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GAUSSIAN NULL SETS AND DIFFERENTIABILITY OF LIPSCHITZ MAP ON BANACH SPACES

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In this note we introduce and briefly study the notion of a Gaussian null set in a real separable Banach space E. As a corollary to recent work of Aronszajn we then show that a locally Lipschitz mapping from E into a Banach space with the Radon-Nikodym property is Gateaux differentiable outside of a Gaussian null set. This is an infinite dimensional generalization of Rademacher's classical theorem that such mappings from R^n to R^m are differentiable almost everywhere (Lebesgue). This approach will be compared with another generalization of Rademacher's theorem due independently to Christensen and Kaier and to Mankiewicz.

In order to prove an extension of Rademacher's theorem, one first needs a generalization of the notion of a set of Lebesgue measure zero. Since we are interested in a question involving continuous functions, we restrict our attention to Borel sets. What we want to define, then, is a class of Borel sets (which will eventually be considered as the class of null sets) which is closed with respect to countable unions and translations. Moreover, we want a Borel subset of a null set to be itself a null set, and we want our class to coincide with the Borel sets of Lebesgue measure zero in finite dimensional spaces. We also require that a nonempty open set *not* be a null set. Now, it is well known that there is no analogue to Lebesgue measure in infinite dimensional spaces; in fact, there does not even exist a positive σ-finite measure on l_2 whose null sets are translation invariant [14, p. 108]. Thus, we cannot simply use the class of null sets of some fixed measure. We *can*, however, use the common null sets of a *family* of measures, namely, the family of nondegenerate Gaussian measures.

DEFINITION. A nondegenerate Gaussian measure μ on the real line R is one having the form

$$(*) \qquad \mu(B) = (2\pi b)^{-1/2}\int_B \exp[-(2b)^{-1}(t-a)^2]dt$$

where B is a Borel subset of R and the constant b is positive. The point $a \in R$ is called the *mean* of μ.

Such measures are obviously mutually absolutely continuous with respect to Lebesgue measure (on the Borel sets) and the product of n such measures is equivalent to Lebesgue measure on R^n. As we will see, any nondegenerate Gaussian measure on E is (essentially) a countable product of such measures. First, we recall the following definition.

DEFINITION. A probability measure λ on the Borel subsets of the real Banach space E is said to be a *nondegenerate Gaussian measure of mean* $x_0 \in E$ if for each $f \in E^*$, $f \neq 0$, the measure $\mu = \lambda \circ f^{-1}$ has the form (*) (above), where $a = f(x_0)$.

The lemma which follows is proved by combining a number of known results, but does not itself appear to be stated explicitly in the literature.

LEMMA 1. *Suppose that μ is a nondegenerate Gaussian measure of mean* 0 *on the separable infinite dimensional Banach space E. Then there exists a sequence $\{e_n\} \subseteqq E$ and a one-to-one continuous linear map $T: E \to R^N$ (the countable product of lines with the product topology) with the following properties*:

(i) *The linear span E_∞ of $\{e_n\}$ is dense in E.*

(ii) *For each n, the image of e_n under T is δ_n, the sequence having* 1 *in the* nth *place,* 0 *elsewhere.*

(iii) *For each Borel subset $B \subseteqq E$, the set TB is a Borel subset of R^N.*

(iv) *The measure $\nu = \mu \circ T^{-1}$ is a countable product of nondegenerate Gaussian measures ν_n on R of mean* 0.

Proof. Kuelbs [8, 9] has shown the following: Let $\{x_n\}$ be dense in E, choose $\{f_n\} \subseteqq E^*$ such that $f_n(x_n) = \|x_n\|$ and $\|f_n\| = 1$ for each n and define

$$(x, y) = \Sigma 2^{-n} f_n(x) f_n(y) , \quad x, y \in E .$$

Then this defines an inner product on E with associated norm $\|x\|_2 = (x, x)^{1/2} \leqq \|x\|$. The completion of $(E, \|\,..\,\|_2)$ is a separable Hilbert space H and the natural embedding $T_1: E \to H$ carries Borel subsets of E into Borel subsets of H. Moreover, the density of T_1E in H implies that $\mu_1 = \mu \circ T_1^{-1}$ is a nondegenerate Gaussian measure on H. From Chapter 1 of Skorohod [14] it follows that there exists an orthonormal basis $\{e_n\}$ for H and a sequence $\{\nu_n\}$ of nondegenerate Gaussian measures on the line with the following properties. First, the natural continuous linear embedding $T_2: H \to R^N$ [defined by $T_2(\Sigma t_n e_n) = (t_n)$] maps Borel subsets of H into Borel subsets of R^N. [The basis $\{e_n\}$ is the sequence of normalized eigenvectors for the strictly positive nuclear correlation operator associated with μ_1.] Second, if $\nu = \Pi\nu_n$, then $\nu = \mu_1 \circ T_2^{-1}$; this makes sense, since R^N is separable and metrizable, so the Borel sets coincide with the usual product space σ-algebra generated by the cylinder sets based on Borel sets. Now, Kuelbs [8] has shown that the sequence $\{e_n\}$ is actually contained in T_1E, i.e., in E. Moreover, he has proved that given $f \in E^*$ there exists for each n a Borel measurable function P_n, defined μ-almost everywhere on E with values in span $\{e_1, \cdots, e_n\}$, such that the sequence $f \circ P_n$ converges μ-almost everywhere to f. Suppose that the span E of $\{e_n\}$ were not dense in E; then there would exist $f \in E^*$, $f \neq 0$, such that $f(E_\infty) = 0$. In particular, this would imply that $f \circ P_n = 0$ a.e. μ and hence that $f = 0$ a.e.μ, contradicting the fact that $\mu \circ f^{-1}$ is nondegenerate on R. This proves part (i). To prove the remaining parts we simply let $T = T_2 \circ T_1$.

DEFINITION. A Borel subset B of the separable Banach space E will be called a *Gaussian null set* if $\mu(B) = 0$ for every nondegenerate Gaussian measure μ on E. The family of all Gaussian null sets will be denoted by $\mathscr{G}$.

PROPOSITION 2. *The family $\mathscr{G}$ of Gaussian null sets has the following properties*:

(i) *The countable union of members of $\mathscr{G}$ is an element of $\mathscr{G}$ and a Borel subset of a member of $\mathscr{G}$ is in $\mathscr{G}$.*

(ii) *For all $B \in \mathscr{G}$ and $x \in E$, the translate $x + B$ is in $\mathscr{G}$.*

(iii) *If $U \subseteq E$ is open and nonempty, then $U \notin \mathscr{G}$.*

(iv) *If $S: E \to E$ is an isomorphism (that is, one-one, linear, continuous and onto), then $S(B) \in \mathscr{G}$ for every $B \in \mathscr{G}$.*

(v) *If E is finite dimensional (hence isomorphic to R^n for some n), then a Borel set B is in $\mathscr{G}$ if and only if B has Lebesgue measure zero.*

Proof. Part (i) is immediate from the definition. Part (ii) follows from the easily verified fact that a translate of a nondegenerate Gaussian measure is again such a measure (not necessarily absolutely continuous with respect to the original [10]). Part (iii) is shown in Corollary 4 below. Part (iv) is a consequence of the observation that if μ is a nondegenerate Gaussian measure, then so is the measure $\mu \circ S$; this fact is immediate from the definition. Finally, part (v) is proved by noting [14] that a Gaussian measure on a finite dimensional space is mutually absolutely continuous with Lebesgue measure.

LEMMA 3. *If the sequence $\{w_n\} \subseteq E$ has dense linear span and* *satisfies* $\|w_n\| \to 0$, *then its symmetric closed convex hull* K *is compact and is not a Gaussian null set.*

Proof. Define $L: l_2 \to E$ by setting, for $x = (x_n) \in l_2$,

$$Lx = \Sigma 2^{-n} x_n w_n .$$

It is clear that L is linear and has dense range. Let U denote the unit ball of l_2. If $x \in U$, then $|x_n| \leqq 1$ for all n, hence $Lx \in K$. Since K is the closed convex hull of the compact set $\{\pm w_n\} \cup \{0\}$, it is compact. In particular, it is bounded, which shows that T is continuous. It follows from the definition that if μ is any nondegenerate Gaussian measure on l_2, then $\lambda = \mu \circ T^{-1}$ is a nondegenerate Gaussian measure on E. Moreover, if $\mu(U) > 0$, then, since $T^{-1}K \supseteq U$, we have $\lambda(K) = \mu(T^{-1}K) \geqq \mu(U) > 0$. It is known (see, e.g., [13]) that any nondegenerate Gaussian measure μ on l_2 assigns positive measure to any nonempty open set, so K is not Gaussian null.

COROLLARY 4. *If* U *is any nonempty open subset of* E, *then* U *is not a Gaussian null set.*

Proof. Since E is separable, it can be expressed as a countable union of translates of U, hence would itself be Gaussian null if U were. Thus, we merely need to show the existence of at least one nondegenerate Gaussian measure on E, and this is a consequence of Lemma 3.

Note that the above result implies that the complement of a Gaussian null set is dense in E.

We next define Aronszajn's class $\mathscr{A}$ of exceptional sets and compare them with the Gaussian null sets.

DEFINITION. Let $\{a_n\} \subset E$ be a sequence of nonzero elements which has dense linear span in E. Define $\mathscr{A}\{a_n\}$ to be the family of all Borel sets of the form $\bigcup_{n=1}^{\infty} A_n$, where each A_n is a Borel set with the property that for each $x \in E$, the set

$$(A_n + x) \cap Ra_n$$

has Lebesgue measure zero in the line Ra_n. Finally, let $\mathscr{A}$ be the intersection of the families $\mathscr{A}\{a_n\}$, over all possible such sequences $\{a_n\}$.

Aronszajn [1] has shown that the family $\mathscr{A}$ has all the properties listed (for the Gaussian null sets) in Proposition 2.

PROPOSITION 5. *If a Borel subset A of E belongs to Aronszajn's class $\mathscr{A}$ of exceptional sets, then it is a Gaussian null set.*

Proof. Since $\mathscr{A}$ is translation invariant it suffices to prove that $\mu(A) = 0$ for each nondegenerate Gaussian measure μ of mean 0 on E. Choose $\{e_n\} \subseteq E$ and $T: E \to R^N$ as in Lemma 1. By hypothesis, it is possible to write $A = \cup A_n$ where each Borel set A_n has the property that $A_n \cap (x + Re_n)$ is Lebesgue null for all $x \in E$. We need only show $\mu(A_n) = 0$ for each n. By Lemma 1, $B = TA_n$ is a Borel subset of R^N and $\nu = \mu \circ T^{-1}$ is a product of one dimensional nondegenerate Gaussian measures ν_n of mean 0. Thus, we want $\nu(B) = 0$. Since T is one-one, we have $T(A_n \cap (x + Re_n)) = B \cap (Tx + R\delta_n)$. The latter has Lebesgue measure zero in the line $R\delta_n$, since T maps Re_n linearly onto $R\delta_n$. Thus, $B \cap (y + R\delta_n)$ is Lebesgue null for all $y \in TE$. Let $g = \chi_B$. By the pointwise Fubini-Jessen theorem [7, p. 209], $\nu(B) = \int g d\nu = \lim_{k\to\infty} g_k(x)$ for ν-almost all $x \in R^N$, where, for $x = (x_n) \in R^N$,

$$g_k(x) = \int \cdots \int g(y_1, \cdots, y_k, x_{k+1}, x_{k+2}, \cdots) d\nu_1(y_1) \cdots d\nu_k(y_k) .$$

Since $\nu(TE) = 1$, we can fix $x \in TE$ such that the above limit exists. It clearly suffices to show that $g_k(x) = 0$ if $k \geqq n$. But for any such k, the integrand appearing in the definition of $g_k(x)$ can be expressed (using Fubini's theorem) as the integral with respect to $d\nu_1 \times \cdots \times d\nu_{n-1} \times d\nu_{n+1} \times \cdots \times d\nu_k$ of

$$\int g(y_1, \cdots, y_n, \cdots, y_k, x_{k+1}, x_{k+2}, \cdots) d\nu_n(y_n) .$$

The integrand in this last integral is the characteristic function of the set

$$B \cap [(y_1, \cdots, y_{n-1}, 0, y_{n+1}, \cdots, y_k, x_{k+1}, x_{k+2}, \cdots) + R\delta_n]$$

which is of the form $B \cap (z + R\delta_n)$, where

$$z = (y_1 - x_1, \cdots, y_{n-1} - x_{n-1}, - x_n, y_{n+1} - x_{n+1}, \cdots, y_k - x_k, 0, 0, 0, \cdots) + x .$$

Clearly, the finitely nonzero sequences are in TE and $x \in TE$, so $z \in TE$ and therefore this set has one dimensional Lebesgue measure zero. It follows that $g_k(x) = 0$ and the proof is complete.

We do not know whether $\mathscr{A}$ is a proper subset of $\mathscr{G}$.

We now give the definitions which are needed to apply the notion of Gaussian null sets to differentiability of Lipschitz mappings.

DEFINITIONS. Suppose that G is a nonempty open subset of the real Banach space X and that $T: G \to Y$ is a mapping from G into the real Banach space Y. We say that T is *Gateaux differentiable* at $x \in G$ if for each $u \in X$ the limit

$$\lim_{t \to 0} \frac{T(x + tu) - T(x)}{t} \tag{d}$$

exists in the norm topology of Y and defines a linear (in u) map which is continuous from X to Y.

We say that T is *locally Lipschitz* if for each $x \in G$ there exist positive constants M and δ such that

$$\|T(y) - T(z)\| \leqq M\|y - z\|$$

whenever $y, z \in G$ and $\|y - x\| < \delta$ and $\|z - x\| < \delta$.

Finally, a real Banach space Y is said to have the *Radon-Nikodym property* (RNP) provided every function of bounded variation from $[0, 1]$ into Y is differentiable almost everywhere.

The name for this class of spaces (which contains separable dual spaces and reflexive spaces) arises from the fact that a Radon-Nikodym theorem is valid for vector measures with values in such spaces. For convenience we have chosen as our definition one of a large number of known characterizations of the RNP; see [5] or [6] for a comprehensive survey.

THEOREM 6. (*Aronszajn*) *Suppose that X is a separable real Banach space, that Y is a real Banach space with the Radon-Nikodým property and that G is a nonempty open subset of X. If $T: G \to Y$ is a locally Lipschitz mapping, then T is Gateaux differentiable at all points x of a Borel subset of G whose complement is in $\mathscr{A}$. In particular, it is Gateaux differentiable outside of a Gaussian null subset of G.*

This is essentially Theorem 1 of Chapter 2 of [1]. In the latter, the space Y was assumed to be a separable dual space or a reflexive space (hence a space with the RNP). Since $T(G)$ is separable, so is its closed linear span, hence one can assume that Y is separable. It is an open question of some standing whether a separable space with the RNP can be embedded in a separable dual space [5, 6]. If the answer is affirmative, then Aronszajn's proof is obviously adequate as it stands. In any event, the hypothesis that the separable space Y be a dual space is only used in two places in Chapter 2 of [1]: In Lemma 1 it is used to show that a Lipschitz map from a real interval into Y is differentiable almost everywhere, and this is immediate from our definition of the *RNP*. In Lemma 2 it is used to produce a dense sequence $\{b_k\}$ in the predual of Y, but the proof is valid if $\{b_k\}$ is merely a total sequence in Y^*. With these changes in [1], then, the above theorem is valid as stated. (We caution the reader that our terminology differs from that of [1]. For instance Aronszajn says T has a "Gateaux differential at x" if the limit in (d) exists for all u and for positive t, with no assumption about linearity or continuity of the resulting map. Moreover, where we say "T is Gateaux differentiable at x", Aronszajn writes "$DT(x)$ is a differential.")

A different class of Borel null sets has been defined by Christensen [2, 4] and used by him and Kaier [3] and (independently) by Mankiewicz [11, 12] to prove differentiability theorems for Lipschitz maps in Fréchet spaces. Christensen calls a Borel subset B of an abelian Polish topological group G a *Haar zero set* if there exists a Borel probability measure μ on G such that $\mu(B + x) = 0$ for all $x \in G$. The family of all such sets is clearly closed under translations. It is less obvious (but true) that it is closed under countable unions. Moreover, it is defined in any separable Fréchet space and agrees with Lebesgue null Borel sets in finite dimensional spaces. In an infinite dimensional space every compact set is a Haar zero set but (by Lemma 3) this is not true of the Gaussian null sets. On the other hand, it is immediate from the definitions that any Gaussian null set is a Haar zero set. Consequently, to say that a map is differentiable outside of a Haar zero set is weaker than saying that it is differentiable outside of an Aronszajn exceptional set.

The following example shows that the Gaussian null sets (and hence Aronszajn's exceptional sets) fail to have a useful property possessed by Lebesgue measurable sets. Recall that any pairwise disjoint family of sets of positive Lebesgue measure is at most countable. Christensen [2] has posed the question as to whether the analogous property is valid for the Haar zero sets; this seems still to be open.[1]

EXAMPLE. There exists an uncountable collection of pairwise disjoint compact subsets of l_2, each of which is not a Gaussian null set.

Proof. For each sequence $s = (s_n)$ such that $s_n = \pm 1$ for each n let

[1] Added in proof. Christensen has informed us that Ryll-Nardjewski has constructed a simple example in the countable product of lines which shows that the answer is negative.

$$K(s) = \{(x_n) \in l_2 \colon |x_n - s_n 2^{-n}| \leqq 4^{-n},\ n = 1, 2, 3, \cdots\}\,.$$

Each $K(s)$ is a compact convex Hilbert cube centered at the point $c(s) = (s_n 2^{-n})$. If $s \neq s'$, then there exists an index m such that $|s_m - s'_m| = 2$. Consequently, if $x \in K(s) \cap K(s')$, then

$$2 \cdot 2^{-m} = |s_m 2^{-m} - s'_m 2^{-m}| \leqq |x_m - s_m 2^{-m}| + |x_m - s'_m 2^{-m}| \leqq 2 \cdot 4^{-m}\,,$$

a contradiction which shows that the sets $K(s)$ form a pairwise disjoint family, which is clearly uncountable. To show that each $K(s)$ is not a Gaussian null set it suffices to show that each translate

$$K(s) - c(s) \equiv \{x \in l_2 \colon |x_n| \leqq 4^{-n}\,,\quad n = 1, 2, 3, \cdots\}$$

is not Gaussian null. If e_n denotes the canonical nth basis vector in l_2, then the sequence $w_n = 4^{-n} e_n$ satisfies the hypothesis of Lemma 3. Since its symmetric closed convex hull is contained in $K(s) - c(s)$, the latter is not a Gaussian null set.

As has been noted in [1], there is no possibility of replacing Gateaux differentiability in Theorem 6 by Fréchet differentiability (where the limit in (d) is assumed to exist uniformly for $\|u\| \leqq 1$). The simplest example seems to be the norm in l_1, which is Gateaux differentiable at each point with all nonzero coordinates, but is nowhere Fréchet differentiable. For a discussion of Gateaux differentiability of continuous convex functions (which are necessarily locally Lipschitz) we refer the reader to [1], where it is shown that for such functions the conclusion to Theorem 6 can be strengthened a bit.

J. Diestel has informed us that a modification of the discussion in [5, p. 107] shows that if E has the property that every Lipschitz map from [1] into E is differentiable a.e., then E has the RNP. Thus, the latter class of spaces is the most general one for which Theorem 6 remains valid. For a discussion of the relationship between differentiability of Lipschitz functions and the isomorphic classification of Banach spaces, see [5, p. 118].

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