Cl_{\beta X|X}(\mathscr{M}_\alpha)$, so f is unbounded on $\mathscr{M}_\alpha$. By 1.20 of [1], $\mathscr{M}_\alpha$ contains a copy of $\boldsymbol{N}$ which is C-embedded in $\beta X \backslash X$. This contradicts Lemma 3.6, and hence $Cl_{\beta X}(\mathscr{M}_\alpha) \subseteqq \beta X \backslash X$ for every $\alpha X \in K(X)$. Lemma 3.3 shows that $K(X)$ is a lattice.

COROLLARY 3.8. *If X is a space of point countable type and $\beta X \backslash X$ is C^*-embedded in βX then $K(X)$ is a lattice.*

THEOREM 3.9.[1] *If $\beta X \backslash X$ is C^*-embedded in βX and if $\alpha X \backslash X$ is a P-space for some $\alpha X \in K(X)$, then $K(X)$ is a lattice.*

Proof. We claim that $f_\alpha(\mathscr{M}_\alpha)$ is finite. For if $f_\alpha(\mathscr{M}_\alpha)$ is infinite then it contains a countably infinite subset A. By $4K$ of [1], we see that A is a copy of $\boldsymbol{N}$, which is C-embedded in $\alpha X \backslash X$. Let $f \in C(\alpha X \backslash X)$ such that $f(A) = \boldsymbol{N} \subseteqq \boldsymbol{R}$. Hence, $f \circ f_\alpha \in (\beta X \backslash X)$ is unbounded on $f_\alpha^{-1}(A) \subseteqq \mathscr{M}_\alpha$. Thus $f_\alpha^{-1}(A)$ contains a copy of $\boldsymbol{N}$ which is C-embedded in $\beta X \backslash X$. Since $\beta X / X$ is C^*-embedded in βX, this contradicts Lemma 3.6. Therefore, $f_\alpha(\mathscr{M}_\alpha)$ is finite. Let $\gamma X \in K(X)$.

[1] Yusuf Ünlü proved independently in his doctoral thesis [7] that $K(X)$ is a lattice if either (1) $\beta X \backslash X$ is realcompact and C^*-embedded in βX, or (2) $\beta X \backslash X$ is a P-space and $\mathscr{C}l_{\beta X}(\beta X \backslash X)$ is an F-space.

Claim. $f_\alpha(\mathscr{M}_\gamma \backslash \mathscr{M}_\alpha)$ is finite. Suppose $f_\alpha(\mathscr{M}_\gamma \backslash \mathscr{M}_\alpha)$ is infinite then $\mathscr{M}_\gamma / \mathscr{M}_\alpha$ contains a copy of N which is C-embedded in $\beta X \backslash X$. This is a contradiction. $\mathscr{M}_\alpha = \cup \{f_\alpha^{-1}(p) \mid p \in f_\alpha(\mathscr{M}_\alpha)\}$ so that $\mathscr{M}_\alpha$ is a finite union of closed (hence compact) subsets of βX. Thus $\mathscr{M}_\alpha$ is compact. Similarly, $\mathscr{M}_\gamma \subseteqq \cup \{f_\alpha^{-1}(p) \mid p \in f_\alpha(\mathscr{M}_\gamma \backslash \mathscr{M}_\alpha)\} \cup \mathscr{M}_\alpha$ and both of these sets are compact. Therefore, $Cl_{\beta X}(\mathscr{M}_\gamma) \subseteqq \beta X \backslash X$. Since this is for an arbitrary $\gamma X \in K(X)$, the theorem follows from Lemma 3.3.

We summarize the major results of this section in the following theorem:

THEOREM 3.10. *If $\beta X \backslash X$ is C^*-embedded in βX then any of the following conditions implies that $K(X)$ is a lattice*:

(i) *$\alpha X \backslash X$ is realcompact for some $\alpha X \in K(X)$,*

(ii) *$\alpha X \backslash X$ is a P-space for some $\alpha X \in K(X)$,*

(iii) *X is of countable type,*

(iv) *X is of point-countable type.*

Note that the class of spaces X for which $\beta X \backslash X$ is C^*-embedded in βX and for which $\alpha X \backslash X$ is realcompact for some αX in $K(X)$ contains the class of locally compact spaces. ($\beta X \backslash X$ is compact so that it is both realcompact and C^*-embedded in βX.) Likewise, the class of spaces X for which $\beta X \backslash X$ is C^*-embedded in βX and for which $\alpha X \backslash X$ is a P-space for some αX in $K(X)$ contains the class of locally compact spaces. ($\beta X \backslash X$ is C^*-embedded in βX since it is compact and $\omega X \backslash X = \{p\}$ is a P-space.) Thus our results here can be considered as generalizations of those of Lubben [3].

4. **Examples.** Let Ω denote the class of ordinals. For $\alpha \in \Omega$, $W(\alpha) = \{\alpha \in \Omega \mid \sigma < \alpha\}$. ω will denote the smallest member of Ω with infinitely many predecessors: $W(\omega)$ is infinite and for all $\alpha < \omega$, $W(\alpha)$ is finite. ω_1 will denote the smallest member of Ω with uncountably many predecessors.

THEOREM 4.1 ([1], page 138). *If X is compact, with $|X| < \aleph_\alpha$, $\alpha \neq 0$, then $\beta(X \times W(\omega_\alpha)) = X \times W(\omega_\alpha + 1)$.*

Proof. See ([10], page 92).

THEOREM 4.2 ([1], page 89). *$X \subseteqq Y \subseteqq \beta X$, then $\beta Y = \beta X$.*

LEMMA 4.3. *For $\alpha Y \in K(Y)$, there exists X such that $Y = \beta X \backslash X$ and $Cl_{\beta X}(Y) = \alpha Y$.*

Proof. Let $\lambda \neq 0$ be choosen, so that $|\alpha Y| < \aleph_\lambda$. By Theorem

4.1, we have $\beta(\alpha Y \times W(\omega_\lambda)) = \alpha Y \times W(\omega_\lambda + 1)$. Let $X = \beta(\alpha Y \times W(\omega_\lambda))\backslash(Y \times \{\omega_\lambda\})$, then $\alpha Y \times W(\omega_\lambda) \subseteqq X \subseteqq \beta(\alpha Y \times W(\omega_\lambda))$ and hence $\beta X = \beta(\alpha Y \times W(\omega_\lambda)) = \alpha Y \times W(\omega_\lambda + 1)$. Since $\alpha Y \times \{\omega_\lambda\}$ is compact and contains $Y \times \{\omega_\lambda\} = Y$ as a dense subspace, X is the space desired.

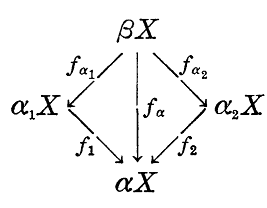
COROLLARY 4.4. *For any space Y there is an X such that $\beta X\backslash X = Y$ and Y is C^*-embedded in X.*

THEOREM 4.5.[2] *Given any two spaces X and Y, there is an $\alpha X \in K(X)$ such that Y is homeomorphic to $Cl_{\alpha X}(\alpha X\backslash X)$ iff there is a continuous map h from $Cl_{\beta X}(\beta X\backslash X)$ onto Y such that $h(\beta X\backslash X) \subseteqq Y\backslash h(R(X))$ and h is one-to-one on $R(X)$, where $R(X)$ is the set of points at which X is not locally compact.*

EXAMPLE 4.6. (1) Let ωN be the one-point compactification of N. Then there exists X such that $\beta X\backslash X = N$ and $Cl_{\beta X}(N) = \omega N$. There exists a sequence, namely N, which converges to $(\omega, \omega_1) \in X$. Thus $K(X)$ is not a lattice by 2.8.

In the above example, $\beta X\backslash X$ is realcompact and a P-space but not C^*-embedded in βX.

EXAMPLE 4.7. (2) If $Y = W(\omega_1)$, then $\beta Y = W(\omega_1 + 1)$. Let $X = (\beta Y \times \beta Y)\backslash(Y \times \{\omega_1\})$, then $\beta X\backslash X = Y$. Let $\mathscr{D}$ be the collection of subsets of βX of the form $\{(\lambda + 2j, \omega_1), (\lambda + 2j + 1, \omega_1)\}$ for λ a limit ordinal, $j = 0, 1, 2, \cdots$, and all other singletons. Then $\mathscr{D}$ is a decomposition space of X. Let $P\colon X \to \mathscr{D}$ be the quotient map, then $\mathscr{D}$ can be considered as the quotient space of X induced by P. Clearly $P(Cl_{\beta X} Y)$ is compact Hausdorff. By 4.5 we have $\mathscr{D} = \alpha_1 X \in K(X)$. Similarly, let $\mathscr{D}'$ be the collection of subsets of βX of the form $\{(\alpha + 2j - 1, \omega_1), (\alpha + 2j, \omega_1)\}$ for α a limit ordinal, $j = 1, 2, \cdots$, and all other singletons, then $\mathscr{D}' = \alpha_2 X \in K(X)$. If $\alpha X \in K(X)$ and $\alpha X \leqq \alpha_1 X, \alpha_2 X$, then the following diagram commutes:



Thus, if $f_\alpha((\lambda, \omega)) = y$, for some λ a limit ordinal then $f((\lambda + j,$

[2] This theorem is a modified version of theorem due to Rayburn [5].

$\omega_1)) = y$ for all $j \in N$. Therefore, $f_\alpha(W \times \{\omega_1\}) = y = f_\alpha((\omega_1, \omega_1))$, which is a contradiction since $f_\alpha(\beta X\backslash X) \subseteqq f_\alpha(\beta X)\backslash f_\alpha(X)$. Hence $K(X)$ is not a lattice.

In this example, the subspace $\beta X\backslash X$ is C^*-embedded but not realcompact nor a P-space. We also claim that $\alpha X\backslash X$ is not a P-space for any $\alpha X \in K(X)$. For if $\alpha X\backslash X$ is a P-space, then $\alpha X\backslash X$ contains a C-embedded copy of N, which implies Y contains a C-embedded copy of N. But this is not possible since Y is pseudocompuct.

EXAMPLE 4.9. (3) Let Y be the subspace of $W(\omega_2)$ obtained by deleting all nonisolated points having a countable base, then Y is a P-space that is not realcompact ([1], page 138).

Let X be chosen so that $\beta X\backslash X = Y$ ahd Y C^*-embedded in βX, then $K(X)$ is a lattice by Theorem 3.9, $\beta X\backslash X$ is not realcompact.

EXAMPLE 4.3. (4) Let Q be the set of rationals. Choose X so that $\beta X\backslash X = Q$ and Q is C^*-embedded in βX. Since Q is realcompact, $K(X)$ is a lattice. We claim that $\alpha X\backslash X$ is not a P-space for any $\alpha X \in K(X)$. For if $\alpha X\backslash X$ is a P-space, then $f_\alpha(\mathscr{M}_\alpha)$ contains a C-embedded copy of N which contradicts Lemma 3.6.

EXAMPLE 4.10. (5) $E = \{2n \mid n \in N\}$ and $0 = \{2n+1 \mid n \in N\}$. Then $N = E \cup 0$ and $E \cap 0 = \phi$. Define $t\colon N \to N$ by $t(2n) = 2n + 1$ and $t(2n + 1)2n$, $n \in N$. Thus, $t(E) = 0$ and $t(0) = E$. For each $p \in \beta N\backslash N$, there exists a unique free ultrafilter $\mathscr{U}_p$ on N such that $\mathscr{U}_p \to p$. Let $\overline{\mathscr{U}} = \{\mathscr{U}_p\}_{p \in \beta N/N}$. It is clear that $\overline{\mathscr{U}}$ is exactly the set of free ultrafilters on N. Define $\overline{\mathscr{U}}_E = \{\mathscr{U}_p \in \overline{\mathscr{U}} \mid E \in \mathscr{U}_p\}$ and $\overline{\mathscr{U}}_0 = \{\mathscr{U}_p \in \overline{\mathscr{U}} \mid 0 \in \mathscr{U}_p\}$. Obviously, $\mathscr{U}_E$ and $\mathscr{U}_0$ form a partition of $\overline{\mathscr{U}}$. If $\overline{\mathscr{U}}_p \in \mathscr{U}_E$, then $t(\mathscr{U}_p)$ the ultrafilter generated by $\{t(u) \mid u \in \mathscr{U}_p\}$ is identical to $\{t(u) \mid u \in \mathscr{U}_p\}$, furthermore, $t(\mathscr{U}_p) \in \overline{\mathscr{U}}_0$. Similarly, $t(\mathscr{U}_p) \in \overline{\mathscr{U}}_E$ if $\mathscr{U}_p \in \overline{\mathscr{U}}_0$. Thus, t induces a one-to-one correspondence between $\overline{\mathscr{U}}_E$ and $\overline{\mathscr{U}}_0$. Each p in $\beta N\backslash N$ corresponds to unique $\mathscr{U}_p$ in $\overline{\mathscr{U}}$, therefore the partition $\overline{\mathscr{U}} = \overline{\mathscr{U}}_E \cup \overline{\mathscr{U}}_0$, $\overline{\mathscr{U}}_E \cap \overline{\mathscr{U}}_0 = \phi$ induces a partition on $\beta N\backslash N$. The induced partition is $\beta N\backslash N = (Cl_{\beta N}(E)\backslash E) \cup (Cl_{\beta N}(0)\backslash 0)$ with $(Cl_{\beta N}(E)\backslash E) \cap (Cl_{\beta N}(0)\backslash 0) = \phi$. Define a relation $\sim$ on βN as follows: $p_1 \sim p_2$ if and only if $p_1 = p_2$ or $t(\mathscr{U}_{p_1}) = \mathscr{U}_{p_2}$. Then $\sim$ is an equivalence relation on βN. Let $\mathscr{D}$ be the identification space $\beta N/\sim$ with the quotient map P. Clearly $\mathscr{D}$ is compact and T_1. We want to show $\mathscr{D}$ is Hausdorff. For $x \in P(N)$, $P^{-1}(x)$ is a singleton in N, so $P^{-1}(x)$ is both open and closed in βN. It follows that $\{x\}$ is both open and closed in $\mathscr{D}$. Thus x can be separated from any other point by open sets in $\mathscr{D}$. Let $p, q \in P(\beta N\backslash N)$. Then

$P^{-1}(p) = \{p_1, p_2\}$, and $P^{-1}(q) = \{q_1, q_2\}$ for $p_1, q_1 \in Cl_{\beta N}(E)\backslash E$ and $p_2, q_2 \in Cl_{\beta N}(0)\backslash 0$. Let u, v be open in βN such that $u, v \subseteqq Cl_{\beta N}(E)$, $p_1 \in u$, $q_1 \in v$ and $u \cap v = \phi$. Let $\bar{t}$ be the extension of t from βN to βN. Obviously, $\bar{t}$ is a homeomorphism, so $\bar{t}(u)$ and $\bar{t}(v)$ are open in βN, moreover $\bar{t}(u), \bar{t}(v) \subseteqq Cl_{\beta N}(0)$ and $p_2 \in \bar{t}(u)$, $q_2 \in \bar{t}(v)$. Let $G = P(u \cup (\bar{t}(u)))$, $H = P(v \cup (\bar{t}(v)))$. Clearly, $P^{-1}(G) = u \cup (\bar{t}(u))$ and $P^{-1}(H) = v \cup (\bar{t}(v))$, so G and H are open in $\mathscr{D}$. Since $p \in G$, $q \in H$, $G \cap H = \phi$, so p, q can be separated by open sets. Thus $\mathscr{D}$ is Hausdorff. Thus there is a $\gamma N \in K(N)$ such that $\gamma N = \mathscr{D}$.

Let X be obtained as in Lemma 4.3 such that $\beta X\backslash X = N$ and $Cl_{\beta X}(N) = \gamma N$. For $\alpha X \in K(X)$, we claim αX has the following properties.

(1) $\mathscr{S}_1^\alpha = \{F \in \mathscr{E}_\alpha \mid |F| \geqq 3\}$ is finite,

(2) $\mathscr{S}_2^\alpha = \{F \in \mathscr{E}_\alpha \mid |F| = 2, F \subseteqq E\}$ and
$\mathscr{S}_3^\alpha = \{F \in \mathscr{E}_\alpha \mid |F| = 2, F \subseteqq E\}$ are finite,

(3) Let $\mathscr{C}_\alpha = \{F \in \mathscr{E}_\alpha \mid |F \cap E| = |F \cap 0| = 1\}$, then
$\mathscr{S}_4^\alpha = \{F \in \mathscr{C}_\alpha \mid F \neq \{2n, 2n+1\}$ for any $n \in N\}$ is finite.

Proof of (1). If $\mathscr{S}_1^\alpha$ is infinite, then $\mathscr{M}_\alpha$ contains three copies of N, say $\{A_i\}_{i=1}^3$, which are C-embedded in $N \subseteqq \beta X$ such that $\{A_i\}_{i=1}^3$ is a section partition induced by f_α. Clearly, $\{f_\gamma^{-1}(A_i)\}_{i=1}^3$ is a section partition induced by $g_\alpha \circ f_\gamma$ where g_α is the restriction of f_α to $Cl_{\beta X}(N) = \gamma N$. Let $(a_\lambda^{(1)})$ be an ultranet in A_1 and $a_\lambda^{(1)} \to a_1 \in \beta N\backslash N$. Let $(a_\lambda^{(2)}) \subseteqq A_2$, $(a_\lambda^{(3)}) \subseteqq A_3$ be ultranets induced by the section correspondences which are induced by $g_\alpha \circ f_\gamma$ on $(g_\alpha \circ f_\gamma)(A_1)$. Let $a_\lambda^{(2)} \to a_2$, $a_\lambda^{(3)} \to a_3$, where $a_2, a_3 \in \beta N\backslash N$. Obviously a_1, a_2, a_3 are distinct. By the definition of γN, $|f_\gamma(\{a_i\}_{i=1}^3)| \geqq 2$. $f_{\gamma|X}$ is one-to-one, so $|(g_\alpha \circ f_\gamma)(\{a_i\}_{i=1}^3)| \geqq 2$. This is not possible, since $(g_\alpha \circ f_\gamma)(a_\lambda^{(1)}) = (g_\alpha \circ f_\gamma)(a_\lambda^{(2)}) = (g_\alpha \circ f_\gamma)(a_\lambda^{(3)})$ for all λ which implies $|(g_\alpha \circ f_\gamma)(\{a_i\}_{i=1}^3)| = 1$. Thus (1) holds.

Proof of (2). It is sufficient to show $\mathscr{S}_2^\alpha$ cannot be infinite. Suppose $\mathscr{S}_2^\alpha$ is infinite, then E contains two copies of N, say A_1 and A_2, which are C-embedded in $N = \beta X\backslash X$ such that $\{A_1, A_2\}$ is a section partition induced by f_α. This is not possible, since no two-points in $Cl_{\beta N}(E)$ are equivalent with respect to $\sim$, and $f_{\gamma|X}$ is one-to-one. Thus (2) holds.

Proof of (3). If $\mathscr{S}_4^\alpha$ is infinite, then there exists $A = \{a_n\}_{n=0}^\infty \subseteqq E$, $B = \{b_n\}_{n=0}^\infty \subseteqq 0$ such that $\{A, B\}$ is a section partition induced by f_α, $\{a_n, b_n\} \in \mathscr{S}_4^\alpha$ for $n \in N$, and $t(A) \cap B = \phi$. Let $a \in Cl_{\beta N}$, then $t(\mathscr{U}_a) \to \bar{t}(a) \notin Cl_{\beta N}(B)$, since $B \notin t(\mathscr{U}_a)$. Let (a_λ) be the ultranet in A based on $A \cap \mathscr{U}_a$ such that $a_\lambda \to a$. Let (b_λ) be the ultranet in B induced by the map $a_n \to b_n$. Then $b_\lambda \to b \in Cl_{\beta N}(B)N$. a and b are not equivalent with respect to $\sim$. Thus $f_\gamma(a) \neq f_\gamma(b)$. However, $(g_\alpha \circ f_\alpha)(a) = (g_\alpha \circ f_\gamma)(b)$. This is a contradiction. Hence (3) holds.

Let $\mathscr{S}_\alpha = \{F \in \mathscr{C}_\alpha | F = \{2n, 2n+1\}$ for some $n \in N\}$, $G_\alpha = \{x \in \mathscr{M}_\alpha | x \in F$ for some $F \in \mathscr{S}_\alpha\}$. Let $K_\alpha = \{x \in \mathscr{M}_\alpha | x \in \bigcup_{i=1}^{4} \mathscr{S}_i^\alpha\}$. Then $\mathscr{M}_\alpha = G_{\alpha_i} \cup K_\alpha$.

Using these notations, for $\alpha_1 X$, $\alpha_2 X \in K(X)$, we write $\mathscr{M}_{\alpha_i} = G_{\alpha_i} \cup K_{\alpha_i}$, $i = 1, 2$. We want to show that $\alpha_1 X$ and $\alpha_2 X$ have a lower bound in $K(X)$. Let τX be obtained by idetifying subsets of βX of the form $\{2n, 2n+1\}$ to a point for each $n \in N$. It is clear that $\tau X \in K(X)$. Let $K = f_\tau(K_{\alpha_1} \cup K_{\alpha_2})$. Obtain αX by identifying K to a point, then $\alpha X \in K(X)$. Each set in $\mathscr{F}(\alpha_i X)$ is a subset of a set in $\mathscr{F}(\alpha X)$, thus $K(X)$ is a lattice by Theorem 2.2.

This example shows that the condition $Cl_{\beta X}(\mathscr{M}_\alpha) \subseteqq \beta X \backslash X$ for every $\alpha X \in K(X)$ in Lemma 3.3 is not necessary for $K(X)$ to be a lattice.

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