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KLINGENBERG STRUCTURES AND PARTIAL DESIGNS II: REGULARITY AND UNIFORMITY

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We continue our study of the c-K-structures introduced in Part I of this paper; these are triples (ϕ, Π, Π') where $\phi: \Pi \rightarrow \Pi'$ is a well-behaved incidence structure epimorphism. This paper is concerned with uniformity and regularity of c-K-structures. We obtain connections with PBIBD's ARBD's and transversal designs.

Introduction. This paper is a continuation of our paper "Klingenberg structures and partial designs I" [6]. In the previous paper, we considered generalizations of projective Klingenberg and Hjelmslev planes (*PK*-, resp., *PH*-planes) and investigated congruence relations and solutions. The present paper is devoted to the study of the notions of uniformity and regularity.

It is well-known that a *PH*-plane Π is uniform if and only if Π induces an ordinary affine plane in each point neighborhood (Lüneburg [17]). This formulation of the idea of uniformity is herein generalized (in two ways) to c-K-structures: the induced incidence structures are now required to be "$(s, r; \mu)$-nets. These $(s, r; \mu)$-nets (which simultaneously generalize the classes of nets and affine resolvable designs) are considered in the first section of the present paper (numbered §5). These structures are, in fact, the duals of transversal designs (λ not necessarily $= 1$) which were investigated by Hanani in [10]. Uniformity and pre-uniformity are then studied in §6.

In §7, we generalize the notion of a regular *PK*-plane (Jungnickel [13], [16]) to that of a regular c-K-structure and are led thereby to the use of difference methods. We obtain a theorem characterizing the invariants of all regular, balanced, minimally uniform H-structures that is similar to the main result of Part I (Theorem 3.9). Finally, the notions of the preceding sections are combined in §8 where we study regular pre-uniform K-structures. We obtain a complete characterization of the invariants in a special case, but there remains an interesting open problem.

The notation used in this paper agrees with that of Part I and hence in general with that of Dembowski [5]. We decided to continue the numbering of Part I: thus the first section of the paper is called §5, and references like "Theorem 3.9" or "(2.5)" refer without saying to Part I.

5. $(s, r; \mu)$-nets. In this section we examine $(s, r; \mu)$-nets, a common generalization of the familiar concepts of net (Bruck [3]) and affine resolvable design (Bose [2]). These $(s, r; \mu)$-nets are just the duals of the transversal designs of Hanani [10]; they will be used in the next section to study certain c-K-structures.

DEFINITION 5.1. An incidence structure with parallelism $\Sigma = (\mathfrak{P}, \mathfrak{B}, I, \|)$ is called a *net* of *order* s, *degree* r and *type* μ (briefly, an $(s, r; \mu)$-net) if it satisfies the following axioms:

(5.1) each point is on precisely one line of each parallel class;

(5.2) some parallel class has precisely $s \geqq 2$ lines, and there are $r \geqq 3$ parallel classes;

(5.3) lines from distinct parallel classes meet in precisely $\mu \geqq 1$ points.

If furthermore

(5.4) there exists a natural number λ such that any two points are joined by either 0 or λ lines,

then Σ is called an *affine resolvable partial plane* of order s, degree r, type μ and *index* λ (briefly, an $(s, r; \mu)$-ARPP (of index λ)).

PROPOSITION 5.2. *Let Σ be an $(s, r; \mu)$-net. Then Σ has $v := s^2\mu$ points and $b := sr$ lines; each parallel class contains s lines; and each line has $k := s\mu$ points. Also Σ is a design if and only if it is a point cohesive ARPP if and only if it is an affine resolvable design (ARBD).*

Proof. Let s_1, s_2, s_3 be the numbers of lines in three distinct parallel classes. Count all double flags (p, G, H) as G and H vary over two distinct parallel classes to obtain $s_1s_2\mu = s_1s_3\mu$. The rest of the argument is routine.

The following result was proved by Hanani [10, Lemma 5]. Lenz has given an alternate unpublished proof which utilizes linear algebra. Here we present a third proof which seems somewhat simpler to us; in addition, some of our intermediate steps will be used in the proof of Proposition 5.7.

PROPOSITION 5.3 (Hanani). *Let Σ be an $(s, r; \mu)$-net. Then*

$$r \leqq \frac{s^2\mu - 1}{s - 1}\,. \tag{5.5}$$

If we have equality in (5.5), *Σ is called* **complete**. *Then $(s-1)\mid(\mu-1)$ whenever Σ is complete.*

Proof. Choose any point p of Σ, and label the remaining $v-1=s^2\mu-1$ points by q_i $(i=1,\cdots,v-1)$. Let $x_i{:}=[p, q_i]$. Counting in two ways the flags (q_i, G) with $p, q_i IG$ yields

$$\sum_i x_i = r(s\mu - 1)\,. \tag{5.6}$$

Counting the flags (q_i, H) with $p, q_i IG, H$ where G is a fixed line through p yields

$$\begin{aligned}\sum_{q_i IG} x_i &= (s\mu - 1) + \sum_{q_i IG}(x_i - 1)\\ &= (s\mu - 1) + (r-1)(\mu - 1)\,.\end{aligned} \tag{5.7}$$

Summing (5.7) over all lines G through p, we obtain

$$\sum_i x_i^2 = r[s\mu - 1 + (r-1)(\mu-1)]\,; \tag{5.8}$$

but, in general, $(\sum_{i=1}^m a_i)^2 \leqq m(\sum_{i=1}^m a_i^2)$ (cf., e.g., [11, p. 245]). Hence by (5.6) and (5.8),

$$r^2(s\mu - 1)^2 \leqq (s^2\mu - 1)r[s\mu - 1 + (r-1)(\mu - 1)]$$

which reduces to

$$r \leqq \frac{s^2\mu - 1}{s-1} = s\mu + \mu + \frac{\mu - 1}{s-1}\,;$$

hence, if Σ is complete, $(s-1)\,|\,(\mu-1)$.

COROLLARY 5.4. *In an $(s, r; \mu)$-ARPP, $\lambda(s\mu - 1) = s\mu - 1 + (r-1)(\mu-1)$ holds, so the index λ is determined by s, r, and μ.*

Proof. Apply (5.7).

EXAMPLES 5.5.

(a) Nets in the usual sense (of order t and degree r) are here $(t, r; 1)$-nets. It is well-known that the maximum value of r is $t+1$, which agrees with (5.5).

(b) In Proposition 4.9, we exhibited $(q, q^2+q+1; q)$-nets; these are complete by (5.5).

(c) For each μ for which there exists a Hadamard matrix of order 4μ, there is a complete $(2, 4\mu-1; \mu)$-net (Hanani [10, 2.1]).

(d) There exists an $(s, 7; \mu)$-net whenever $\mu \geqq 2$ (Hanani [10, Theorem 3]).

PROPOSITION 5.6 (Bose). *Let Σ be an ARBD with parameters v, b, k, r, λ, μ. Then there exist natural numbers s and m such that*

(5.9) $$v = s^2(1 + ms - m)\,, \quad b = ms^3 + s^2 + s\,, \quad k = s(1 + ms - m)\,,$$
$$r = 1 + s + ms^2\,, \qquad \lambda = 1 + ms\,, \qquad \mu = 1 + ms - m\,.$$

Considered as a net, Σ has parameters $(s, r; \mu)$.

Proof. Substitute s for n and m for t in [2, (2.44)], and use the fact that $\mu = k^2/v$ [2, p. 108].

PROPOSITION 5.7. *Let Σ be an $(s, r; \mu)$-net. Then Σ is complete if and only if it is an ARBD.*

Proof. Let Σ be an ARBD. Then by (5.9), $r = 1 + s + ms^2 = s(1+ms-m) + (1 + ms-m) + (m(s-1))/(s-1) = s\mu + \mu + (\mu-1)/(s-1)$; thus Σ is complete by 5.3. Conversely, assume that Σ is complete. Then $r = (s^2\mu - 1)/(s - 1)$ by 5.3. Let $\bar{\lambda}$ be the average number of lines joining a given point p of Σ to the remaining points q_i ($i = 1, \cdots, v - 1$). Using the same notation as in the proof of 5.3, we see that $\sum_i x_i = \bar{\lambda}(s^2\mu - 1)$. Hence by (5.6), $\bar{\lambda} = (s\mu - 1)/(s - 1)$. Computing the invariance

$$\sigma = \sum_i (\bar{\lambda} - x_i)^2\,,$$

one gets 0 (using (5.6) and (5.8)). Hence we have $x_i = \bar{\lambda} = (s\mu - 1)/(s - 1)$ for all i; thus Σ is a design and hence (by 5.2) an ARBD.

THEOREM 5.8 (Bose [2], Shrikhande [20], [21]). *An ARBD with parameters given by* (5.9) *exists in at least the following cases*:

(i) *s a prime power, $m = 0$;*

(ii) *s a prime power, N a natural number and $m = 1 + s + \cdots + s^{N-1}$;*

(iii) *$s = 2$, $m = 1$ or 2.*

An ARBD with parameters given by (5.9) *cannot exist in the following cases*:

(i) *there is a symmetric $(ms^2 + s + 1, ms + 1, m)$-design but no symmetric $(ms^3 + s^2 + s + 1, ms^2 + s + 1, ms + 1)$-design* (e.g., $m = s = 3$);

(ii) *s and m are odd; in addition, either k is not a square, or $sm \equiv 1 \pmod 4$ and the squarefree part of s contains a prime $\equiv 3 \pmod 4$;*

(iii) *m is even, and s is odd; in addition, either μ is not a square, or $s + m \equiv 1 \pmod 4$ and the squarefree part of s contains a prime $\equiv 3 \pmod 4$;*

(iv) *$s \equiv 2 \pmod 4$, the squarefree part of s contains a prime $\equiv 3 \pmod 4$.*

PROPOSITION 5.9. *Denote by $N(s)$ the maximum number of pairwise orthogonal Latin squares of order s. If $N(s) \geqq r-2$, then there exists an $(s, r; \mu)$-net Σ for every $\mu \geqq 1$. If $r = s+1$, Σ may be required to be point cohesive.*

Proof. The assertion is well-known for $\mu = 1$. Let $p_1, \cdots, p_v$ be the points of such an $(s, r; 1)$-net. Replicate each point μ times; if a line of the original net contains the points $p_{i_1}, \cdots, p_{i_s}$, define a line of the new net by taking all the corresponding replicated points.

PROPOSITION 5.10. *There exists an $(s, r; \mu)$-net Σ if and only if there exists a μ-set (see Definition 4.3) of r symmetric $(s^2\mu \times s^2\mu)$-matrices $M_1, \cdots, M_r$ satisfying*

$$M_i^2 = s\mu M_i \quad and \quad m_{ii}^j = 1 \quad for\ all \quad i, j\,. \tag{5.10}$$

A point cohesive $(s, r; \mu)$-net exists if and only if there exists a μ-set with the above listed properties which also satisfies

$$\textstyle\sum_i M_i^2 \geqq J\,. \tag{5.11}$$

Proof. Assume first the existence of an $(s, r; \mu)$-net. Define the M_i as in the proof of Proposition 4.8. Then row i of M_j will be the incidence vector of the uniquely determined line in parallel class $\mathfrak{P}_j$ through point p_i. By definition, the M_i are symmetric. Since Σ satisfies (5.3), $\{M_1, \cdots, M_r\}$ is a μ-set. If the (i, k)-entry of M_j is 1, rows i and k of j are identical; hence their inner product will be $s\mu$ If the (i, k)-entry of M_j is 0, p_i, and p_k are on distinct lines from parallel class $\mathfrak{P}_j$; as these lines do not meet, the inner product of rows i and k will be zero. Thus $M_i^2 = s\mu M_i$.

Assume conversely the existence of a μ-set of symmetric $(s^2\mu \times s^2\mu)$-matrices $M_1, \cdots, M_r$ satisfying (5.10). Take as points of Σ the symbols $p_1, \cdots, p_v$ with $v := s^2\mu$. Take as lines in 'parallel class' $\mathfrak{P}_i$ the point sets given by the rows of M_i. By (5.10), each row of M_i must contain precisely $s\mu$ entries 1. Again by (5.10), two distinct rows of M_i which meet at all must in fact be identical; thus they induce the same line of Σ. Hence Σ will satisfy (5.1). Also, lines from distinct parallel classes will meet precisely μ times, as $\{M_1, \cdots, M_r\}$ is a μ-set of matrices; then Σ satisfies (5.3). Finally, there are $s^2\mu$ rows in M_i, and each line of $\mathfrak{P}_i$ is induced by $s\mu$ rows of M_i. Hence each parallel class contains precisely s lines, and Σ is an $(s, r; \mu)$-net. Now the truth of the second assertion should also be easy to see.

COROLLARY 5.11. *An ARBD with parameters s, m as described in 5.6 exists if and only if there exits a $(1 + ms - m)$-set of sym-*

metric matrices M_i *of order* $s^2(1 + ms - m)$ *which satisfies*

$$M_i^2 = s(1 + ms - m)M_i, \quad \text{and} \quad m_{jj}^i = 1 \quad \text{for all} \quad i, j\,; \tag{5.12}$$

$$\textstyle\sum_i M_i^2 = s(1 + ms)(1 + ms - m)J + s^2(ms - m + 1)^2 I\,. \tag{5.13}$$

Proof. Apply 5.6 and the proof of 5.10.

COROLLARY 5.12. *Assume the existence of* $r - 2$ *mutually orthogonal Latin squares of order* s. *Then for all* $\mu \geqq 1$ *there exists* μ*-set of* r *symmetric matrices* $s^2\mu$ *which satisfies* (5.10).

Proof. Apply 5.9 and 5.10.

6. Uniform c**-**K**-structures.** In this section we consider two special classes of c-K-structures: pre-uniform and uniform c-K-structures. These are generalizations of the notion of uniformity for ordinary PH-planes. The existence of pre-uniform, resp., uniform c-K-structures over given gross structures is equivalent to the existence of certain $(s, r; c)$-nets, resp., ARPP's (Theorem 6.14, Proposition 6.19, Corollary 6.20).

DEFINITION 6.1. A c-K-structure Π with parameter $t \neq c$ is called *pre-uniform* provided that

(6.1) $p \sim q$, $G \sim H$, pIH and p, qIG always imply qIH.

Π is called *uniform* (of *index* λ) if, for some natural number λ, the following property also holds:

(6.2) if $p \sim q$, $p \neq q$ and $[p, q] \neq 0$, then $[p, q] = \lambda t$ and dually.

Property (6.1) is one of the standard definitions for uniformity in the case of PH-planes; in this special case, (6.2) is automatically satisfied with $\lambda = 1$ (cf. Proposition 6.2).

PROPOSITION 6.2. *A pre-uniform K-structure is uniform of index* 1.

Proof. Let $p \sim q \neq p$ and $[p, q] \neq 0$, say p, qIG. There are precisely t neighbor lines of G through p (which by (6.1) all contain q). Since Π is a 1-K-structure, nonneighbor lines intersect in at most one point of a given point neighborhood. Hence a line H with $H \not\sim G$ cannot contain both p and q. Thus $[p, q] = t$; and by duality, $[G, H] = t$ or 0 for $G \sim H \neq H$.

PROPOSITION 6.3. *The classes of pre-uniform and uniform c-K-*

structures are self-dual and thus satisfy the principle of duality. Again we refrain from writing dual statements of the results obtained.

DEFINITION 6.4. Let Π be a c-K-structure, p be a point of Π. Denote the set of all points neighbor to p by p'. For any line G, let $G(p)$ be the set of all points on G which are in p'. Let $\mathfrak{B}(p)$ consist of all $G(p)$ with $|G(p)| \neq 0$. Then we call $\Pi(p) := (p', \mathfrak{B}(p), \varepsilon)$ the *incidence structure induced in the neighborhood* of p. $\Pi(G)$ is defined dually.

PROPOSITION 6.5. *Let Π be a c-K-structure with parameter $t \neq c$. Then the following assertions are equivalent:*

(i) *Π is pre-uniform;*

(ii) *c divides t (write s for t/c), and the incidence structure $\Pi(p)$ induced in the neighborhood of p is an $(s, [p']; c)$-net for all points p of Π.*

Proof. (i) $\Rightarrow$ (ii): Let Π be pre-uniform. Call the induced lines $G(p)$ and $H(p)$ parallel if and only if $G \sim H$. As Π is pre-uniform, each point of $\Pi(p)$ is on precisely one line of each parallel class. Each line $G(p)$ contains precisely t points, and $\Pi(p)$ has exactly t^2/c points; thus each of the $[p'] \geqq 3$ parallel classes consists of $t/c = s$ lines. Finally, lines from distinct parallel classes intersect in precisely c points of $\Pi(p)$, since Π is a c-K-structure. Thus $\Pi(p)$ is an $(s, [p']; c)$-net.

(ii) $\Rightarrow$ (i). Now let $\Pi(p)$ be an $(s, [p']; c)$-net for all points p of Π. Let $p \sim q$, $G \sim H$, pIH and p, qIG. We want to show that $G(p) = H(p)$, hence that qIH. We assert that lines from different neighbor classes induce lines from different parallel classes in $\Pi(p)$. Thus let $K \not\sim L$ and $[K(p), L(p)] \neq 0$. As $K', L'Ip'$, there are precisely c points $r \sim p$ with rIK, L. So $[K(p), L(p)] = c < t$; thus $K(p)$ and $L(p)$ are not parallel by (5.1). Hence the $[p']$ neighbor classes of lines incident with p already induce at least $[p']$ different parallel classes of $\Pi(p)$. As $\Pi(p)$ has precisely $[p']$ parallel classes, we must have $K(p) \| L(p)$ whenever $K \sim L$. Hence, in particular, $G(p) \| H(p)$; and thus by (5.1), $G(p) = H(p)$.

COROLLARY 6.6. *Let Π be a c-K-structure with parameter $t \neq c$. Then the following assertions are equivalent:*

(i) *Π is uniform of index λ;*

(ii) *c divides t (let s denote t/c), and $\Pi(p)$ is an $(s, [p']; c)$-ARPP of index λ for all points p of Π.*

COROLLARY 6.7. *Let Π be a K-structure with parameter $t \neq 1$. Then the following assertions are equivalent*:

(i) *Π is uniform.*

(ii) *$\Pi(p)$ is a $(t, [p']; 1)$-net for all points p.*

Proof. Apply Propositions 6.5 and 6.2.

Corollary 6.7 was proved in [12, Theorem 2.40] in the special case that Π is a *PK*-plane.

COROLLARY 6.8 (Lüneburg [17, Sätze 2.12, 2.13]). *Let Π be a (t, r)-PH-plane with $t \neq 1$. Then the following assertions are equivalent*:

(i) *Π is uniform.*

(ii) *$\Pi(p)$ is an affine plane for all points p.*

(iii) *$t = r$.*

COROLLARY 6.9 (Jungnickel [12, Corollary 2.42]). *Let Π be an (r, r)-PK-plane. Then Π is in fact a uniform PH-plane.*

Proof. It is well-known (see, e.g., [7, Proposition 2.6]) that the average number of lines joining a point p with a neighbor point $q \neq p$ is $t(r+1)/(t+1)$, i.e., here (where $t = r$) r. Since $p \sim q \neq p$ implies $[p, q] \leqq t = r$, we have $[p, q] = r \geqq 2$. By duality, Π is a *PH*-plane. By Corollary 6.8, Π is uniform.

PROPOSITION 6.10. *Let Π be a uniform c-K-structure with parameter $t \neq c \neq 1$. Then Π' is in fact a tactical configuration with $k = r = (\lambda - 1)(t - 1)/(c - 1) + 1$.*

Proof. By Corollary 6.6, c divides t; and $\Pi(p)$ is an $(s, [p']; c)$-ARPP of index λ for all points p of Π (with $s = t/c$). Let r denote $[p']$. Then Corollary 5.4 asserts that $\lambda(sc - 1) = sc - 1 + (r - 1)(c - 1)$. Hence r is indeed independent of p and is given by the formula in the assertion. A dual argument proves the assertion for lines.

DEFINITION 6.11. A *c-K*-structure will be called a *c-H-structure* provided that

$$p \sim q \quad \text{implies} \quad [p, q] > c\,, \tag{6.3}$$

$$G \sim H \quad \text{implies} \quad [G, H] > c\,. \tag{6.4}$$

(Cf. Definition 3.4.) A 1-*H*-structure will, of course, be abbreviated to *H-structure*.

PROPOSITION 6.12. *Let Π be a neighbor cohesive, uniform c-K-structure with parameter $t \neq c$. Then Π is a c-H-structure, and Π' is a tactical configuration with $k = r = (s^2c - 1)/(s - 1)$ (where s denotes t/c). Furthermore, either $c = 1$ (and $s = r - 1$) or $s \leqq (1/2)[-1 + \sqrt{(4r-3)}] < \sqrt{r}$.*

Proof. As Π is neighbor cohesive and uniform, we have $[p, q] = \lambda t \geqq t > c$ for any two neighbor points p, q, and dually. Thus Π is a c-H-structure. By Corollary 6.6 and Proposition 5.7, $\Pi(p)$ is a complete $(s, r; c)$-net for the appropriate r for all points p of Π. Hence by Proposition 5.3, $r = (s^2c - 1)/(s - 1) = k$. Clearly $s = r - 1$ if $c = 1$. By Propositions 5.7 and 5.6, there exists an integer m with $r = 1 + s + ms^2$. Thus for $c \neq 1$, one has $s^2 \leqq r - 1 - s$, which yields the desired conclusion.

EXAMPLE 6.13.

(a) Let a tactical configuration Π' with $k = r = 11$ be given. If Π' is the gross structure of some neighbor cohesive, uniform c-K-structure for which $t \neq c$, then by 6.12, either $c = 1$ (hence $s = 10$, i.e., by 6.10, the induced structures $\Pi(p)$ would be affine planes of order 10) or $s \leqq 3$; in fact, the only possible pair for (s, m) then is (2, 2) (which is conceivable by 5.8 and can, in fact, be realized, as will be shown in Corollary 6.20).

(b) Similarly, assume $k = r = 22$. Either $s = 21$, $c = 1$ (corresponding to an affine plane of order 21, which does not exist by the Bruck-Ryser theorem [4]); or $s \leqq 4$. The only possible solution pair (s, m) would be (3, 2) (note that $s \geqq 2$ by 6.6, since $t = sc$ and $t \neq c$), which is excluded by 5.8 (iii). Hence no uniform c-H-structure Π over a tactical configuration Π' with $k = r = 22$ exists.

THEOREM 6.14. *Let Π' be a connected incidence structure with at least 3 points per line and at least 3 lines per point. Define*

$$(6.5)\qquad r := \max\{[p'] : p' \in \Pi'\} \quad \text{and} \quad k := \max\{[G'] : G' \in \Pi'\}\,.$$

Assume that $r \geqq k$. Then there exists a pre-uniform c-K-structure Π with gross structure Π' and parameter $t \neq c$ if and only if there exists a $(t/c, r; c)$-net.

Proof. The necessity is given by Proposition 6.5. Now assume the existence of an $(s, r; c)$-net Σ with $s = t/c$. Construct from Σ a c-set of $(s^2c \times s^2c)$-matrices $M_1, \cdots, M_r$ as in the proof of Proposition 5.10. Let B be an incidence matrix for Π'; then each line sum of B is at most r. Decompose B as the sum of r matrices $P_1, \cdots, P_r$ having at most one 1 per line (cf. [18, Theorem 11.1.6]). Replace each 1 in P_i by M_i and each 0 by a zero matrix. Let A be the sum of these enlarged matrices, and take A as an incidence matrix of Π. Then Π is a c-K-structure over Π' if neighborhood is defined in the obvious way. That Π is pre-uniform and has parameter $t = sc$ follows from the fact that the M_i's are constructed from Σ. We leave the details of the proof to the reader.

COROLLARY 6.15. *Let Π' satisfy the conditions of Theorem* 6.14. *In addition, assume that* $7 \geqq r \geqq k \geqq 3$. *Then, for all* $s, c \geqq 2$, *there exists a pre-uniform c-K-structure over Π' with parameter* $t = sc$.

Proof. Apply 5.5(d) and Theorem 6.14.

COROLLARY 6.16. *Let Π' be as in* 6.14, *k and r as in* (6.5). *Then there exists a uniform K-structure Π with parameter t and gross structure Π' if and only if there exists a* $(t, r; 1)$-*net.*

COROLLARY 6.17 (Drake/Lenz [7, Theorem 3.1], Jungnickel [12, Theorem 4.22]). *Let Π' be a projective plane of order r. Then the following assertions are equivalent*:

(i) *there exists a (t, r)-PK-plane Π with gross structure Π'*;

(ii) *there exists a $(t, r + 1; 1)$-net*;

(iii) *there exists a uniform (t, r)-PK-plane Π with gross structure Π'.*

Proof. By [7, Proposition 2.8], the truth of (i) implies that of (ii).

COROLLARY 6.18. *Assume the existence of a (t, r)-PH-plane with $t \neq r$. Then there also exists a uniform (t, r)-PK-plane which is not a PH-plane.*

Proof. Apply Corollaries 6.17 and 6.8.

PROPOSITION 6.19. *Let Π' be a connected incidence structure with at least* 3 *points per line and dually. Let* $c \neq 1$. *Then there exists a c-K-structure Π over Π' with parameter $t \neq c$ which is uniform of index λ if and only if*: *Π' is a tactical configuration with* $k = r = [(\lambda - 1)(t - 1)/(c - 1)] + 1$, *and there exists a $(t/c, r; c)$-$\widetilde{A}RPP$ Σ.*

Proof. The necessity is given by 6.6 and 6.10. Construct Π from Σ as in the proof of 6.14. Then Π is pre-uniform with the desired parameter t. Note that in this special case all line sums of B are r; hence, the P_i's are permutation matrices, and each M_i has been inserted into each line of B precisely once. Now let p, q be two joined neighbor points of Π; then p, q correspond to the same point of Π', i.e., to the same row of B. As each M_i has been used exactly once in that row and the M_i's come from an ARPP of index λ, p and q will be joined by precisely λt lines. The dual argument for neighbor lines completes the proof that Π is uniform.

COROLLARY 6.20. *Let Π' be a connected incidence structure with at least 3 points per line and dually. Then there exists a uniform c-H-sturucture Π with gross structure Π' and parameter $t \neq c$ if and only if*

(i) *c divides t (let s denote t/c),*

(ii) *Π' is a tactical configuration with parameters $k = r = (s^2c - 1)/(s - 1)$,*

(iii) *there exists an ARBD with parameters s and $\mu = c$.*

Proof. The necessity is given by 6.6 and 6.12. Conversely, construct Π as in the proofs of 6.14 and 6.19 to obtain a uniform c-K-structure with the desired parameters. Since we begin with an ARBD, $\Sigma M_i M_i^T \geqq J$ holds; hence Π is neighbor cohesive; hence, by Proposition 6.12, Π is a c-H-structure.

The following result on partial designs is similar to Propositions 2.16 and 2.18.

PROPOSITION 6.21. *Let (ϕ, Π, Π') be a uniform c-H-structure with parameter t. Suppose that Π' is a partial design on d classes. Then Π is a partial design on $d+1$ classes. If Π' is divisible or symmetric, so too is Π.*

Proof. Let the parameters of Π' be $n_0, \cdots, n_{d-1}$; $\lambda_0, \cdots, \lambda_{d-1}$; p_{ij}^h. Then Π has the following parameters:

(6.6) $$m_i = n_i t^2 \quad \text{for all} \quad i < d\,, \qquad m_d = t^2 - 1\,;$$

(6.7) $$q_{ij}^h = p_{ij}^h \cdot t^2 \quad \text{for} \quad h, i, j < d\,, \qquad q_{ii}^d = m_i \quad \text{for} \quad i < d\,, \qquad q_{di}^i = t^2 - 1 \quad \text{for} \quad i < d\,, \qquad q_{dd}^d = t^2 - 2\,;$$

(The remaining q_{ij}^h are 0.)

(6.8) $$\mu_i = c\lambda_i \quad \text{for} \quad i < d\,,$$

$\mu_d = \lambda t = \dfrac{sc-1}{s-1}\cdot t$ (where we have written s for t/c. By 6.5 and 5.2, $\Pi(p)$ is an $(s, [p']; c)$-ARBD for each p in Π. That $\lambda = (sc-1)/(s-1)$ now follows from 5.6).

EXAMPLES 6.22.

(i) Using 6.20, 5.8, and 6.21, one can duplicate the first two steps of the construction given in Examples 4.10. Specifically, one begins with a symmetric design Π' whose parameters are $(v, k, \lambda) = (15, 7, 3)$. Using $s = c = 2$, one obtains a uniform 2-H-structure Π'' over Π' whose parameters are $(v, k, \lambda_0, \lambda_1) = (120, 28, 6, 12)$; then (with $c = 1$, $s = 27$) a uniform 1-H-structure Π over Π'' whose parameters are $(v, k, \lambda_0, \lambda_1, \lambda_2) = (120\cdot 27^2, 28\cdot 27, 6, 12, 27)$.

(ii) Let Π' be a (23, 11, 5)-design. By 5.8(iii), there exists an ARBD with $s = 2$ and $m = 2$, i.e., with $\mu = 3$. Then $(s^2\mu - 1)/(s-1) = 11$. Thus, by 6.20, one obtains a uniform 3-H-structure Π (over Π') whose parameter $t = s\mu = 6$. By 6.21, Π is a symmetric divisible partial design with parameters $(v, k, \lambda_0, \lambda_1) = (276, 66, 15, 30)$.

7. Regular c-K-structures.

We now introduce difference methods; equivalently, we restrict our study to "regular" c-K-structures, these being generalizations of the regular PK-planes introduced in [13]. There the gross structure was assumed to be a cyclic projective plane; here we need suitable gross structures which may be described by difference methods. The gross structures of the c-K-structures introduced in §1 had only to satisfy a few comparatively weak properties (cf. (1.7)); the main assumption to be added here is the existence of a nice automorphism group. The resulting gross structures ("regular" incidence structures) can all be described by generalizations of ordinary difference families. The main result of this section will be an analogue of Theorem 3.9, yielding the existence of many regular symmetric divisible partial designs.

DEFINITION 7.1. A finite incidence structure $\Omega = (\mathfrak{P}, \mathfrak{B}, I)$ is called *regular* if

(7.1) Ω is connected, and each block contains at least 3 points;

(7.2) Ω admits an abelian automorphism group Z acting regularly on $\mathfrak{P}$ and semiregularly on $\mathfrak{B}$.

This definition is quite general: it includes, for example, cyclic projective planes, regular block designs, regular PK-planes (as in [13]) and the "regular planes of index λ" of [14].

DEFINITION 7.2. Let Z be an abelian group. A *generalized* *difference family* $\mathfrak{D}$ in Z is a collection of subsets $D_1, \cdots, D_u$ of Z satisfying

(7.3) $|D_i| \geqq 3$ for $i = 1, \cdots, u$;

(7.4) Z is generated by $\Delta_1 \cup \cdots \cup \Delta_u$ where Δ_i is the set of differences from D_i; i.e.,

$$\Delta_i = \{d_{im} - d_{in} : d_{im}, d_{in} \in D_i\} .$$

We denote by $k(\mathfrak{D})$ the number of elements in the D_i's minus 1, i.e., $k(\mathfrak{D}) := \sum_{i=1}^{u} |D_i| - 1$.

As examples, we mention ordinary difference sets and (v, k, λ)-difference families (cf., e.g., [9] and [23]), the (t, r)-difference sets of [13] and the $(v, k, \lambda; u)$-difference families of [14].

PROPOSITION 7.3. *Let $\mathfrak{D} = \{D_1, \cdots, D_u\}$ be a collection of nonempty subsets of a finite abelian group Z. One obtains an incidence structure $\Omega = (\mathfrak{P}, \mathfrak{B}, I)$ from $\mathfrak{D}$ as follows: $\mathfrak{P} = Z$, $\mathfrak{B} = \{D_i + x: i = 1, \cdots, u; x \in Z\}$, $I \equiv \varepsilon$. (Here blocks $D_i + x$ and $D_j + y$ are to be considered distinct unless the pairs (i, x) and (j, y) are identical: in particular, repeated blocks may occur.) Then Ω is regular if and only if $\mathfrak{D}$ is a generalized difference family. Every regular incidence structure Ω may be described in the preceding manner.*

We will not give a detailed proof, as the arguments needed are largely routine. To help the reader, however, we indicate here the method of constructing a suitable $\mathfrak{D}$ from a regular Ω. One fixes a point p and gives to each point x of Ω the coordinate z from Z if $(p)z = x$. Since Z acts regularly on the points of Ω, the union of the blocks in any block orbit covers the points of Ω. Number the block orbits by $1, 2, \cdots, u$; from orbit i, select any block B which contains p, and put D_i equal to the set of all coordinates of points in B.

For future use, we cite the following well-known result. For a proof, see e.g., [23, Lemma 2].

LEMMA 7.4. *A regular incidence structure is a (v, k, λ)-design if and only if (in the representation of Proposition 7.3) $\mathfrak{D}$ is a (v, k, λ)-difference family*; i.e., *$|D_i| = k$ for $i = 1, \cdots, u$ and each nonzero $x \in Z$ has precisely λ representations $x = d_{im} - d_{in}$.*

We are now in a position to generalize the notion of regularity for PK-planes (see [13]) to arbitrary c-K-structures.

DEFINITION 7.5. Let Π be a c-K-structure. Π is called *regular* if it admits an abelian collineation group $G = Z \oplus N$ such that

(7.5) Π' is regular with respect to Z;

(7.6) N acts regularly on the elements (points or lines) of each neighbor class of Π.

If Π has parameter t and Π' is a (v, k, λ)-design, we shall sometimes call Π a *regular* c-$(t; v, k, \lambda)$-*K-design*. As usual, we omit the prefix c- when $c = 1$.

One notes that G acts regularly on the points of Π and semi-regularly on the lines of Π.

DEFINITION 7.6. Let $G = Z \oplus N$ be an abelian group of order vt^2/c. Then a family $\mathfrak{D} = (D_i)$, $i = 1, \cdots, u$, of subsets D_i of G, where $D_i = \{(d_{ih}, d_{ihm})\colon h = 1, \cdots, k_i; m = 1, \cdots, t\}$, each $k_i \geqq 3$, is called a *generalized* (c, t)-*difference family* if the following two conditions are satisfied:

(7.7) $\mathfrak{D}' := \{D_i'\colon i = 1, \cdots, u\}$ with $D_i' := \{d_{ih}\colon h = 1, \cdots, k_i\}$ is a generalized difference family in Z;

(7.8) if (i, h) and (j, k) are unequal ordered pairs, then every element y of N has precisely c representations of the form $y = d_{ihm} - d_{jkn}$.

$\mathfrak{D}$ is called *cohesive* if furthermore

(7.9) for each nonzero $y \in N$, there is at least one representation $y = d_{ihm} - d_{ihn}$.

If for each nonzero $y \in N$, there are at least $c + 1$ such representations of y, $\mathfrak{D}$ is called *special*. If $\mathfrak{D}'$ is an ordinary (v, k, λ)-difference family, we call $\mathfrak{D}$ a c-$(t; v, k, \lambda)$-*difference family*. Again, the prefix c- will be omitted for $c = 1$.

PROPOSITION 7.7. *Let $\phi\colon \Pi \to \Pi'$ be an incidence structure epimorphism. Then (ϕ, Π, Π') is a regular c-K-structure with parameter t if and only if it can be described as follows:*

(i) *$\mathfrak{P} = G = Z \oplus N$ for some abelian group G;*

(ii) *$\mathfrak{B} = \{D_i + (x, y)\colon i = 1, \cdots, u; (x, y) \in G\}$ where $\mathfrak{D} = \{D_1, \cdots, D_u\}$ is a generalized (c, t)-difference family;*

(iii) *$I \equiv \varepsilon$;*

(iv) *Π' is a regular incidence structure relative to $\mathfrak{D}'$;*

(v) *ϕ is defined by $(x, y)^\phi = x$, $(D_i + (x, y))^\phi = D_i' + x$. (Again blocks $D_i + (x, y)$ and $D_j + (x', y')$ are to be considered distinct un-*

less the ordered triples (i, x, y) *and* (j, x', y') *are identical.*)

Π *is a* c-$(t; v, k, \lambda)$-*K-design if and only if* $\mathfrak{D}$ *is a* c-$(t; v, k, \lambda)$-*difference family. Furthermore,* Π *is neighbor point cohesive if and only if* $\mathfrak{D}$ *is cohesive, and* Π *satisfies* (6.3) *if and only if* $\mathfrak{D}$ *is special. Finally, if* $u = 1$, *then* Π *is neighbor cohesive, resp., a c-H-struc-ture if and only if* $\mathfrak{D}$ *is cohesive, resp., special.*

Proof. Let $\mathfrak{D}$ be a generalized (c, t)-difference family, (ϕ, Π, Π') be constructed from $\mathfrak{D}$ in the manner indicated. Clearly, properties (7.5) and (7.6) hold. Thus it suffices to show that ϕ is a c-K-epimorphism. Then let $(x, y)^\phi \neq (x', y')^\phi$, i.e., $x \neq x'$, and assume that $x, x'\ I(D_i' + a)$, say $x = d_{ih} + a$, $x' = d_{ik} + a$. Then (x, y), $(x', y')I(D_i + (a, b))$ if and only if there exist indices m, n such that $y = d_{ihm} + b$, $y' = d_{ikn} + b$; i.e., precisely when indices m, n exist such that $d_{ihm} - d_{ikn} = y - y'$. By (7.8), there are c respresentations of this form, and the verification of (1.4) is complete. The proof of the dual double flag lifting property is similar. Then Π is a regular c-K-structure (which clearly has the desired parameter t).

To prove the converse, one constructs $\mathfrak{D}$ from (ϕ, Π, Π') as in the proof of Proposition 7.3. Clearly, $\mathfrak{P}$, $\mathfrak{B}$, I satisfy (i), (ii), and (iii), through we still must verify that $\mathfrak{D}$ is a generalized (c, t)-difference family. By (7.5) and (7.6), we may so "coordinatize" Π' that (iv) and (v) are satisfied. By 7.3, $\mathfrak{D}'$ is a generalized difference family; i.e., $\mathfrak{D}$ satisfies (7.7). Let (d_{ih}, d_{ihm}) be any element of any D_i. Then (7.6) and Theorem 1.25 imply that D_i contains precisely t elements (d_{ih}, d_{ihx}) and that $|N| = t^2/c$. Thus it suffices to verify (7.8) in order to conclude that $\mathfrak{D}$ is a generalized (c, t)-difference family. Suppose then that $(i, h) \neq (j, k)$ and y are given. Then $0 \in (D_i' - d_{ih}) \cap (D_j' - d_{jk})$. Since these two lines are unequal, there are precisely c pre-images of 0 (i.e., points of the form $(0, z)$) in the intersection of $D_i + (-d_{ih}, 0)$ and $D_j + (-d_{jk}, y)$. For each of these c points, $z = d_{ihm} = d_{jkn} + y$ for some m, n. This yields c representations for y of the form $d_{ihm} - d_{jkn}$, and there are no others.

The assertion on c-K-designs follows from Lemma 7.4. Now consider neighbor points (x, y) and (x, y'). They will be joined by a line $D_i + (a, b)$ if and only if $x = d_{ih} + a$, $y = d_{ihm} + b$, $y' = d_{ihn} + b$ for some indices h, m, n; i.e., if and only if $y - y' = d_{ihm} - d_{ihn}$. Then each pair of neighbor points will have some joining line $D_i + (a, b)$ if and only if (7.9) holds; i.e., if and only if $\mathfrak{D}$ is cohesive. Now consider neighbor lines, and assume that $u = 1$. Let the single difference set in $\mathfrak{D}$ be denoted by $D = \{(d_h, d_{hm})\}$. Then neighbor lines $D + (x, y)$ and $D + (x, y')$ will possess a common point (a, b) if and only if $a = d_h + x$, $b = d_{hm} + y = d_{hn} + y'$ for some indices h,

m, n; i.e., if and only if $y - y' = d_{hn} - d_{hm}$ for some h, m, n; i.e., if and only if $\mathfrak{D}$ is cohesive. The arguments for the case that $\mathfrak{D}$ is special are similar.

REMARK 7.8. If $u > 1$, then (7.9) is not sufficient to imply neighbor line cohesiveness. The reader may wonder why we did not modify (7.9) so that the restriction $u = 1$ would be unnecessary in the statement of 7.7. This would also necessitate a suitable modification of (7.12) below to assure the validity of Proposition 7.10. We have not done so, because we have been unable to find nontrivial examples of the so modified "K-matrices." Furthermore, in the case $c = 1$, no such examples can exist (as follows from [15, Lemma 2.5]).

DEFINITION 7.9. Let N be an abelian group of order t^2/c. A *c-(t, k)-K-matrix* over N is a matrix $A = (a_{ij})$ $(i = 0, \cdots, k;\ j = 1, \cdots, t)$ with entries from N such that the following conditions are satisfied.

(7.10) If $a_{ih} = a_{im}$, then $h = m$.

(7.11) For each pair (i, j) with $i \neq j$ and $i, j \in \{0, \cdots, k\}$, the list of differences $a_{ih} - a_{jm}$ $(h, m = 1, \cdots, t)$ contains each element of N exactly c times.

A is called *cohesive* (briefly, a *c-(t, k)-CK-matrix*) if furthermore

(7.12) each nonzero element of N occurs at least once among the $(k + 1)t(t - 1)$ differences $a_{ih} - a_{im}$ with $h \neq m$.

If each nonzero element of N occurs at least $c + 1$ times in this way, A is called a *c-(t, k)-H-matrix*. When $c = 1$, we write simply *(t, k)-K-matrix* or *(t, k)-H-matrix* as in [13]. When we do not wish to express t and k, we shall abbreviate to *c-K-matrix* or *c-H-matrix* and, if $c = 1$, to *K-matrix* and *H-matrix*.

PROPOSITION 7.10. *Let $\mathfrak{D}$ be a generalized (c, t)-difference family in $G = Z \oplus N$, $\mathfrak{D}'$ be the associated generalized difference family in Z (see (7.6)), k denote $k(\mathfrak{D}')$ (see 7.2). Then the $k + 1$ sets $\{d_{ihm}$: $m = 1, \cdots, t\}$ $(i = 1, \cdots, u;\ h = 1, \cdots, k_i)$ form a c-(t, k)-K-matrix A over N. Conversely, given a generalized difference family $\mathfrak{D}'$ in Z with $k = k(\mathfrak{D}')$ and a c-(t, k)-K-matrix A over N, one obtains a generalized (c, t)-difference family $\mathfrak{D}$ in $G = Z \oplus N$ with sets D_i as follows: for $1 \leqq i \leqq u$, $D_i := \{d_{ih} \times A_{ih}: h = i, \cdots, k_i\}$, where $\{A_{ih}\}$ is the set of rows of A.*

$\mathfrak{D}$ is cohesive if and only if A is a c-CK-matrix; $\mathfrak{D}$ is special if and only if A is a c-H-matrix.

The proof is obvious from the definitions.

EXAMPLES 7.11.

(i) Assume the existence of a generalized difference family $\mathfrak{D}$ with $k = k(\mathfrak{D})$. Let t be a natural number for which $k \leqq q_i$ for all i, where $t = q_1 \cdots q_n$ is the prime power factorization of t. Then there exists a (t, k)-K-matrix ([13, Corollary 3.4]) and thus a regular K-structure with parameter t.

(ii) Assume the existence of a (v, k, λ)-difference set. There exists a $(t, k-1)$-H-matrix and hence a regular $(t; v, k, \lambda)$-H-design in at least the following cases:

(a) $t = (k-1)^n$, where $k-1$ is a prime power [13, Corollary 4.3];

(b) $t = q^n(k-1)$, where q and $k-1$ are prime powers with $2k \leqq q+1 \leqq k(k-1)$ [13, Corollary 5.8];

(c) $(t, k-1)$ a special Lenz pair (cf. Definition 2.13) [15, Theorem 2.17].

For further existence results on K- and H-matrices the reader should consult [13] and [16].

(iii) Applying (ii) to an (11, 5, 2)-difference set (a regular biplane), we get regular $(4^n; 11, 5, 2)$-H-designs and regular $(4q^n; 11, 5, 2)$-H-designs for $q = 9, 11, 13, 17, 19$ (cf. Examples 4.7). Similarly, a (19, 9, 4)-difference set can be used to obtain regular $(8^n; 19, 9, 4)$-H-designs.

PROPOSITION 7.12. *Assume the existence of a (t, r)-K-matrix A. Then there exists a c-(ct, r)-K-matrix for any natural number c. If A is a CK-, resp., an H-matrix, then there also exists a c-(ct, r)-CK-, resp., a c-(ct, r)-H-matrix.*

Proof. Let A be the given matrix (with entries from N, say) and M be any abelian group of order c. If row i of A is A_i, let row i of the new matrix B be $M \times A_i$. It is easily checked that B has the desired properties.

COROLLARY 7.13. *Let t be a natural number, $t = q_1 \cdots q_n$ the prime power factorization of t. If $r \leqq q_i$ for $i = 1, \cdots, n$, then there exists a c-(ct, r)-K-matrix for all natural numbers c.*

Proof. Apply 7.12 and [13, Corollary 3.4].

We now want to consider regular balanced K-structures. We give a definition which is taken from [15, Definition 2.6], but is here extended to the case of K-matrices rather than H-matrices.

DEFINITION 7.14. Let $A = (a_{ik})$ be a (t, r)-K-matrix over N. A is called *balanced of type n* if there exist distinct subgroups $E = U_n < U_{n-1} < \cdots < U_2 < U_1 = N$ of N such that the following conditions are satisfied:

(7.13) there are distinct natural numbers λ_i $(i = 1, \cdots, n-1)$ such that each x in $U_i \backslash U_{i+1}$ occurs precisely λ_i times as a difference of the form $a_{hj} - a_{hk}$;

(7.14) there are natural numbers $q_2, \cdots, q_n$ such that $|U_i| = (q_n q_{n-1} \cdots q_{i+1})^2 =: u_i^2$;

(7.15) for all $i = 1, \cdots, n-1$; $j = 0, \cdots, r$; $k = 1, \cdots, t$; $|\{a_{jm} : a_{jm} \equiv a_{jk} \bmod U_i\}| = u_i$.

PROPOSITION 7.15.

(i) *Let A be a (t, r)-K-matrix, balanced of type n. Let $t_i := q_2 q_3 \cdots q_i$. Then there is a (t_i, r)-K-matrix A_i that is balanced of type i, obtained from A by identifying elements* mod U_i $(i = 1, \cdots, n-1)$.

(ii) *If A is a (t, r)-K-matrix, balanced of type i, then $q_2 = t_2 = \lambda_1 = r$. Hence $r | t$. Also,*

$$\lambda_i = \frac{(r+1)q_2 \cdots q_{i+1}}{1 + q_{i+1}} \, ; \quad \textit{so} \quad \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} \, .$$

Hence A is in fact an H-matrix.

(iii) *r is a prime power.*

Proof. The arguments in [15, 2.7–2.11] hold almost without change to prove (i) and (ii). Then (iii) is just [15, Corollary 3.3].

PROPOSITION 7.16. *Let (ϕ, Π, Π') be a regular neighbor cohesive K-structure with $u = 1$* (cf. *Proposition* 7.7). *Then Π is balanced if and only if its K-matrix A is balanced* (cf. *Propositions* 7.7 *and* 7.10); *in this case, Π is an H-structure and Π' is a tactical configuration with $k = r$.*

Proof. Routine modifications in the proofs of [15, 2.12 and 2.13] yield the equivalence of balance in Π and in A. For the convenience of the reader, however, we indicate here the method of constructing the K-structures Π_j $(j = 1, \cdots, n)$ from the matrices A_j (obtained in 7.15): if D is the "base line" of $\Pi =: \Pi_n$, say $D = \{(d_i, d_{ih})\}$, one takes the sets $D^{(j)} := \{(d_i, d_{ih} + U_j)\}$ as base line for Π_j, and the development occurs with respect to the group $Z \oplus N/U_j$. To obtain the remaining conclusions of 7.16, one applies 7.15(ii) and Corollary 2.8.

DEFINITION 7.17 (See [15, 4.1]). Let A be a (t, r)-H-Matrix that is balanced of type n. Denote by S_{jk}^i the set $\{a_{jm} + U_{i+1}: a_{jm} \equiv a_{jk} \bmod U_i\}$. Then A is called *uniformly balanced* if the following condition holds:

(7.16) for all $i = 1, \cdots, n-1$; $j = 0, \cdots, r$; $k, k' = 1, \cdots, t$; either $S_{jk}^i - S_{jk'}^i = U_i/U_{i+1}$ or there exists an element x in U_i such that $S_{jk}^i = S_{jk'}^i + (x + U_{i+1})$.

PROPOSITION 7.18. *Let Π be a regular H-structure that is balanced of type n. Then Π is minimally uniform* (*see Definition* 2.9) *if and only if its H-matrix A is uniformly balanced.*

Proof. Make routine modifications in the proof of [15, Theorem 4.3].

THEOREM 7.19 [15, Corollary 4.10]. *There exists a uniformly balanced (t, r)-H-matrix if and only if (t, r) is a special Lenz pair.*

THEOREM 7.20. *Let Π' be a regular incidence structure with* $u=1$ (cf., 7.3). *There exists a regular, balanced, minimally uniform H-structure Π with parameter t over Π' if and only if $(t, k-1)$ is a special Lenz pair* (*where k is the cardinality of the difference set for Π'*).

Proof. Apply 7.19, 7.16, and 7.18.

We urge the reader to compare the preceding result with Theorem 3.9. Regarding partial designs, we obtain

COROLLARY 7.21. *Assume the existence of a (v, k, λ)-difference set, and let $t = q_2 \cdots q_n$ be a special Lenz number of type n based on k. Let* $u_i := \prod_{j=i+1}^{n} q_j$ (cf. (2.4)), $u_n := 1$. *Then there exists a regular symmetric divisible partial design Π on n classes with parameters*

(7.17) $$n_0 = (v-1)t^2, \qquad n_i = u_i^2 - u_{i+1}^2 \quad \textit{for} \quad 1 \leqq i \leqq n-1;$$

(7.18) $$\lambda_0 = \lambda, \lambda_i = \frac{q_2 \cdots q_{i+1} k}{1 + q_{i+1}} \quad \textit{for} \quad 1 \leqq i \leqq n-1;$$

(7.19) $$p_{00}^0 = (v-2)t^2; \qquad p_{ii}^h = n_i \quad \textit{for} \quad h \geqq 1,\ h > i;$$
$$p_{hi}^i = n_h \quad \textit{for} \quad h \geqq 1,\ h > i;$$
$$p_{ii}^i = u_i^2 - 2u_{i+1}^2 \quad \textit{for} \quad i \geqq 1;$$
the remaining p_{ij}^h are 0.

In particular, such partial designs always exist for

$$v = \frac{q^{m+1}-1}{q-1}, \qquad k = \frac{q^m-1}{q-1}, \qquad \lambda = \frac{q^{m-1}-1}{q-1}$$

where $m \geqq 2$ *and* q *is a prime power, provided that* $k-1$ *is a prime power.*

Proof. The existence of the (v, k, λ)-difference set implies the existence of a regular symmetric (v, k, λ)-design Π' (i.e., a symmetric divisible partial design of class number 1) by Lemma 7.4. By Theorem 7.20, we obtain a regular balanced $(t; v, k, \lambda)$-H-design Π over Π'. By Propositions 2.16 and 2.18, Π is a regular symmetric divisible partial design on n classes. The parameters of Π follow from formulas (2.8) to (2.15) in the proof of Proposition 2.16. The last assertion is a consequence of Singer's theorem [22].

For numerical examples, the reader should compare 4.7; the parameters obtained there can be obtained here, too, but for *regular* partial designs.

8. Regular uniform c-K-structures. In this final section we combine the notions introduced in §§6 and 7 to study uniform, regular c-K-structures. In particular, we will determine the parameters of all regular pre-uniform c-K-structures with a given gross structure, subject to the condition $(c, t/c) = 1$; a comparable result is obtained for regular uniform c-H-structures. In the latter case, one has necessarily $1 = c = u$. We will also obtain some more regular symmetric divisible partial designs.

DEFINITION 8.1. Let A be a c-(t, r)-K-matrix over N. A is called *pre-uniform* if each row of A is a coset of some subgroup of N. A is called *uniform (of index γ)* if A is pre-uniform and if there exists a natural number γ such that each element $y \neq 0$ of N occurs either $t\gamma$ or 0 times as a difference of the form $a_{ij} - a_{ik}$.

One immediately obtains

PROPOSITION 8.2. *A pre-uniform (t, r)-K-matrix is uniform (of index* 1). *If A is a pre-uniform c-(t, r)-K-matrix, then c divides t. We put $s := t/c$.*

To provide examples of uniform c-H-matrices, we now prove the following result. Further examples of pre-uniform c-K-matrices will be provided by results 8.6, 8.7, and 8.8.

PROPOSITION 8.3. *Let q be a prime power, k be a natural number. Then there are uniform $q^k-(q^{k+1}, r)$-H-matrices with $r = q^{k+1} + q^k + \cdots + q$. This is the maximum r possible for a pre-uniform $q^k-(q^{k+1}, r)$-K-matrix.*

Proof. We construct the desired matrix in the elementary abelian group of order q^{k+2}. Consider the projective geometry in the $(k+2)$-dimensional vectorspace V over $GF(q)$. Take the $q^{k+1} + q^k + \cdots + 1$ hyperplanes as the rows of a matrix A. As any two of these intersect in a k-dimensional subspace of V, we obtain a $q^k - (q^{k+1}, q^{k+1} + q^k + \cdots + q)$-$K$-matrix, which is clearly pre-uniform. Now consider any $x \in V$ with $x \neq 0$. If it occurs from a given row of A at all, it has to occur precisely q^{k+1} times from this row. But x will be on precisely $q^k + \cdots + q + 1$ hyperplanes; hence A is in fact a uniform q^k-H-matrix of index $\gamma = q^k + \cdots + q + 1$.

Now assume that B is a pre-uniform $q^k - (q^{k+1}, r)$-K-matrix over some abelian group N of order q^{k+2}. We construct an $(s, r; \mu)$-net Σ in the following way: points of Σ are the elements of N, and lines of Σ are the cosets of the lines of B. Then Σ is a $(q, r+1; q^k)$-net, as is easily checked. By Proposition 5.3, $r + 1 \leqq q^{k+1} + q^k + \cdots + 1$.

THEOREM 8.4. *Let Π be a regular c-K-structure, A the c-K-matrix of Π* (cf., *Propositions* 7.7 *and* 7.10). *Then the following conditions are equivalent:*

(i) *Π is pre-uniform;*

(ii) *for each flag (p, G), there exists a subgroup U of N acting regularly on $N(p, G) := \{q: qIG, q \sim p\}$ and $N(G, p) := \{H: pIH, G \sim H\}$;*

(iii) *A is pre-uniform.*

Furthermore A is uniform of index γ if and only if Π is uniform of index γ.

Proof. Let $D_1, \cdots, D_u$ be the "base lines" of Π (cf. the construction in Proposition 7.7).

(i) $\Rightarrow$ (ii). Let (p, G) be any flag of Π. As Π is regular, we may assume that $G = D_i$ for some i and that $p = (d_{ih}, d_{ihb})$ for some h, b. Then $N(p, G) = \{(d_{ih}, d_{ihm}): m = 1, \cdots, t\}$. Let $U := \{d_{ihm} - d_{ihb}: m = 1, \cdots, t\}$. We assert that U is a subgroup of N. Consider any fixed element of U, say $d_{ihn} - d_{ihb}$. We have pID_i, $D_i + (0, d_{ihb} - d_{ihn})$. As Π is pre-uniform, we have $N(p, D_i) = N(p, D_i + (0, d_{ihb} - d_{ihn}))$ and thus $U + d_{ihb} = (U + d_{ihb}) + (d_{ihb} - d_{ihn})$, i.e., $U = U - (d_{ihn} - d_{ihb})$. As this holds for all elements of U, U is indeed a subgroup of N. Obviously, U acts regularly on $N(p, G) = N((d_{ih},$

$d_{ihb}), D_i)$. But $N(G, p)$ consists precisely of the lines $D_i + (0, y)$ where $y \in U$ (if $pI(D_i + (0, y))$, we have $d_{ihb} = d_{ihm} + y$ for some m); thus U acts regularly on $N(G, p)$ too.

(ii) $\Rightarrow$ (iii). Consider any row of A, say the row corresponding to d_{ih}, i.e., $\{d_{ih1}, \cdots, d_{iht}\}$ (cf. Proposition 7.10). Let $p := (d_{ih}, d_{ih1})$. By hypothesis, there is a subgroup U of N acting regularly on $N(p, D_i) = \{(d_{ih}, d_{ihm}): m = 1, \cdots, t\}$. As p is mapped onto the point (d_{ih}, d_{ihm}) by the element $d_{ihm} - d_{ih1}$ of N and as N is regular on the neighbors of p, we must have $U = \{d_{ihm} - d_{ih1}: m = 1, \cdots, t\}$. Thus the elements of the given row of A are a coset of the subgroup U of N, i.e., A is pre-uniform.

(iii) $\Rightarrow$ (i). Let $p \sim q$, $G \sim H$, p, qIG, pIH. As Π is regular, we may assume $G = D_i$ for some i. Then we will have $H = D_i + (0, x)$ for some $x \neq 0$, $p = (d_{ih}, d_{ihb}) = (d_{ih}, d_{ihn}) + (0, x)$ for some h, b, n and $q = (d_{ih}, d_{ihm})$ for some m. As the elements of the row of A corresponding to d_{ih} are a coset of some subgroup U of N, we will have $x = d_{ihb} - d_{ihn} \in U$. Hence $d_{ihm} - x \in U + d_{ihm}$; as $\{d_{ih1}, \cdots, d_{iht}\} = U + d_{ihm}$, we conclude $d_{ihm} - x = d_{ihp}$ for some p. Therefore $q = (d_{ih}, d_{ihm}) = ((d_{ih}, d_{ihp}) + (0, x))I(D_i + (0, x)) = H$. Thus Π is pre-uniform.

The truth of the assertion on uniformity is now clear.

LEMMA 8.5. *Let A be a pre-uniform c-(sc, r)-K-matrix. Then there exists a pre-uniform cc'-(scc', r)-K-matrix B for every natural number c'. If A is in fact a c-H-matrix, then B may also be required to be a cc'-H-matrix.*

Proof. Let G be any abelian group of order c'. Let row i of the matrix B be $A_i \times G$, where A_i is row i of A.

COROLLARY 8.6. *There are pre-uniform $q^{n+kb} - (q^{n+(k+1)b}, q^{(k+1)b} + q^{kb} + \cdots + q^b)$-$H$-matrices for all prime powers q and all natural numbers b, n.*

Proof. Apply 8.3 (writing q^b in the place of q) and 8.5 (with $c' = q^n$).

PROPOSITION 8.7. *Let $t = q_1 \cdots q_n$ be the prime power factorization of t. Then there exists a uniform (t, r)-K-matrix whenever $r \leqq q_i$ for $i = 1, \cdots, n$.*

Proof. In [13, Theorem 3.1], a (q, q)-K-matrix A was constructed for each prime power q. As q is a prime power, Singer's theorem [22] assures the existence of an ordinary difference set in the cyclic group of order $q^2 + q + 1$; thus our (q, q)-K-matrix corresponds to a (q, q)-PK-plane (cf. [13, Corollary 2.8]), which is uniform by Corollary 6.9. Hence A is uniform by Theorem 8.4. Now assume $r \leqq q$. Then by omitting $q - r$ rows of A, we obtain a uniform (q, r)-K-matrix, since $c = 1$. By the preceding argument, we are assured of the existence of uniform (q_i, r)-K-matrices for $i = 1, \cdots, n$, say $A_1, \cdots, A_n$. Let A_i^j denote row j of A_i. Let B be the matrix whose jth row is $A_1^j \times A_2^j \times \cdots \times A_n^j$. Then it is easily seen that B is a (t, r)-K-matrix (cf. [13, Theorem 3.3]). As each A_i is uniform, A_i^j will be a coset of some subgroup U_i of the group N_i over which A_i is defined; but then clearly row j of B is a coset of $U_1 \oplus \cdots \oplus U_n$; i.e., B is uniform.

COROLLARY 8.8. *Let $s = q_1 \cdots q_n$ be the prime power factorization of s. Then there exists a pre-uniform c-(cs, r)-K-matrix (for every natural number c) whenever $q_i \geqq r$ for $i = 1, \cdots, n$.*

Proof. Apply 8.7 and 8.5.

We now consider the special case where $(c, s) = 1$ and show that, in this case, the sufficient condition of Corollary 8.8 is in fact necessary.

LEMMA 8.9 (Jungnickel [15, Lemma 2.3]). *Let A be a (t, r)-K-matrix with $t \neq 1$. Then $r \leqq t$.*

PROPOSITION 8.10. *Let C be a pre-uniform cc'-$(cc'ss', r)$-K-matrix over N where $(cs, c's') = 1$. Then there exist pre-uniform c-(cs, r)- and c'-$(c's', r)$-K-matrices A and B such that row i of C is $A_i \times B_i$ (where A_i and B_i are rows i of A and B, respectively).*

Proof. N has order $s^2c \cdot s'^2c'$. As $(sc, s'c') = 1$, N splits into groups K, M of orders s^2c, s'^2c', respectively.

Consider any row C_i of C. It has $sc \cdot s'c'$ elements, and we may assume w.l.o.g. that it is a subgroup N_i of N (it is the coset of a subgroup by 8.1, and adding an element to a row of a d-K-matrix again yields a d-K-matrix). Hence C_i splits into subgroups K_i, M_i of K, M, respectively (of orders sc and $s'c'$, respectively). Let K_i be row A_i of matrix A and M_i be row B_i of matrix B. Thus $C_i = A_i \oplus B_i$. Consider any $(x, y) \in K \oplus M$, and any rows i, j of C with $i \neq j$. (x, y) occurs cc' times from $(A_i \oplus B_i) - (A_j \oplus B_j)$ by (7.11). Let x occur u times from $A_i - A_j$, and y occur v times from $B_i - B_j$. Clearly $uv = cc'$. In particular, this holds for $y = 0$.

Hence, if 0 occurs precisely v times from $B_i - B_j$, each element x of K has to occur precisely cc'/v times from $A_i - A_j$. As $A_i - A_j$ gives rise to s^2c^2 differences and as K contains s^2c elements, each $x \in K$ occurs (on the average and hence precisely) c times from $A_i - A_j$. Similarly, each $y \in M$ occurs precisely c' times from $B_i - B_j$. Thus A and B are indeed c-(cs, r)- resp. c'-$(c's', r)$-K-matrices which are obviously pre-uniform.

COROLLARY 8.11. *Let A be a pre-uniform c-(cs, r)-K-matrix with $(c, s) = 1$. Let $s = q_1 \cdots q_n$ be the prime power factorization of s. Then $r \leqq q_i$ for $i = 1, \cdots, n$.*

Proof. By repeated application of 8.10, A may be written as the "product" of a c-(c, r)-K-matrix and (q_i, r)-K-matrices. The assertion follows by Lemma 8.9.

THEOREM 8.12. *Let c, s, r be natural numbers with $(c, s) = 1$, $s = q_1 \cdots q_n$ the prime power factorization of s. Then there exists a pre-uniform c-(cs, r)-K-matrix if and only if $r \leqq q_i$ for $i = 1, \cdots, n$.*

Proof. Apply 8.8 and 8.11.

In [15, Theorem 3.1], the following result was obtained by applying Andre's work on congruence partitions [1].

COROLLARY 8.13 (Jungnickel). *A uniform (r, r)-H-matrix exists if and only if r is a prime power.*

Proof. Apply 8.12 and 8.2 to obtain the result for (r, r)-K-matrices. Then construct an (r, r)-PK-plane and apply 6.9.

The results 7.7, 7.10, 7.3, 8.12 and 8.4 together now yield one of the main results of this section.

THEOREM 8.14. *Let Π' be a regular incidence structure corresponding to the generalized difference family $\mathfrak{D}'$; let $k: = k(\mathfrak{D}')$, c, s be natural numbers with $(c, s) = 1$, $s = q_1 \cdots q_n$ be the prime power factorization of s. Then there exists a pre-uniform regular c-K-structure Π with parameter $t = sc$ over Π' if and only if $k \leqq q_i$ for all i.*

COROLLARY 8.15. *Let r be a prime power, t be a natural number with prime power factorization $t = q_1 \cdots q_n$. Then there exists* *a regular uniform (t, r)-PK-plane if and only if $r \leqq q_i$ for $i = 1, \cdots, n$.*

We urge the reader to compare 8.14 and 8.15 to 6.14 and 6.16, respectively.

THEOREM 8.16. *Let Π' be a regular incidence structure with $u = 1$ (cf. 7.3), c, s be natural numbers with $(c, s) = 1$. Then there exists a regular uniform c-H-structure Π with parameter $t = cs$ over Π' if and only if the following conditions hold*:

(i) $c = 1$ *and* $k = s + 1$ (*where k denotes the block size in Π'*);

(ii) s *is a prime power.*

Proof. Assume the existence of such an H-structure Π. By Corollary 6.20, Π' is a tactical configuration with $k = r = (s^2c - 1)/(s - 1)$. By Theorem 8.4, there is a uniform c-$(sc, k - 1)$-H-matrix corresponding to Π. If $c \neq 1$, we have $k > (s^2 - 1)/(s - 1) = s + 1$, i.e., $k - 1 > s$, which is contrary to Corollary 8.11. Hence $c = 1$ and $k = s + 1$. But then s is a prime power by Corollary 8.13.

Conversely, for all prime powers s, there is a uniform (s, s)-H-matrix by 8.13; the assertion follows by 7.10, 7.7, and Theorem 8.4.

This is the second major result of §8; the reader should compare it to Corollary 6.20. We have now obtained complete characterizations of the parameters of regular pre-uniform c-H-structures and of regular uniform c-H-structures over a given image structure for the special case $(c, s) = 1$. These results are no longer true if $(c, s) \neq 1$, as the examples in 8.3 and 8.6 show. By applying 8.10, one could get a complete characterization in the general case, if one could determine the precise value of the maximum possible r for pre-uniform $p^i - (p^{i+j}, r)$-K-matrices when p is a prime. Our Proposition 8.3 only covers the case where i is a multiple of j. The general case seems to be quite difficult.

Finally we mention that the results obtained in this section give the possibility of constructing some more regular symmetric divisible partial designs. We have

PROPOSITION 8.17. *Assume the existence of a regular symmetric (divisible) partial design Π' on d classes with parameters v' and k'. Assume furthermore the existence of a uniform c-$(sc, k' - 1)$-H-matrix A. Then there exists a regular symmetric (divisible) partial design Π on $d + 1$ classes with parameters as in* (6.6) *to* (6.8) *and*

$v = s^2cv'$, $k = sck'$.

Proof. Since Π' is symmetric, $u = 1$. Hence one may use 7.7 and 7.10 to construct a regular c-H-structure Π with parameter sc from Π' and A; Π will be uniform by Theorem 8.4. The remaining assertions follow by Proposition 6.21 and Theorem 1.25.

EXAMPLES 8.18. Start with a regular (15, 7, 3)-design, i.e., a (15, 7, 3)-difference set (cf., e.g., the tables in [9]); and use a uniform 2-(4, 6)-H-matrix (see Proposition 8.3) to obtain a regular, symmetric divisible partial design with $v = 15\cdot 8 = 120$, $k = 7\cdot 4 = 28$, $\lambda_0 = 3\cdot 2 = 6$, $\lambda_1 = 3\cdot 4 = 12$. Using 7.20 and a Lenz number based on 28, the construction can be iterated. (Cf. Examples 4.10 and 6.22.(i).)

REFERENCES

1. J. André, *Über nicht-desarguessche Ebenen mit transitiver Translationsgruppe*, Math. Z., **60** (1954), 156–186.
2. R. C. Bose, *A note on the resolvability of balanced incomplete block designs*, Sankhya **6** (1942), 105–110.
3. R. H. Bruck, *Finite nets I. Numerical invariants*, Canad. J. Math., **3** (1951), 94–107.
4. R. H. Bruck and H. J. Ryser, *The nonexistence of certain finite projective planes*, Canad. J. Math., **1** (1949), 88–93.
5. P. Dembowski, *Finite Geometries*, Berlin-Heidelberg-New York: Springer 1968.
6. D. A. Drake and D. Jungnickel, *Klingenberg structures and partial designs I: congruence relations and solutions*, J. Stat. Planning and Inference, **1** (1977), 265–281.
7. D. A. Drake and H. Lenz, *Finite Klingenberg planes*, Abh. Math. Sem. Hamburg, **44** (1975), 70–83.
8. M. Hall, *Cyclic projective planes*, Duke Math. J., **14** (1947), 1079–1090.
9. ———, *Combinatorial Theory*, Waltham-Toronto-London: Blaisdell 1967.
10. H. Hanani, *On transversal designs*, In: Proceedings of the Advanced Study Institute on Combinatorics (Breukelen 1974), Part 1, pp. 42–52. Amsterdam: Mathematical Centre 1974.
11. D. R. Hughes and F. C. Piper, *Projective Planes*, Berlin-Heidelberg-New York: Springer 1973.
12. D. Jungnickel, *Klingenberg and Hjelmslev planes*, Diplomarbeit, Freie Universität, Berlin 1975.
13. ———, *Regular Hjelmslev planes*, J. Combinatorial Theory, A., to appear.
14. ———, *Composition theorems for difference families and regular planes*, Discrete Math., to appear.
15. ———, *On balanced regular Hjelmslev planes*, Geometriae Dedicata, **241** (1978), 321–330.
16. ———, *Regular Hjelmslev planes II*, Trans. Amer. Math. Soc., to appear.
17. H. Lüneburg, *Affine Hjelmslev-Ebenen mit transitiver Translationsgruppe*, Math. Z., **79** (1962), 260–288.
18. L. Mirsky, *Transversal Theory*, New York-London: Academic Press 1971.
19. D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, New York: Wiley 1971.
20. S. S. Shrikhande, *On the nonexistence of affine resolvable balanced incomplete block designs*, Sankhya, **11** (1951), 185–186.

21. S. S. Shrikhande, *The nonexistence of certain affine resolvable balanced incomplete block designs*, Canad. J. Math., **5** (1953), 413-420.
22. J. Singer, *A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc., **43** (1938), 377-385.
23. R. M. Wilson, *Cyclotomy and difference families in elementary abelian groups*, J. Number Theory, **4** (1972), 17-47.

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