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ON THE GEOMETRY OF COMBINATORIAL MANIFOLDS

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On a smooth manifold there are classical relations between vector fields and derivations of the smooth function algebra, and between differential forms and alternating linear maps of vector field tuples. In this paper similar relations are obtained for combinatorial manifolds. As an application of these results the existence of connexions and parallel translation on combinatorial manifolds is established.

0. Introduction. The basic theme of (6) and (7) is that there is a striking similarity between the geometry of smoth manifolds and the geometry of simplicial complexes. The purpose of this paper is to continue this theme for smooth manifolds and combinatorial manifolds. (Note: Henceforth a *combinatorial n-manifold M* is the geometric realization of a simplicial complex for which the closed star of each point can be mapped by a homeomorphism onto a combinatorial n-ball in R^n in such a manner that each simplex of M is mapped affinely to a simplex in R^n. Furthermore all combinatorial manifolds are assumed to have no boundary. See (1) and (9) for related definitions.)

Section 1 is devoted to a brief review of some of the terminology and results of (6) and (7). The goal of §2 is the characterization of continuous vector fields on combinatorial manifolds. The main technical results of this paper are proved in §3; these results are complied in the following statement.

THEOREM. *Let M be a combinatorial n-manifold, $A(M)$ the ring of piecewise smooth real-valued functions on M, $\mathscr{X}(M)$ the $A(M)$-module of continuous vector fields on M, and $E(M)$ the $A(M)$-module of piecewise smooth 1-forms on M. Then*:

(1) *there is an $A(M)$-module isomorphism between $\mathscr{X}(M)$ and the module $\mathscr{D}(M)$ of derivations of $A(M)$; consequently $\mathscr{X}(M)$ is a Lie algebra over R with respect to*

$$[X, Y]f = X(Yf) - Y(Xf)$$

for $X, Y \in \mathscr{X}(M)$ and $f \in A(M)$,

(2) *there is an $A(M)$-module isomorphism between $E(M)$ and the module $\mathrm{Hom}_{A(M)}(\mathscr{X}(M), A(M))$ of $A(M)$-linear maps from $\mathscr{X}(M)$ to $A(M)$,*

(3) *there is an $A(M)$-module isomorphism between $\Lambda^q E(M)$*

and the module $\mathrm{Alt}_{A(M)}(\times^q \mathscr{X}(M), A(M))$ *of* $A(M)$*-linear alternating maps from* $\times^q \mathscr{X}(M)$ *to* $A(M)$*, and*

(4) *if* $d: \Lambda^* E(M) \to \Lambda^* E(M)$ *is the differential of the de Rham complex of* M *and* $\theta \in \Lambda^q E(M)$*, then* $d\theta \in \Lambda^{q+1} E(M)$ *is given by the formula*

$$d\theta(X_1, \cdots, X_{q+1}) = \sum_j (-1)^{j+1} X_j \theta(X_1, \cdots, \hat{X}_j, \cdots, X_{q+1}) + \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_i, \cdots, \hat{X}_1, \cdots, \hat{X}_j, \cdots, X_{q+1})$$

for $X_1, \cdots, X_{q+1} \in \mathscr{X}(M)$.

In §4 these results are used to show that connexions exist on combinatorial manifolds, and that such connexions can be described (or defined equivalently in terms of differential forms (Cartan connexions), vector fields (Koszul connexions), or compatible collections of connexions defined on individual simplices. Finally, such connexions are interpreted geometrically in terms of parallel translation.

The main results of this paper are motivated by analogous smooth results; these smooth analogs are presented in (2) and (8), for example. Results similar to some of those presented in §3 appear in (3); these results, however, are obtained from a much different point of view. The present work is motivated by related work of Professor Howard Osborn (see [4] and [5]); I am deeply indebted to him for all he taught me.

1. Review. In this section, and throughout the sequel, simplicial complexes will be locally finite and finite dimensional. A *simplicial complex* is understood to be a space K together with a fixed triangulation of K by simplices.

Let K be a simplicial complex. A *small open neighborhood* U of $x_0 \in K$ is the intersection of the open star of x_0 with any other open neighborhood of x_0. The intersection U_α of a small open neighborhood U of x_0 with any simplex $\sigma_\alpha \subseteqq K$ which contains x_0 is a *wedge* of U. Each such wedge U_α is affinely homeomorphic to an open subset V_α of

$$s^n = \{x = (x_i) \in R^n \mid x_i \geqq 0 \forall i\} ,$$

where $n = n_\alpha = \dim \sigma_\alpha$; a map $\psi_\alpha: U_\alpha \to V_\alpha \subseteqq s^n$ establishing this homeomorphism is a *simplicial chart*. A *small open subset* $U \subseteqq K$ is a subset which is a small open neighborhood of some point $x_0 \in K$. The set of small open subsets of K forms a basis for the topology of K.

A *piecewise smooth* real-valued function f defined on a small open subset U is a continuous function $f: U \to R$ whose restriction to each wedge U_α of U is smooth. If U is a small open neighborhood of x_0, a *coordinate system* on U with origin x_0 is a collection $u = \{u_i: U \to R\}$ of piecewise smooth real-valued functions defined on U such that for each wedge U_α of U and any simplicial chart $\psi_\alpha: U_\alpha \to V_\alpha$,

$$\{u_i \circ \psi_\alpha^{-1}: V_\alpha \longrightarrow R \mid u_i \in u \quad \text{and} \quad u_i|_{U_\alpha} \neq 0\}$$

is a coordinate system on V_α with origin $\psi_\alpha(x_0)$. For each $u_i \in u$ we let St u_i denote the support of u_i.

The rules for change of coordinates are:

(1) If u is a coordinate system on U with origin x_0 and $W \subseteq U$ is a small open neighborhood of $x_1 \in U$, then there is an induced coordinate system on W with origin x_1 given by

$$w = \{u_i - u_i(x_1): W \longrightarrow R \mid u_i \in u \quad \text{and} \quad u_i|_W \neq 0\}$$

where $u_i(x_1)$ is a constant function.

(2) If u and w are coordinate systems on U with origin x_0 then there are real constants c_{ij} for which $u_i = \sum_j c_{ij} w_j$ for each $u_i \in u$ where $w_j \in w$. In fact if the first p functions of both u and w coordinatize the p-simplex in whose interior x_0 is contained, then the matrix (c_{ij}) is of the form

$$(c_{ij}) = \left(\begin{array}{c|c} \begin{matrix}\text{invertible} \\ p \times p \text{ matrix}\end{matrix} & 0 \\ \hline * & \begin{matrix}\text{invertible} \\ \text{diagonal} \\ \text{matrix}\end{matrix} \end{array}\right).$$

There is a category, called the category of *simplicial bundles*, and a functor defined on the category of simplicial complexes with values in the category of simplicial bundles, called the *tangent functor*, which associates to each simplicial complex K a tangent bundle $\tau(K): TK \to K$.

For example, the tangent bundle $\tau(K): TK \to K$ of the simplicial complex K obtained by pasting two 1-simplices together at a vertex is illustrated in Diagram 1. Observe that in this example fibers are vector spaces, but $\tau(K)$ is not a vector bundle since fiber dimensions may vary from point to point. This bundle is the canonical example of a simplicial bundle, and the property of varying fiber dimensions is characteristic of simplicial bundles.

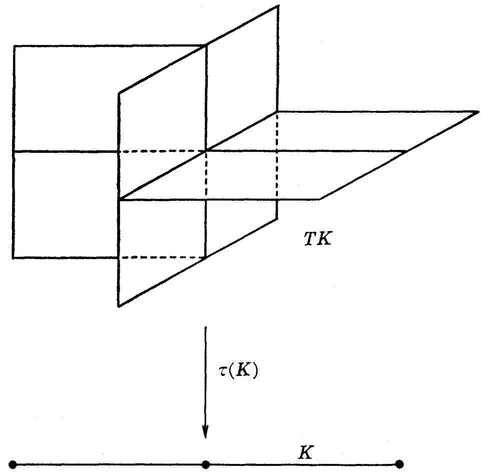


DIAGRAM 1

Tangent bundles are constructed as follows:

First let $U \subseteq K$ be a small open subset and u a coordinate system on U with origin x_0. Let $V(u)$ be the real vector space with basis $\{(\partial/\partial u_i)_{x_0} \mid u_i \in u\}$, and let $T(U; u)$ be the subspace of $U \times V(u)$ consisting of all $(x, \sum_i c_i(\partial/\partial u_i)_{x_0})$ such that c_i is nonzero only if there is a wedge $U_\alpha \subseteq U$ for which $x \in U_\alpha$ and $u_i|_{U_\alpha}$ is a coordinate function on U_α. Projection onto the first factor gives a map $\tau(U; u)$: $T(U; u) \to U$.

Next let $TK = \amalg\, T(U; u)/\sim$, the free union of $T(U; u)$ over all possible combinations of U and u modulo the equivalence relation $\sim$: For $(x_0, v_0) \in T(U_0; u_0)$ and $(x_1, v_1) \in T(U_1; u_1)$, $(x_0, v_0) \sim (x_1, v_1)$ iff $x_0 = x_1$ and, after inducing coordinate systems w_0 and w_1 on a small open neighborhood W of $x_0 = x_1$, $v_1 = \sum_j c_j(\partial/\partial w_j)$ for $w_j \in w_1 \Rightarrow v_0 = \sum_{ij} c_j c_{ij}(\partial/\partial w_i)$ for $w_i \in w_0$ where (c_{ij}) is the transition matrix from w_0 to w_1. There is a map $\tau(K)$: $TK \to K$ induced by the maps $\tau(U; u)$, and this is the tangent bundle of K.

2. Continuous vector fields. A vector field on the simplicial complex K is a function (not necessarily a continuous function) $X: K \to TK$ for which $\tau(K) \circ X = \mathrm{id}_K$ and which satisfies a certain piecewise smoothness condition (see [7]). We are now interested in studying continuous vector fields $X: K \to TK$. Under the assumption of continuity, the piecewise smoothness of a vector field X may be described as follows.

DEFINITION 2.1. A *continuous vector field* on the simplicial complex K is a (continuous) map $X: K \to TK$ for which $\tau(K) \circ X = \mathrm{id}_K$ and such that for each $x_0 \in K$, for each small open neighborhood U of x_0, and for each coordinate system u on U with origin x_0,

$$X: U \longrightarrow T(U; u) \subseteqq U \times V(u)$$

may be written in the form

$$X(x) = \left(x, \sum_i f_i(x)(\partial/\partial u_i)_{x_0}\right)$$

where each $f_i: U \to R$ is piecewise smooth.

Observe that each f_i in Definition 2.1 is necessarily identically zero off St u_i because of the definition of $T(U; u)$. Thus if X satisfies the condition of Definition 2.1 over the small open neighborhood U of x_0 with respect to the coordinate system u, then X automatically satisfies the condition over any small open neighborhood $W \subseteqq U$ of $x_1 \in U$ with respect to the coordinate system on W with origin x_1 induced by u. Furthermore if X satisfies the condition of Definition 2.1 with respect to the coordinate system u, then X automatically satisfies the condition with respect to any other coordinate system on U with origin x_0. Consequently, if U is any small open subset of K, then a continuous vector field

$$X: U \longrightarrow TU = \amalg T(U; u)/\sim .$$

Furthermore X may be written with respect to a coordinate system u on U in the form $X = \sum_i f_i(\partial/\partial u_i)$, the value of X at $x \in U$ being given by $X(x) = \sum_i f_i(x)(\partial/\partial u_i)_x$.

The set $\mathscr{X}(U)$ of continuous vector fields on a small open subset U of a simplicial complex K clearly forms a module over the ring $A(U)$ of piecewise smooth real-valued functions on U with respect to pointwise operations. The set $\mathscr{X}(K)$ of globally defined continuous vector fields on K similarly forms a module over the ring $A(K)$ of piecewise smooth real-valued functions on K.

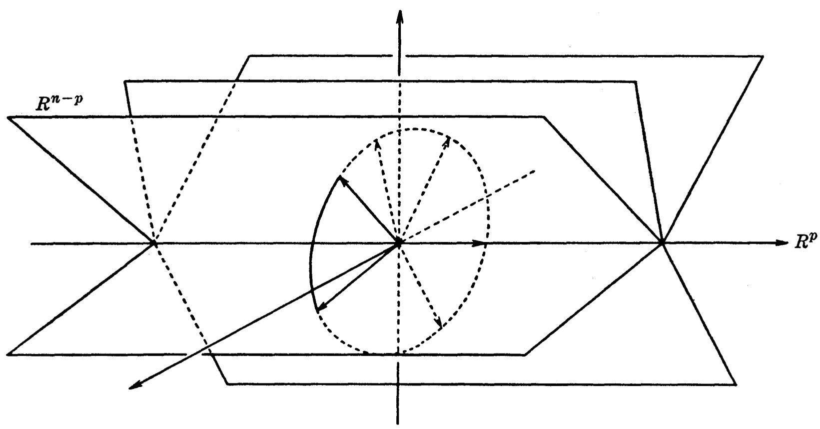
We now focus on combinatorial manifolds. The reason for this is that we may use the following result.

PROPOSITION 2.2. *Let M be a combinatorial n-manifold. Let $x_0 \in M$, let U be a small open neighborhood of x_0, and let $u = \{u_1, \cdots, u_N\}$ be a coordinate system on U with origin x_0 for which $\{u_1, \cdots, u_p\}$ coordinatizes $U_\alpha = \sigma_\alpha \cap U$ where x_0 is contained in the interior of the p-simplex $\sigma_\alpha \subseteqq M$. Then the support* St u_i *of u_i is U if and only if $i = 1, \cdots, p$.*

(*Note*: The set $\{u_1, \cdots, u_p\}$ may be empty; this is the case, in fact, iff x_0 is a vertex. In the event that x_0 is a vertex, Proposition 2.2 states, in particular, that St $u_i \neq U$ for every i.)

Proof. The "if" part is clear; it is, in fact, true for simplicial complexes in general.

For the "only if" part, using the standard metric on R^n we write $R^n = R^p \oplus R^{n-p}$. Let $h: U \to V \subseteq R^n$ be a piecewise linear chart on M for which the image of each wedge of U is a wedge of V and for which $h(U_\alpha) \subseteq R^p$. Without loss of generality we may assume that for $i = p + 1, \cdots, N$, $h_*(\partial/\partial u_i)_{x_0}$ is in $T_{h(x_0)}R^{n-p}$ and has unit length. (See Diagram 2.)



In this case $n = 3$, $p = 1$, and $N = 6$.
DIAGRAM 2

If S^{n-p-1} is the unit sphere in R^{n-p}, then the triangulation of M determines a triangulation of S^{n-p-1} with respect to which S^{n-p-1} is a combinatorial manifold: The vertices v_i of the triangulation are the endpoints of the vectors $h_*(\partial/\partial u_i)_{x_0}$, for $i = p + 1, \cdots, N$, and $(v_{i_1}, \cdots, v_{i_k})$ is a k-simplex of the triangulation iff

$$\{u_1, \cdots, u_p, u_{i_1}, \cdots, u_{i_k}\}$$

coordinatizes a $(p + k)$-wedge of U. To verify that this is a triangulation of S^{n-p-1} one must verify that the intersection of any two simplices is a simplex, and that S^{n-p-1} is covered by simplices. The first assertion is obvious. The second assertion follows since for any $y \in S^{n-p-1}$, the half line

$$\{h(x_0) + t(y - h(x_0)) \mid t \in R^+\}$$

must intersect V in some point y' (since V is an open neighborhood of $h(x_0)$); if $y' \in h(U_\beta)$ for U_β a wedge of U, and U_β is coordinatized by

$$\{u_1, \cdots, u_p, u_{i_1}, \cdots, u_{i_k}\}$$

then $y \in (v_{i_1}, \cdots, v_{i_k})$.

Now if $i = p + 1, \cdots, N$ and $\operatorname{St} u_i = U$, then the closed star of v_i is S^{n-p-1}. But this is impossible since the closed star of every vertex of (the combinatorial manifold)S^{n-p-1} has boundary, and S^{n-p-1} does not.

The main results of this paper are stated for combinatorial manifolds. There are simplicial complexes other than combinatorial manifolds for which many of the following results are still valid, however: One could just as well work with any simplicial complex for which the conclusion of Proposition 2.2 holds. An example of such a simplicial complex (which is not a combinatorial manifold) is pictured in Diagram 3: There are five vertices and six 1-simplices.

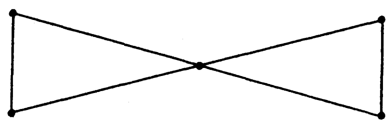
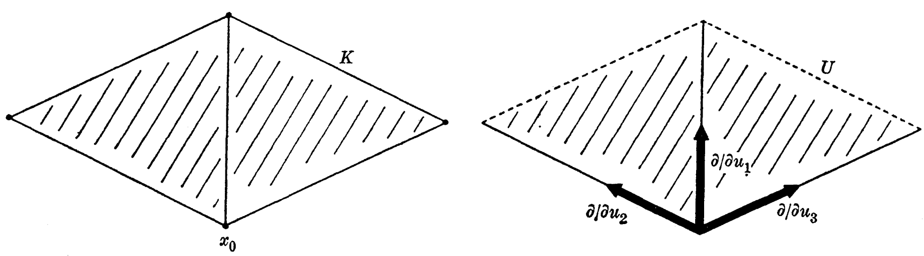


DIAGRAM 3

Recall (see Introduction) that all combinatorial manifolds in this paper are assumed to have no boundary. Observe (see Diagram 4) that Proposition 2.2 is no longer valid if one considers combinatorial manifolds with boundary: In this case U is the open star of the vertex x_0 in K, there are *no* coordinate functions which coordinatize $U_\alpha = \{x_0\}$, and the support $\mathrm{St}\, u_1$ of the coordinate function u_1 is U (see the Note following the statement of Proposition 2.2). Even so, many later results can still be proved for combinatorial manifolds with boundary (and in fact for simplicial complexes in general) by using appropriate modifications of techniques presented here.



(a) (b)

DIAGRAM 4

The following result characterizes continuous vector fields on combinatorial manifolds locally.

PROPOSITION 2.3. *Let M be a combinatorial manifold. Let $U \subseteq M$ be a small convex open neighborhood of x_0, and let $u = \{u_1, \cdots, u_N\}$ be a coordinate system on U with origin x_0 for which $\{u_1, \cdots, u_p\}$ coordinatizes $U_\alpha = \sigma_\alpha \cap U$ where x_0 is contained in the interior of* *the p-simplex $\sigma_\alpha \subseteqq M$. Then $X: U \to TU$ is a continuous vector field iff X is of the form*

$$X = \sum_{i=1}^{p} f_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial/\partial u_i)$$

where $f_i \in A(\mathrm{St}\, u_i) = A(U)$ for $i = 1, \cdots, p$, and $g_i \in A(\mathrm{St}\, u_i)$ for $i = p + 1, \cdots, N$, and $\mathrm{St}\, u_i$ again denotes the support of u_i.

Note. In this proposition, and frequently throughout the sequel, for $g \in A(\mathrm{St}\, u_i)$, $i = p + 1, \cdots, N$, we consider $u_i g$ as an element of $A(U)$ by $(u_i g)x = u_i(x)g(x)$ if $x \in \mathrm{St}\, u_i$, and $(u_i g)x = 0$ if $x \notin \mathrm{St}\, u_i$.

Before proving Proposition 2.3, let us first recall (see [6]) that one can think of elements f of the ring $A(U)$ of piecewise smooth real-valued functions on U as compatible tuples $(f_\alpha) \in \times_\alpha A(U_\alpha)$ of smooth real-valued functions $f_\alpha \in A(U_\alpha)$ defined on the wedges $U_\alpha \subseteqq U$; here "compatible" means that if U_α and U_β are wedges of U then $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$. If functions are expressed in this manner then for $f = (f_\alpha)$ and $g = (g_\alpha)$ in $A(U)$, $f + g = (f_\alpha + g_\alpha)$ and $f \cdot g = (f_\alpha \cdot g_\alpha)$.

Also recall that for each $u_i \in u$ there is a derivation $\partial/\partial u_i$: $A(U) \to A(\mathrm{St}\, u_i)$ defined by a compatible collection of derivations

$$\{\partial/\partial u_\alpha^i\colon A(U_\alpha) \longrightarrow A(U_\alpha)\}$$

defined for the wedges $U_\alpha \subseteqq U$ for which $u_i|_{U_\alpha} = u_\alpha^i$ is a coordinate function on U_α; here "compatible" means that if U_α and U_β are wedges of U for which $\partial/\partial u_\alpha^i$ and $\partial/\partial u_\beta^i$ are defined, then

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta} \Longrightarrow \frac{\partial f_\alpha}{\partial u_\alpha^i}\bigg|_{U_\alpha \cap U_\beta} = \frac{\partial f_\beta}{\partial u_\beta^i}\bigg|_{U_\alpha \cap U_\beta} .$$

Observe that if $i = 1, \cdots, p$ then $\mathrm{St}\, u_i = U$ so that $\partial/\partial u_i$: $A(U) \to A(U)$.

Proof (of Proposition 2.3). If $X = \sum_i f_i(\partial/\partial u_i)$, then $f_i\colon U \to R$ is identically zero off $\mathrm{St}\, u_i$ for each $u_i \in u$. Thus Proposition 2.2 implies that for $i = p + 1, \cdots, N, f_i(x) = 0$ if $u_i(x) = 0$. Now by working wedgewise and then checking for compatibility, we find that the following calculation makes sense: If we consider f_i as a function of one variable, namely u_i (i.e., holding $u_1, \cdots, \hat{u}_i, \cdots, u_N$ fixed), we have

$$D_i f_i(u_1, \cdots, u_N) = D_i \int_{t=0}^{t=u_i} D_i f_i(u_1, \cdots, t, \cdots, u_N) dt ,$$

where D_i is differentiation with respect to the ith variable, so that

$$f_i(u_1, \cdots, u_N) + C = \int_{t=0}^{t=u_i} D_i f_i(u_1, \cdots, t, \cdots, u_N) dt$$

on $\mathrm{St}\, u_i$. Evaluating at $u_i = 0$ we find that $C = 0$ so that after a change of variable (namely $t = u_i s$),

$$f_i(u_1, \cdots, u_N) = u_i \int_{s=0}^{s=1} D_i f_i(u_1, \cdots, s u_i, \cdots, u_N) ds .$$

Before stating the following corollary of Proposition 2.3, let us first recall (see [6]) that for $i = p + 1, \cdots, N$, any $f \in A(\mathrm{St}\, u_i)$ can be extended to an $\bar{f} \in A(U)$; such extensions are not unique.

COROLLARY 2.4. *The $A(U)$-module $\mathscr{X}(U)$ is finitely generated by $\partial/\partial u_i$ for $i = 1, \cdots, p$, and $u_i(\partial/\partial u_i)$ for $i = p + 1, \cdots, N$. In fact for f in $A(U)$,*

$$X = \sum_{i=1}^{p} f_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial/\partial u_i)$$

and

$$Y = \sum_{i=1}^{p} h_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i k_i(\partial/\partial u_i)$$

in $\mathscr{X}(U)$ we have

$$f \cdot X = \sum_{i=1}^{p} f f_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i(f|_{\mathrm{St} u_i}) g_i(\partial/\partial u_i) ,$$

and

$$X + Y = \sum_{i=1}^{p} (f_i + h_i)(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i(g_i + k_i)(\partial/\partial u_i) .$$

Observe that $\mathscr{X}(U)$ is not, in general, free over $A(U)$ since for $i = p + 1, \cdots, N$, $f', f'' \in A(U)$ may agree on $\mathrm{St}\, u_i$, so that

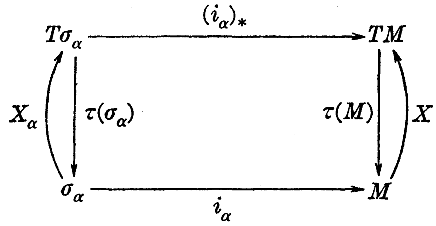
$$\begin{aligned} f' \cdot (u_i(\partial/\partial u_i)) &= u_i(f'|_{\mathrm{St} u_i})(\partial/\partial u_i) \\ &= u_i(f''|_{\mathrm{St} u_i})(\partial/\partial u_i) = f'' \cdot (u_i(\partial/\partial u_i)) , \end{aligned}$$

although f' and f'' clearly need not agree on all of U.

Continuous vector fields on combinatorial manifolds "lie along simplices" in the following sense.

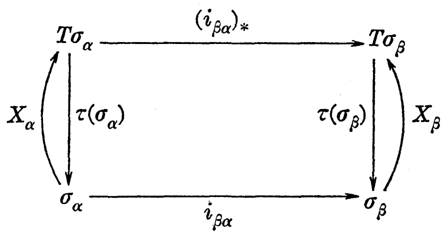
COROLLARY 2.5. *Let M be a combinatorial manifold. If $X \in \mathscr{X}(M)$, then*

(1) *for every simplex $\sigma_\alpha \subseteq M$ there is a smooth vector field $X_\alpha: \sigma_\alpha \to T\sigma_\alpha$ defined on σ_α for which the following diagram commutes*



where i_α denotes inclusion and $(i_\alpha)_*$ denotes the Jacobian of i_α, and

(2) *for simplices $\sigma_\alpha, \sigma_\beta \subseteqq M$ for which $\sigma_\alpha \subseteqq \sigma_\beta$, the following diagram commutes naturally*



where $i_{\beta\alpha}$ denotes inclusion and $(i_{\beta\alpha})^*$ denotes the Jacobian of $i_{\beta\alpha}$.

Conversely, given any collection $\{X_\alpha: \sigma_\alpha \rightarrow T\sigma_\alpha\}$ of vector fields defined on the simplices $\sigma_\alpha \subseteqq M$ which satisfies Condition 2 above, there is a unique continuous vector field $X: M \rightarrow TM$ on M which induces $X_\alpha: \sigma_\alpha \rightarrow T\sigma_\alpha$ on each simplex $\sigma_\alpha \subseteqq M$ and for which Condition 1 holds.

In particular, a continuous vector field on a combinatorial manifold M has a zero at every vertex of M.

To state the next corollary, let us first recall (see [7]) that a piecewise smooth flow F on a simplicial complex K is a piecewise smooth right action $F: K \times R \rightarrow K$ of the additive group of reals on K. Also recall that there is a distinguished type of vector field on a simplicial complex, namely the integrable vector fields, and there is a correspondence between integrable vector fields on K and piecewise smooth flows on K.

COROLLARY 2.6. *Let M be a combinatorial manifold. Then every continuous vector field on M is integrable. Furthermore there is a bijection between continuous vector fields on M and piecewise smooth flows $F: M \times R \rightarrow M$ for which $F: \sigma_\alpha \times R \rightarrow \sigma_\alpha$ for each simplex $\sigma_\alpha \subseteqq M$.*

Proof. This is an immediate consequence of Corollary 2.5 and the following lemma.

LEMMA 2.7. *Let U and W be open neighborhoods of*

$$0 \in s^{n,p} = \{x = (x_i) \in R^n \mid x_i \geqq 0 \quad for \quad i = p+1, \cdots, N\}$$

for which the closure of W is contained in U. Then for every smooth vector field

$$X = \sum_{i=1}^{p} f_i(\partial/\partial x_i) + \sum_{i=p+1}^{N} x_i g_i(\partial/\partial x_i)$$

on U there is a unique smooth flow $F = (F_i): W \times I \to U$ for which

$$\left.\frac{\partial F_i}{\partial t}\right|_{t=0} = \begin{cases} f_i & \text{for} \quad i = 1, \cdots, p \\ x_i g_i & \text{for} \quad i = p+1, \cdots, N \end{cases}$$

and $F: (s_\alpha \cap W) \times I \to s_\alpha \cap U$ for each face s_α of $s^{n,p}$.

This completes the proof of Corollary 2.6.

3. The main results.

We will now prove the main results of this paper.

THEOREM 3.1. *Let M be a combinatorial manifold. There is an $A(M)$-module isomorphism between the module $\mathscr{X}(M)$ of continuous vector fields on M and the module $\mathscr{D}(M)$ of derivations of $A(M)$.*

Proof. It suffices to prove this result locally, so let U be a small convex open neighborhood of $x_0 \in M$, and let $u = \{u_1, \cdots, u_N\}$ be a coordinate system on U for which $\{u_1, \cdots, u_p\}$ coordinatizes $U_\alpha = \sigma_\alpha \cap U$ where x_0 is contained in the interior of the p-simplex $\sigma_\alpha \subseteqq M$. We define

$$F: \mathscr{X}(U) \longrightarrow \mathscr{D}(U)$$

by associating to the vector field (see Proposition 2.3)

$$X = \sum_{i=1}^{p} f_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial/\partial u_i)$$

the derivation

$$F(X): f \longmapsto \sum_{i=1}^{p} f_i(\partial f/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial f/\partial u_i)\,.$$

This makes sense: Since f_i and $\partial f/\partial u_i$ are in $A(U)$ for $i = 1, \cdots, p$, $f_i(\partial f/\partial u_i) \in A(U)$. Since g_i and $\partial f/\partial u_i$ are in $A(\mathrm{St}\, u_i)$ for $i = p+1, \cdots, N$, $g_i(\partial f/\partial u_i) \in A(\mathrm{St}\, u_i)$; thus $u_i g_i(\partial f/\partial u_i) \in A(U)$.

Clearly $F: \mathscr{X}(U) \to \mathscr{D}(U)$ is an $A(U)$-module homomorphism. Moreover it is injective: If $X \in \mathscr{X}(U)$ is a continuous vector field as above and $F(X) = 0$, then $(F(X))u_i = 0$ for every i. If $i = 1, \cdots, p$, this means that $f_i = 0$ on U, and if $i = p + 1, \cdots, N$, this means that $u_i g_i = 0$ on U. Thus $X = 0$.

Finally we show that $F: \mathscr{X}(U) \to \mathscr{D}(U)$ is surjective as follows: Given a derivation $D: A(U) \to A(U)$, let $f_i = Du_i$ for $i = 1, \cdots, N$. Then since D is a derivation it is easy to see that for $i = p + 1, \cdots, N$, $f_i(x) = 0$ for $x \notin \mathrm{St}\, u_i$. Consequently $f_i = u_i g_i$, for $i = p + 1, \cdots, N$, where $g_i \in A(\mathrm{St}\, u_i)$ (see the proof of Proposition 2.3). The surjectivity of F will follow from showing that

$$D = \sum_{i=1}^{p} f_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial/\partial u_i)$$

so that $D = F(X)$ where

$$X = \sum_{i=1}^{p} f_i(\partial/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial/\partial u_i) .$$

To show this we need the following lemma.

LEMMA 3.2. *Let W be a small open neighborhood of x_1 which is star shaped with respect to x_1, and let w be a coordinate system on W with origin x_1. Then for every $f \in A(W)$ and $w_i \in w$ there is a piecewise smooth function $\varphi_i \in A(\mathrm{St}\, w_i)$ such that $\varphi_i(x_1) = (\partial f/\partial w_i)x_1$ and $f = f(x_1) + \sum_i w_i \varphi_i$ on W (where $f(x_1)$ is a constant function on W).*

This lemma is proved by first verifying a similar result on each wedge of W (see (2) for the classical smooth analog), and then verifying that these similar results are compatible with respect to restriction.

Now to finish the proof of surjectivity (again compare with (2)), let $x_1 \in U$, let $W \subseteqq U$ be a small star shaped open neighborhood of x_1, and let

$$w = \{u_i - u_i(x_1): W \to R \mid u_i \in u \quad \text{and} \quad u_i|_W \neq 0\}$$

be the coordinate system on W with origin x_1 induced by u. Observe that for $i = 1, \cdots, p$, $w_i = u_i - u_i(x_1)$ is in w.

We let I denote the index set of w for convenience.

Given any $f \in A(U)$ we apply Lemma 3.2 to $f|_W$ and observe that for each $i \in I$, $(\partial f/\partial w_i)x_1 = (\partial f/\partial u_i)x_1$ to obtain

$$\begin{aligned}(Df)x_1 &= (D(f|_W))x_1 \\ &= \Big(D\Big(f(x_1) + \sum_{i \in I} w_i \varphi_i\Big)\Big)x_1 \\ &= \sum_{i \in I} D(w_i \bar{\varphi}_i)x_1 \\ &= \sum_{i \in I} ((Dw_i)\bar{\varphi}_i + w_i(D\bar{\varphi}_i))x_1\end{aligned}$$

where $\bar{\varphi}_i = \varphi_i$ if $\mathrm{St}\, w_i = W$, and where $\bar{\varphi}_i \in A(W)$ is an extension of $\varphi_i \in A(\mathrm{St}\, w_i)$ otherwise. Thus,

$$\begin{aligned}(Df)x_1 &= \sum_{i\in I}((D(u_i - u_i(x_1)))\varphi_i)x_1\\ &= \sum_{i\in I}((Du_i)(\partial f/\partial w_i))x_1\\ &= \sum_{i\in I}(f_i(\partial f/\partial u_1))x_1\\ &= \Big(\sum_{i=1}^{p} f_i(\partial f/\partial u_i) + \sum_{i\in I-\{1,\cdots,p\}} f_i(\partial f/\partial u_i)\Big)x_1\\ &= \Big(\sum_{i=1}^{p} f_i(\partial f/\partial u_i) + \sum_{i\in I-\{1,\cdots,p\}} u_i g_i(\partial f/\partial u_i)\Big)x_1\\ &= \Big(\sum_{i=1}^{p} f_i(\partial f/\partial u_i) + \sum_{i=p+1}^{N} u_i g_i(\partial f/\partial u_i)\Big)x_1\ ,\end{aligned}$$

the last equality following since $u_i(x_1) = 0$ if $i \notin I$.

COROLLARY 3.3. *The $A(M)$-module $\mathscr{X}(M)$ of continuous vector fields on a combinatorial manifold M is a Lie algebra over R with respect to the bracket operation*

$$[X, Y]f = X(Yf) - Y(Xf)$$

for $X, Y \in \mathscr{X}(M)$ and $f \in A(M)$.

In fact it is easy to show that if $F, G: M \times R \to M$ are integral flows for $X, Y \in \mathscr{X}(M)$, then $[Y, X]$ is the derivative at $0 \in R^+$ of the map from $M \times R^+$ to M given by

$$(x, t) \longmapsto G(F(G(F(x, \sqrt{t}), \sqrt{t}), -\sqrt{t}), -\sqrt{t})\ .$$

This is completely analogous to the smooth case (see [8]).

The next corollary is an application of Theorem 3.1 to PL manifolds. By a *closed PL n-manifold $\underline{M}$* we mean a closed topological n-manifold $\underline{M}$ together with an equivalence class of triangulations $T: M \to \underline{M}$ of $\underline{M}$ by closed combinatorial n-manifolds M, two triangulations being equivalent iff they have a common subdivision; the equivalence class is the "PL structure" of $\underline{M}$.

If $\underline{M}$ is a closed PL n-manifold, we let $A(\underline{M})$ denote the ring of continuous functions $f: \underline{M} \to R$ such that for some triangulation $T: M \to \underline{M}$ in the PL structure of $\underline{M}$, $f \circ T \in A(M)$.

COROLLARY 3.4. *Let $\underline{M}$ be a closed PL manifold, and let $\underline{D}: A(\underline{M}) \to A)\underline{M})$ be a derivation which satisfies the following property: for every triangulation $T: M \to \underline{M}$* of *$\underline{M}$ by a combinatorial manifold M in the PL structure of $\underline{M}$ there is a derivation $D: A(M) \to A(M)$ for which the following diagram commutes*

$$\begin{array}{ccc} A(M) & \xrightarrow{(T^{-1})^*} & A(\underline{M}) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \underline{D}} \\ A(M) & \xleftarrow[(T^{-1})^*]{} & A(\underline{M}) \end{array}$$

where $(T^{-1})^*f = f\circ T^{-1}\colon \underline{M}\to R$ *for* $f\in A(M)$. *Then* $\underline{D}=0$.

Proof. Let $\underline{f}\in A(\underline{M})$ and $x_0\in\underline{M}$. We will show that $(\underline{D}f)x_0=0$: Let $T\colon M\to\underline{M}$ be a triangulation in the PL structure of $\underline{M}$ for which $f\circ T\in A(M)$ and for which $T^{-1}(x_0)$ is a vertex of M. Then the associated derivation D of $A(M)$ corresponds to a continuous vector field on M which is necessarily zero at x_0. Thus

$$(\underline{D}f)x_0 = \underline{D}(T^{-1})^*(f\circ T)x_0 = (T^{-1})^*D(f\circ T)x_0 = (D(f\circ T))(T^{-1}(x_0)) = 0\ .$$

Now let K be a simplicial complex and let U be a small open subset of K. Recall (see [6]) that the $A(U)$-module $E(U)$ of piecewise smooth 1-forms on U consists of all compatible tuples $(\theta_\alpha)\in \times_\alpha E(U_\alpha)$ of smooth 1-forms $\theta_\alpha\in E(U_\alpha)$ defined on the wedges $U_\alpha\subseteq U$; here "compatible" means that if U_α and U_β are wedges of U then $\theta_\alpha|_{U_\alpha\cap U_\beta}=\theta_\beta|_{U_\alpha\cap U_\beta}$. If 1-forms are expressed in this manner then for $f=(f_\alpha)$ in $A(U)$, $\theta=(\theta_\alpha)$ and $\varphi=(\varphi_\alpha)$ in $E(U)$, $f\cdot\theta=(f_\alpha\cdot\theta_\alpha)$ and $\theta+\varphi=(\theta_\alpha+\varphi_\alpha)$; i.e., operations are wedgewise.

Alternately, if u is a coordinate system on U then $\theta\in E(U)$ may be expressed $\theta=\sum_i f_i du_i$ where each $f_i\in A(\mathrm{St}\,u_i)$, $\mathrm{St}\,u_i$ denoting the support of u_i. If forms are expressed in this manner then for f in $A(U)$, $\theta=\sum_i f_i du_i$ and $\varphi=\sum_i g_i du_i$ in $E(U)$, $f\cdot\theta=\sum_i (f|_{\mathrm{St}u_i})f_i du_i$ and $\theta+\varphi=\sum_i (f_i+g_i)du_i$.

THEOREM 3.5. *Let M be a combinatorial manifold. There is an $A(M)$-module isomorphism between the module $E(M)$ of global piecewise smooth 1-forms on M and the module* $\mathrm{Hom}_{A(M)}(\mathscr{X}(M), A(M))$ *of $A(M)$-linear maps from $\mathscr{X}(M)$ to $A(M)$.*

Proof. It again suffices to prove this result locally, so, with the notation of Theorem 3.1, we define

$$F\colon E(U)\longrightarrow \mathrm{Hom}_{A(U)}(\mathscr{X}(U), A(U))$$

by associating to the 1-form $\theta=\sum_i f_i du_i$ the homomorphism

$$F(\theta)\colon \sum_{i=1}^{p} g_i(\partial/\partial u_i)+\sum_{i=p+1}^{N} u_i h_i(\partial\partial u_i)\longmapsto \sum_{i=1}^{p} f_i g_i+\sum_{i=p+1}^{N} u_i f_i h_i\ .$$

This makes sense: Since f_i and g_i are in $A(U)$ for $i=1,\cdots,p$, clearly $f_i g_i\in A(U)$. Since f_i and h_i are in $A(\mathrm{St}\,u_i)$ for $i=p+1,\cdots,N$, we have $f_i h_i\in A(\mathrm{St}\,u_i)$; thus $u_i f_i h_i\in A(U)$.

Clearly $F: E(U) \to \mathrm{Hom}_{A(U)}(\mathscr{X}(U), A(U))$ is an $A(U)$-module homomorphism. Moreover it is injective: If $\theta = \sum_i f_i du_i$ is in $E(U)$ and $F(\theta) = 0$, then for $i = 1, \cdots, p, f_i = (F(\theta))du_i = 0$ on U, and for $i = p+1, \cdots, N, u_i f_i = (F(\theta))u_i(\partial/\partial u_i) = 0$ on U; thus $\theta = 0$.

Finally we show that $F: E(U) \to \mathrm{Hom}_{A(U)}(\mathscr{X}(U), A(U))$ is surjective as follows: Given $f \in \mathrm{Hom}_{A(U)}(\mathscr{X}(U), A(U))$, let $f_i = f(\partial/\partial u_i)$ for $i = 1, \cdots, p$ and $f_i = f(u_i(\partial/\partial u_i))$ for $i = p+1, \cdots, N$. Since f is $A(U)$-linear, it is easy to see that $f_i = f(u_i(\partial/\partial u_i)) = 0$ off $\mathrm{St}\, u_i$ for $i = p+1, \cdots, N$. Thus, as before (see the proof of Proposition 2.3), $f_i = u_i g_i$, for $i = p+1, \cdots, N$, where $g_i \in A(\mathrm{St}\, u_i)$. Consequently if

$$\theta = \sum_{i=1}^{p} f_i du_i + \sum_{i=p+1}^{N} g_i du_i ,$$

then clearly $F(\theta) = f$.

In order to generalize the previous result, recall (see [6]) that if K is a simplicial complex and U is a small open subset of K then the $A(U)$-module $\Lambda^q E(U)$ of piecewise smooth q-forms on U consists of all compatible tuples $(\theta_\alpha) \in \times_\alpha \Lambda^q E(U_\alpha)$ of smooth q-forms $\theta_\alpha \in \Lambda^q E(U_\alpha)$ defined on the wedges $U_\alpha \subseteqq U$; here "compatibility" is defined as before (i.e., as for $E(U)$), and the module operations are again wedgewise.

Alternately, if u is a coordinate system on U then $\theta \in \Lambda^q E(U)$ may be expressed $\theta = \sum_i f_i du_{i_1} \cdots du_{i_q}$ where the summation is taken over all $i = (i_1, \cdots, i_q)$ for which there is a wedge $U_\alpha \subseteqq U$ such that $u_{i_1}|_{U_\alpha}, \cdots, u_{i_q}|_{U_\alpha}$ are all coordinate functions on U_α and where $f_i \in A(\bigcap_{j=1}^q \mathrm{St}\, u_{i_j})$; for convenience we will use multi-index notation to write $\theta = \sum_i f_i du_i$ where $f_i \in A(\mathrm{St}\, u_i)$. If q-forms are expressed in this manner, then for f in $A(U)$, $\theta = \sum_i f_i du_i$ and $\varphi = \sum_i g_i du_i$ in $\Lambda^q E(U)$, $f \cdot \theta = \sum_i (f|_{\mathrm{St} u_i}) f_i du_i$ ond $\theta + \varphi = \sum_i (f_i + g_i) du_i$.

Observe that since the index of summation for $\theta = \sum_i f_i du_i$ in $\Lambda^q E(U)$ is restricted, $\Lambda^q E(U)$ is *not* the q-fold exterior product of $E(U)$: Actually $\Lambda^q E(U)$ is the q-fold exterior product $E(U)$ with the added relation that $du_{i_1} \cdots du_{i_q} = 0$ if there is no wedge $U_\alpha \subseteqq U$ for which $u_{i_1}|_{U_\alpha}, \cdots, u_{i_q}|_{U_\alpha}$ are all coordinate functions on U_α. We will continue to use the notation $\Lambda^q E(U)$ for the module of piecewise smooth q-forms on U since the q-fold exterior product of $E(U)$ will not be used in the sequel.

THEOREM 3.6. *Let M be a combinatorial manifold. There is an $A(M)$-module isomorphism between the module $\Lambda^q E(M)$ of global piecewise smooth q-forms on M and the module $\mathrm{Alt}_{A(M)}(\times^q \mathscr{X}(M), A(M))$ of $A(M)$-linear alternating maps from the q-fold product $\times^q \mathscr{X}(M)$ of $\mathscr{X}(M)$ to $A(M)$.*

Proof. Once again it suffices to prove the result locally, so again with the notation of Theorem 3.1 we define

$$F: \Lambda^q E(U) \longrightarrow \mathrm{Alt}_{A(U)}(\times^q \mathscr{X}(U), A(U))$$

by defining it on forms $f_i du_i$ and extending to $\Lambda^q E(U)$ by $A(U)$-linearity; the map $F(f_i du_{i_1} \cdots du_{i_q})$ takes $(g_{j_1}(\partial/\partial u_{j_1}), \cdots, g_{j_q}(\partial/\partial u_{j_q}))$ to $(-1)^\pi f_i g_{j_1} \cdots g_{j_q}$ if there is a permutation π taking $i = (i_1, \cdots, i_q)$ to $(j_1, \cdots, j_q)$, and 0 otherwise. (Note that for $j = p + 1, \cdots, N$, we write $u_j h_j(\partial/\partial u_j) = g_j(\partial/\partial u_j)$.) As before it is easy to verify that F is well defined and an isomorphism.

Finally recall (see [6]) that if K is a simplicial complex, U is a small open subset of K, and u is a coordinate system on U, then the differential d of the de Rham complex $(\Lambda^* E(U), d)$ of U is defined wedgewise but may alternately be described by

$$d\theta = d(\sum_i f_i du_{i_1} \cdots du_{i_q}) = \sum_{i_0, i} (\partial f_i/\partial u_{i_0}) du_{i_0} du_{i_1} \cdots du_{i_q} ,$$

the summation taken over all i_0, i for which $(\partial f_i/\partial u_{i_0}) du_{i_0} du_{i_1} \cdots du_{i_q}$ is a $(q + 1)$-form on U.

THEOREM 3.7. *Let M be a combinatorial manifold. If $d: \Lambda^* E(M) \to \Lambda^* E(M)$ is the differential of the de Rham complex of M and $\theta \in \Lambda^q E(M)$, then $d\theta \in \Lambda^{q+1} E(M)$ is given by the formula*

$$d\theta(X_1, \cdots, X_{q+1}) = \sum_j (-1)^{j+1} X_j \theta(X_1, \cdots, \hat{X}_j, \cdots, X_{q+1}) + \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, \hat{X}_{q+1})$$

for $X_1, \cdots, X_{q+1} \in \mathscr{X}(M)$.

Proof. Working locally with the notation of Theorem 3.1 and using the fact that d, θ, and $X_1, \cdots, X_{q+1}$ are all additive, one need only prove the result in the case $\theta = f du_{i_1}, \cdots, du_{i_q}$, and $X_1 = g_{j_1}(\partial/\partial u_{j_1}), \cdots, X_q = g_{j_q}(\partial/\partial u_{j_q})$ (where for $j = p+1, \cdots, N$ we again write $u_j h_j(\partial/\partial u_j) = g_j(\partial/\partial u_j)$). The result in this case is a straightforward calculation.

4 Connexions on combinatorial manifolds. Connexions exist (in tangent bundles) on smooth manifolds, and can be described in various equivalent ways; also such connexions can be interpreted geometrically in terms of parallel translation. The goal of this section is to establish analogous results for combinatorial manifolds. In particular we will establish the existence of connexions on combinatorial manifolds, and show that such connexions can be described (or defined) equivalently in terms of differential forms (Cartan connexions), vector fields (Koszul connexions) or compatible collections of connexions defined on individual simplices. Finally, such connexions are interpreted geometrically in terms of parallel translation.

Throughout this section, M will be a combinatorial n-manifold (without boundary).

DEFINITION 4.1. For every small open subset $U \subseteqq M$, let $E(U) \otimes E(U)$ denote the set of all tuples

$$\omega = (\omega_\alpha) = \left(\sum_i \theta^i_\alpha \otimes \varphi^i_\alpha\right) \in \times_\alpha \left(E(U_\alpha) \bigotimes_{A(U_\alpha)} E(U_\alpha)\right),$$

the product taken over all wedges $U_\alpha \subseteqq U$, for which $\omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha \cap U_\beta}$ for wedges U_α, $U_\beta \subseteqq U$, where

$$\omega_\alpha|_{U_\alpha \cap U_\beta} = \sum_i \theta^i_\alpha|_{U_\alpha \cap U_\beta} \otimes \varphi^i_\alpha|_{U_\alpha \cap U_\beta} \in E(U_\alpha \cap U_\beta) \bigotimes_{A(U_\alpha \cap U_\alpha)} E(U_\alpha \cap U_\beta).$$

Clearly $E(U) \otimes E(U)$ is an $A(U)$-module with respect to wedgewise operations. The construction of $E(U) \otimes E(U)$ is natural with respect to restriction to small open subsets $W \subseteqq U$, and hence defines a presheaf on M. The associated sheaf is fine since piecewise smooth partitions of unity subordinate to covers of M by small open subsets exist on M (see [6]).

DEFINITION 4.2. The $A(M)$-module $E(M) \otimes E(M)$ is the module of global sections of the sheaf over M associated to the presheaf which assigns to each small open subset $U \subseteqq M$ the $A(U)$-module $E(U) \otimes E(U)$.

As in the case of piecewise smooth forms, if u is a coordinate system on the small open subset $U \subseteqq M$, then each $\omega \in E(U) \otimes E(U)$ may be written in the form $\omega = \sum_{i,j} f_{ij} du_i \otimes du_j$ where the summation is taken over all i and j for which there is a wedge $U_\alpha \subseteqq U$ such that $u_{i|U_\alpha}$ and $u_{j|U_\alpha}$ are coordinate functions on U_α, and where $f_{ij} \in A(\mathrm{St}\, u_i \cap \mathrm{St}\, u_j)$, $\mathrm{St}\, u_i \cap \mathrm{St}\, u_j$ the intersection of the supports of u_i and u_j. Furthermore for f in $A(U)$, $\psi = \sum_{i,j} f_{ij} du_i \otimes du_j$ and $\omega = \sum_{i,j} g_{ij} du_i \otimes du_j$ in $E(U) \otimes E(U)$,

$$f \cdot \psi = \sum_{i,j} (f|_{\mathrm{St}u_i \cap \mathrm{St}u_j}) f_{ij} du_i \otimes du_j$$

and

$$\psi + \omega = \sum_{i,j} (f_{ij} + g_{ij}) du_i \otimes du_j$$

where the summations are restricted as above.

Since the index of summation for elements $\omega = \sum_{i,j} f_{ij} du_i \otimes du_j$ of $E(U) \otimes E(U)$ is restricted, $E(U) \otimes E(U)$ is *not* the tensor product of $E(U)$ with itself; actually $E(U)\otimes E(U)$ is the tensor product of $E(U)$ with itself with the added relation that $du_i \otimes du_j = 0$ if there is no wedge $U_\alpha \subseteqq U$ for which $u_{i|U_\alpha}$ and $u_{j|U_\alpha}$ are both coordinate functions on U_α. We will continue to use the notation $E(U)\otimes E(U)$ since the tensor product of $E(U)$ with itself will not be used in the sequel.

We now describe connexions on M by differential forms.

DEFINITION 4.3. A *Cartan connexion on* M is a real linear map

$$D\colon E(M) \longrightarrow E(M)\otimes E(M)$$

for which $D(f\theta) = df \otimes \theta + fD\theta$ for $f \in A(M)$ and $\theta \in E(M)$.

As in the smooth case, a Cartan connexion D on M uniquely determines a Cartan connexion

$$D_U\colon E(U) \longrightarrow E(U)\otimes E(U)$$

for every small open subset $U \subseteqq M$.

Henceforth "$\mathscr{C}$" denotes "set complement", and $\mathrm{St}\, u_i$ again denotes the support of the coordinate function u_i.

THEOREM 4.4. *Let $U \subseteqq M$ be a small open subset, and let u be a coordinate system on U. Then*

$$D\colon E(U) \longrightarrow E(U)\otimes E(U)$$

is a Cartan connexion on U iff there are piecewise smooth functions γ^i_{jk} in $A(\mathrm{St}\, u_j \cap \mathrm{St}\, u_k)$, for each i, j, and k, for which

$$D(du_i) = \sum_{j,k} \gamma^i_{jk} du_k \otimes du_j$$

and for which $\gamma^i_{jk} = 0$ on $\mathscr{C}(\mathrm{St}\, u_i) \cap \mathrm{St}\, u_j \cap \mathrm{St}\, u_k$.

Proof. Given D, the existence of the γ^i_{jk} is obvious. The fact that $\gamma^i_{jk} = 0$ on $\mathscr{C}(\mathrm{St}\, u_i) \cap \mathrm{St}\, u_j \cap \mathrm{St}\, u_k$ follows since D is an $A(U)$-derivation.

Conversely, given γ^i_{jk} in $A(\mathrm{St}\, u_j \cap \mathrm{St}\, u_k)$ for which $\gamma^i_{jk} = 0$ on $\mathscr{C}(\mathrm{St}\, u_i) \cap \mathrm{St}\, u_j \cap \mathrm{St}\, u_k$, define D by

$$\begin{aligned} D\Big(\sum_i f_i du_i\Big) &= D\Big(\sum_i \bar{f}_i \cdot du_i\Big) \\ &= \sum_i (d\bar{f}_i \otimes du_i + \bar{f}_i D(du_i)) \\ &= \sum_i \Big(d\bar{f}_i \otimes du_i + \bar{f}_i\Big(\sum_{j,k} \gamma^i_{jk}\, du_k \otimes du_j\Big)\Big) \end{aligned}$$

where $\bar{f}_i = f_i$ if $\mathrm{St}\, u_i = U$, and $\bar{f}_i \in A(U)$ is an extension of $f_i \in A(\mathrm{St}\, u_i)$

otherwise. To show that this makes sense, first observe that if $\theta \in E(U)$ and $\theta = 0$ on $\mathrm{St}\, u_i$, then $\theta \otimes du_i \in E(U) \otimes E(U)$ is 0. (If $\theta = \sum_j g_j du_j$, then $\theta = 0$ on $\mathrm{St}\, u_i$ implies that either $u_{j\,|\,\mathrm{St}u_i \cap \mathrm{St}u_j} \neq 0$, in which case $g_j = 0$ on $\mathrm{St}\, u_i \cap \mathrm{St}\, u_j$, or $u_{j\,|\,\mathrm{St}u_i \cap \mathrm{St}u_j} = 0$. In the first case

$$g_{j|\mathrm{St}u_i \cap \mathrm{St}u_j} du_j \otimes du_i = 0 \ ,$$

and in the latter $du_j \otimes du_i = 0$.) Thus given extensions $\bar{f}'_i$ and $\bar{f}''_i$ of f_i, $d\bar{f}'_i - d\bar{f}''_i = 0$ on $\mathrm{St}\, u_i$ so that

$$d\bar{f}'_i \otimes du_i - d\bar{f}''_i \otimes du_i = (d\bar{f}'_i - d\bar{f}''_i) \otimes du_i = 0 \ .$$

Second observe that since $\bar{f}_i$ is well defined on u_i and $\gamma^i_{jk} = 0$ on $\mathscr{C}(\mathrm{St}\, u_i) \cap \mathrm{St}\, u_j \cup \mathrm{St}\, u_k$, clearly $(\bar{f}_{i\,|\,\mathrm{St}u_j \cap \mathrm{St}u_k})\gamma^i_{jk}$ is well defined on $\mathrm{St}\, u_j \cap \mathrm{St}\, u_k$. Since this is the support of $du_k \otimes du_j$, $\bar{f}_i D(du_i)$ is independent of the extension $\bar{f}_i$ of f_i.

COROLLARY 4.5. *Cartan connexions exist on M.*

Proof. Let $\{\varphi_i\}_{i \in I}$ be a piecewise smooth partition of unity subordinate to a locally finite cover $\{U_i\}_{i \in I}$ of M by small open subsets. For each $i \in I$, define the Cartan connexion

$$D_i \colon E(U_i) \longrightarrow E(U_i) \otimes E(U_i)$$

on U_i by letting $D_i(du_j) = 0$ for each j. It is easy to verify that the map

$$D \colon E(M) \longrightarrow E(M) \otimes E(M)$$

given by $D\theta = \sum_i \varphi_i D_i(\theta|_{U_i})$ is a Cartan connexion on M.

We now describe connexions on M by continuous vector fields.

DEFINITION 4.6. *A Koszul connexion* on M is a real-bilinear map

$$\nabla \colon \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow \mathscr{X}(M)$$
$$(X, Y) \longmapsto \nabla_X Y$$

such that $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X fY = X(f)Y + f\nabla_X Y$ for $f \in A(M)$ and $X, Y \in \mathscr{X}(M)$.

Again, as in the smooth case, a Koszul connexion ∇ on M determines a Koszul connexion

$$\nabla_U \colon \mathscr{X}(U) \times \mathscr{X}(U) \longrightarrow \mathscr{X}(U)$$

for every small open subset $U \subseteq M$.

THEOREM 4.7. *Let $U \subseteq M$ be small open neighborhood of x_0, let $u = \{u_1, \cdots, u_N\}$ be a coordinate system on U with origin x_0 for which*

$\{u_1, \cdots, u_p\}$ coordinatizes $U_\alpha = \sigma_\alpha \cap U$ where x_0 is contained in the interior of the p-simplex $\sigma_\alpha \subseteqq M$, and let

$$X_i = \begin{cases} \partial/\partial u_i & \text{for } i = 1, \cdots, p \\ u_i(\partial/\partial u_i) & \text{for } i = p+1, \cdots, N . \end{cases}$$

Then

$$\nabla: \mathscr{X}(U) \times \mathscr{X}(U) \longrightarrow \mathscr{X}(U)$$

is a Koszul connexion on U iff there are piecewise smooth functions δ^i_{jk} in $A(\mathrm{St}\, u_i)$, for each i, j, and k, for which

$$\nabla_{X_j} X_k = \sum_i \delta^i_{jk} X_i$$

and for which $\delta^i_{jk} = 0$ on $\mathrm{St}\, u_i \cap \mathscr{C}(\mathrm{St}\, u_j \cap \mathrm{St}\, u_k)$.

Proof. Given ∇, the existence of the δ^i_{jk} is obvious. The fact that $\delta^i_{jk} = 0$ on $\mathrm{St}\, u_i \cap \mathscr{C}(\mathrm{St}\, u_j \cap \mathrm{St}\, u_k)$ follows since $\nabla_X Y$ is $A(U)$-linear in X and an $A(U)$-derivation in Y.

Conversely, given δ^i_{jk} in $A(\mathrm{St}\, u_i)$ for which $\delta^i{}_{jk} = 0$ on $\mathrm{St}\, u_i \cap \mathscr{C}(\mathrm{St}\, u_j \cap \mathrm{St}\, u_k)$, we first let

$$\nabla_{fX_j} X_k = \nabla_{\bar{f}X_j} X_k = \bar{f}\nabla_{X_j} X_k = \bar{f}\Big(\sum_i \delta^i_{jk} X_i\Big)$$

where $\bar{f} = f$ if $\mathrm{St}\, u_j = U$, and $\bar{f} \in A(U)$ is an extension of $f \in A(\mathrm{St}\, u_j)$ otherwise. This makes sense since $\bar{f}$ is well defined on $\mathrm{St}\, u_j$ and $\delta^i_{jk} = 0$ on $\mathrm{St}\, u_i \cap \mathscr{C}(\mathrm{St}\, u_j \cap \mathrm{St}\, u_k)$ so that $\bar{f}\delta^i_{jk} \in A(\mathrm{St}\, u_i)$ is well defined.

Next define

$$\begin{aligned} \nabla_{X_j} f X_k &= \nabla_{X_j} \bar{f} X_k \\ &= X_j(\bar{f}) X_k + \bar{f}\nabla_{X_j} X_k \\ &= X_j(\bar{f}) X_k + \bar{f}\Big(\sum_i \delta^i_{jk} X_i\Big) \end{aligned}$$

where $\bar{f} = f$ if $\mathrm{St}\, u_k = U$, and $\bar{f} \in A(U)$ is an extension of $f \in A(\mathrm{St}\, u_k)$ otherwise. To see that this makes sense, first notice that since X_j is a derivation, $f = 0$ on $\mathrm{St}\, u_i$ implies that $X_j(f) = 0$ on $\mathrm{St}\, u_i$, so that $X_j(f) X_i = 0$ on U; thus given extensions $\bar{f}'_i$ and $\bar{f}''_i$ of f_i,

$$X_j(\bar{f}'_i) X_i - X_j(\bar{f}''_i) X_i = X_j(\bar{f}'_i - \bar{f}''_i) X_i = 0$$

since $\bar{f}'_i - \bar{f}''_i = 0$ on $\mathrm{St}\, u_i$. Second observe that $\bar{f}(\sum_i \delta^i_{jk} X_i)$ is well defined; this is precisely the same argument as above for $\bar{f}\delta^i_{jk}$.

Finally define ∇ on all of $\mathscr{X}(U) \times \mathscr{X}(U)$ by extending by real-linearity.

THEOREM 4.8. *There is a bijection between the set of Cartan connexions on M and the set of Koszul connexions on M.*

Proof. It is not difficult to show that for every $X \in \mathscr{X}(M)$ there is a real-linear map

$$\text{eval}_X: E(M) \otimes E(M) \longrightarrow E(M)$$

which is given locally, on a small open subset $U \subseteqq M$, by

$$\text{eval}_X\Big(\sum_i \theta_i \otimes \varphi_i\Big) = \sum_i \langle X, \varphi_i \rangle \theta_i$$

where "$\langle -, - \rangle$" denotes "evaluation" (see Theorem 3.5). One now defines the bijection between Cartan connexions D on M and Koszul connexions ∇ on M by imposing the contraction formula

$$\langle \nabla_X Y, \theta \rangle = X\langle Y, \theta \rangle + \langle Y, \text{eval}_X D\theta \rangle$$

where $X, Y \in \mathscr{X}(M)$ and $\theta \in E(M)$.

To describe the correspondence of Theorem 4.8 locally, let U be a small open neighborhood of $x_0 \in M$, let $u = \{u_1, \cdots, u_N\}$ be a coordinate system on U with origin x_0 such that $\{u_1, \cdots, u_p\}$ coordinatizes $U_\alpha = \sigma_\alpha \cap U$ where x_0 is contained in the interior of the p-simplex $\sigma_\alpha \subseteqq M$, and let

$$X_i = \begin{cases} \partial/\partial u_i & \text{for} \quad i = 1, \cdots, p \\ u_i(\partial/\partial u_i) & \text{for} \quad i = p+1, \cdots, N \,. \end{cases}$$

If the Cartan connexion D on U given by $D(du_i) = \sum_{j,k} \gamma^i_{jk} du_k \otimes du_j$ and the Koszul connexion ∇ on U given by $\nabla_{X_j} X_k = \sum_i \delta^i_{jk} X_i$ correspond to each other via Theorem 4.8, then the relations between γ^j_{jk} and δ^i_{jk} are given by:

if $i=$	if $j=$	if $k=$	relation
$1, \cdots, p$	$1, \cdots, p$	$1, \cdots, p$	$\delta^i_{jk} = \gamma^i_{jk}$
$1, \cdots, p$	$1, \cdots, p$	$p+1, \cdots, N$	$\delta^i_{jk} = u_k\gamma^i_{jk}$
$1, \cdots, p$	$p+1, \cdots, N$	$1, \cdots, p$	$\delta^i_{jk} = u_j\gamma^i_{jk}$
$1, \cdots, p$	$p+1, \cdots, N$	$p+1, \cdots, N$	$\delta^i_{jk} = u_j u_k\gamma^i_{jk}$
$p+1, \cdots, N$	$1, \cdots, p$	$1, \cdots, p$	$u_i\delta^i_{jk} = \gamma^i_{jk}$
$p+1, \cdots, N$	$1, \cdots, p$	$p+1, \cdots, N$	$u_i\delta^i_{jk} = u_k\gamma^i_{jk}$
$p+1, \cdots, N$	$p+1, \cdots, N$	$1, \cdots, p$	$u_i\delta^i_{jk} = u_j\gamma^i_{jk}$
$p+1, \cdots, N$	$p+1, \cdots, N$	$p+1, \cdots, N$	$\begin{cases} u_i\delta^i_{jk} = u_j u_k \gamma^i_{jk} \text{ if } i \neq j \text{ or } i \neq k \\ \delta^i_{jk} = 1 + u_i\gamma^i_{jk} \text{ if } i = j = k \,. \end{cases}$

The following result is an immediate consequence of Theorem 4.8 and Corollary 4.5.

COROLLARY 4.9. *Koszul connexions exist on M.*

It is easy to show that if ∇ is a Koszul connexion on M then for every $x_0 \in M$ and $X, Y \in \mathscr{X}(M)$, the value of $\nabla_X Y$ at x_0 depends only on X_{x_0} and the values of Y on some curve that fits X_{x_0}.

THEOREM 4.10. *A Koszul connexion ∇ on M is equivalent to a collection*

$$\nabla_\alpha \colon \mathscr{X}(\sigma_\alpha) \times \mathscr{X}(\sigma_\alpha) \longrightarrow \mathscr{X}(\sigma_\alpha)$$

of Koszul connexions ∇_α defined on the simplices $\sigma_\alpha \subseteqq M$ which are compatible in the following sense: If $\sigma_\alpha, \sigma_\beta \subseteqq M$ are simplices for which $\sigma_\alpha \subseteqq \sigma_\beta$, x_0 is in the interior of σ_α, $X \in T_{x_0}\sigma_\alpha$, and Y is a smooth vector field on some smooth curve in σ_α which fits X, then

$$(D_\alpha)_X Y = (D_\beta)_{(i_{\beta\alpha})_* X}(i_{\beta\alpha})_* Y ,$$

$(i_{\beta\alpha})_*$ denoting the Jacobian of the inclusion $i_{\beta\alpha} \colon \sigma_\alpha \to \sigma_\beta$.

Proof. Given a Koszul connexion ∇ on M, let U be a small open neighborhood of $x_0 \in M$ and let $u = \{u_i\}$ be a coordinate system on U with origin x_0. For each wedge $U_\alpha \subseteqq U$, let $u_\alpha = \{u_\alpha^i\}$ denote the coordinate system on U_α induced by u. If

$$D \colon E(M) \longrightarrow E(M) \otimes E(M)$$

is the Cartan connexion on M associated to ∇ and

$$D \colon E(U) \longrightarrow E(U) \otimes E(U)$$

is the induced Cartan connexion on U, given in coordinate form by

$$D(du_i) = \sum_{j,k} \gamma_{jk}^i du_k \otimes du_j ,$$

then for every wedge $U_\alpha \subseteqq U$ there is a unique Koszul connexion

$$\nabla_\alpha \colon \mathscr{X}(U_\alpha) \times \mathscr{X}(U_\alpha) \longrightarrow \mathscr{X}(U_\alpha)$$

given by

$$(\nabla_\alpha)_{\partial/\partial u_\alpha^j} \partial/\partial u_\alpha^k = \sum_i (\gamma_{jk|U_\alpha}^i)(\partial/\partial u_\alpha^i) .$$

This construction is natural, and hence a Koszul connexion

$$\nabla_\alpha \colon \mathscr{X}(\sigma_\alpha) \times \mathscr{X}(\sigma_\alpha) \longrightarrow \mathscr{X}(\sigma_\alpha)$$

is defined on each simplex $\sigma_\alpha \subseteqq M$. The compatibility of these connexions is immediate.

Conversely, given a compatible collection of connexions as described in the hypothesis, let $X, Y \in \mathscr{X}(M)$. To define $D_X Y$ at $x_0 \in M$, let $\sigma_\alpha \subseteqq M$ be the simplex in whose interior x_0 is contained. By Corollary 2.5 there are smooth vector fields $X_\alpha, Y_\alpha \in \mathscr{X}(\sigma_\alpha)$ for which $(i_\alpha)_* X_\alpha = X$ and $(i_\alpha)_* Y_\alpha = Y$ on σ_α. There is clearly a smooth curve in σ_α which fits X_{x_0}, and since Y_α is defined along this curve we may let

$$(\nabla_X Y)_{x_0} = (i_\alpha)_*(D_\alpha)_{(X_\alpha)_{x_0}} Y_\alpha \ .$$

This clearly defines a Koszul connexion on M and the proof is complete.

We will next use Theorem 4.10 to interpret connexions geometrically. First, however, recall (see [7]) that a piecewise smooth curve $f: [a, b] \to M$ in M is a map for which there is a finite subdivision

$$(*) \qquad a = c_0 < c_1 < \cdots < c_N = b$$

of $[a, b]$ such that for each $i = 0, \cdots, N-1$ there is a simplex $\sigma_\alpha \subseteqq M$ for which $f: [c_i, c_{i+1}] \to \sigma_\alpha$ is a smooth curve in σ_α.

DEFINITION 4.11. A *continuous vector field along a piecewise smooth curve* $f: [a, b] \to M$ is a map $Y: [a, b] \to TM$ which is smooth on each subinterval of the subdivision (*) of $[a, b]$ and for which $\tau(M) \circ Y = f$.

Now let ∇ be a Koszul connexion on M. Let $\sigma_\alpha, \sigma_\beta \subseteqq M$ be n-simplices whose intersection σ_γ is a p-simplex of M, and let ∇_α and ∇_β be the (compatible) connexions on σ_α and σ_β induced by ∇. Finally let $f: [a, b] \to M$ be a piecewise smooth curve for which there is a $c \in (a, b)$ such that

$$f_\alpha = f|_{[a,c]}: [a, c] \longrightarrow \sigma_\alpha \ ,$$
$$f_\beta = f|_{[c,b]}: [c, b] \longrightarrow \sigma_\beta \ ,$$

are smooth curves in σ_α and σ_β, respectively.

THEOREM 4.12. *For each $t \in [a, b]$ there is a p-dimensional subspace $V_t \subseteqq T_{f(t)}M$ such that for every $Y_0 \in V_a$ there is a continuous vector field $Y = Y_t$ on f for which $Y_t \in V_t$ for each $t \in [a, b]$, $Y_a = Y_0$, $Y_\alpha = Y|_{[a,c]}$ is parallel along f_α with respect to ∇_α, and $Y_\beta = Y|_{[c,b]}$ is parallel along f_β with respect to ∇_β.*

Proof. With respect to ∇_α, for every $Y_0 \in T_{f(a)}\sigma_\alpha$ there is a unique smooth vector field $Y_\alpha = (Y_\alpha)_t$ on σ_α such that $(Y_\alpha)_0 = Y_0$ and Y_α is parallel along f_α, and a linear isomorphism

$$P^\alpha_{a,t}\colon T_{f(a)} \longrightarrow T_{f(t)}$$

called parallel translation along f_a from $f(a)$ to $f(t)$; similarly for σ_β. With the notation of Corollary 2.5, for $t \in [a, c]$ we let

$$V_t = (i_\alpha)_*(P^\alpha_{t,c})^{-1}(i_{\alpha\gamma})_* T_{f(c)}\sigma_\gamma \subseteqq T_{f(t)}M$$

and for $t \in [c, b]$ we let

$$V_t = (i_\beta)_*(P^\beta_{c,t})(i_{\beta\gamma})_* T_{f(c)}\sigma_\gamma \subseteqq T_{f(t)}M\,;$$

observe that V_t is well defined for $t = c$. For $Y_0 \in V_a$, we define $Y = Y_t$ on f by

$$Y_t = \begin{cases} (i_\alpha)_*(P^\alpha_{a,t})(i_\alpha)^{-1}_* Y_0 & \text{for} \quad t \in [a, c] \\ (i_\beta)_*(P^\beta_{c,t})(i_\beta)^{-1}_*(i_\alpha)_*(P^\alpha_{a,c})(i_\alpha)^{-1}_* Y_0 & \text{for} \quad t \in [c, b]\,. \end{cases}$$

Observe that $P_{a,t}$ is independent of c since ∇_α and ∇_β are compatible.

Thus there is a map $P_{a,t}\colon V_a \to V_t$ defined by $P_{a,t}(Y_0) = Y_t$ which is a linear isomorphism and which can reasonably be called *parallel translation* along f from $f(a)$ to $f(t)$.

Theorem 4.12 is still valid under the hypothesis that $\sigma_\alpha, \sigma_\beta \subseteqq M$ are simplices of arbitrary dimension whose intersection is a p-simplex σ_γ of M.

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