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THE NUMBER OF NONFREE COMPONENTS IN THE DECOMPOSITION OF SYMMETRIC POWERS IN CHARACTERISTIC p

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If G is the group with p (=prime) elements and k a field of characteristic p let $V_1, V_2, \cdots, V_p$ denote the indecomposable $k[G]$-modules of k-dimension $1, 2, \cdots, p$ respectively. Let $e_{n,\nu}$ denote the number of nonfree components of the decomposition of the symmetric power $S^\nu V_{n+1}$. Then the following symmetry relation is proved

$$e_{n,p-n-\nu-1} = e_{n,\nu} .$$

As a corollary we find that $S^r V_{n+1}$ has exactly one nonfree component when $n + r = p - 2$ thus solving a problem in a previous paper by R. Fossum and the author. An explicit formula for $e_{n,\nu}$ expressed in numbers of restricted partitions is obtained.

Let G be the group with p elements where p is a prime number. Let k be a field of characteristic p. Then there are p indecomposable $k[G]$-modules $V_1, V_2, \cdots, V_p$ where

$$V_n \cong k[x]/(x-1)^n .$$

Note that $V_p = k[G]$ is free and $\dim_k V_n = n$.

The symmetric power $S^\nu V_{n+1}$ taken over k is again a $k[G]$-module and can be decomposed into a direct sum of the V_i: s

$$S^\nu V_{n+1} = \bigoplus_{j=1}^{p} c_{\nu,j}(n) V_j$$

where the integer $c_{\nu,j}(n)$ is the number of times V_j is repeated. Let

$$e_{n,\nu} = \sum_{j=1}^{p-1} c_{\nu,j}(n)$$

be the number of nonfree components in $S^\nu V_{n+1}$.

If we write down these numbers in triangular form we get the following pictures where the number in the $(\nu + 1)$th place in the $(n + 1)$th row from below is $e_{n,\nu}$.

P = 11 $\searrow \nu$

1
1 1
1 1 1
1 1 1 1
1 1 2 1 1
1 1 2 2 1 1
1 1 3 3 3 1 1
1 1 2 3 3 2 1 1
1 1 2 2 3 2 2 1 1
1 1 1 1 1 1 1 1 1 1
$n \uparrow$ 1 1 1 1 1 1 1 1 1 1 1

P = 13

1
1 1
1 1 1
1 1 1 1
1 1 2 1 1
1 1 2 2 1 1
1 1 3 3 3 1 1
1 1 3 4 4 3 1 1
1 1 3 4 5 4 3 1 1
1 1 2 3 4 4 3 2 1 1
1 1 2 2 3 3 3 2 2 1 1
1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1

P = 17

1
1 1
1 1 1
1 1 1 1
1 1 2 1 1
1 1 2 2 1 1
1 1 3 3 3 1 1
1 1 3 5 5 3 1 1
1 1 4 6 7 6 4 1 1
1 1 4 6 9 9 6 4 1 1
1 1 4 6 10 10 10 6 4 1 1
1 1 3 6 9 10 10 9 6 3 1 1
1 1 3 5 7 9 10 9 7 5 3 1 1
1 1 2 3 5 6 6 6 6 5 3 2 1 1
1 1 2 2 3 3 4 4 4 3 3 2 2 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

P = 19

1																		
1	1																	
1	1	1																
1	1	1	1															
1	1	2	1	1														
1	1	2	2	1	1													
1	1	3	3	3	1	1												
1	1	3	5	5	3	1	1											
1	1	4	6	8	6	4	1	1										
1	1	4	7	10	10	7	4	1	1									
1	1	5	8	13	14	13	8	5	1	1								
1	1	4	8	13	15	15	13	8	4	1	1							
1	1	4	7	13	15	18	15	13	7	4	1	1						
1	1	3	6	10	14	15	15	14	10	6	3	1	1					
1	1	3	5	8	10	13	13	13	10	8	5	3	1	1				
1	1	2	3	5	6	7	8	8	7	6	5	3	2	1	1			
1	1	2	2	3	3	4	4	5	4	4	3	3	2	2	1	1		
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The first triangle is in [1] III. 4 (compare also Problem VI. 3.10) and the other ones are computed by using methods explained there. The symmetry of the triangles suggests the following result.

THEOREM 1. (1) $e_{n,p-n-\nu-1} = e_{n,\nu}$
(2) $e_{p-n-\nu-1,\nu} = e_{n,\nu}$
(3) $e_{n,\nu} = e_{\nu,n}$
(4) $e_{n,\nu+p} = e_{n,\nu}$.

Proof. The third relation is a consequence of

$$S^{\nu} V_{n+1} \cong S^{n} V_{\nu+1}$$

(see [1] III. 2.7b).

The fourth relation follows from (see [1] III. 2.5)

$$S^{\nu+p} V_{n+1} \cong \text{free} \oplus S^{\nu} V_{n+1} .$$

To prove (1) and (2) we are going to find a formula for $e_{n,\nu}$ or rather for the generating function

$$\eta_n(t) = \sum_{\nu=0}^{\infty} e_{n,\nu} t^{\nu} .$$

The proof is rather technical and will use the method of Fourier series. For the notation see [1] Ch. V. 4.

The number $a_{n,r}$ of *all* components of $S^r V_{n+1}$ is up to $r = p - 1$ given by the p first coefficients of

$$\tilde{\Phi}_n = \psi_n + \sum_{j=1}^{\infty} (u_{n,2pj} + u_{n,2pj+1})$$

where

$$\psi_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi) d\varphi$$

$$u_{n,j}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi) \cot \frac{\varphi}{2} (\sin j\varphi - \sin (j-1)\varphi) d\varphi$$

with

$$g_n(\varphi) = \prod_{\nu=0}^{n} (1 - te^{i(n-2\nu)\varphi})^{-1} .$$

By considering the decomposition of $S^r V_{n+1}$ into the virtual indecomposable $k[G]$-modules W_i for all $i \geqq 0$ (see [1] I. 1.9) we find that the number of *free* components of $S^r V_{n+1}$ (for $r < p$) will be given by the p first coefficients of

$$u_{n,p} + u_{n,p+1} + u_{n,3p} + u_{n,3p+1} + u_{n,5p} + u_{n,5p+1} + \cdots .$$

Hence

$$\eta_n = \psi_n + \sum_{j+1}^{\infty} (u_{n,2jp} + u_{n,2jp+1}) - \sum_{j=0}^{\infty} (u_{n,(2j+1)p} + u_{n,(2j+1)p+1})$$

will give the number $e_{n,r}$ of *nonfree* components for $r = 0, 1, 2, \cdots, p - 1$.

The first part

$$\tilde{\Phi}_n = \psi_n + \sum_{j=1}^{\infty} (u_{n,2jp} + u_{n,2jp+1})$$

is computed in [1] V. 4.7.

The second sum becomes

$$\sum_{j=0}^{\infty} (u_{n,(2j+1)p} + u_{n,(2j+1)p+1})$$
$$= \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi) \cot \frac{\varphi}{2} \sum_{j=0}^{m-1} [\sin ((2j+1)p + 1)\varphi$$
$$- \sin ((2j+1)p - 1)\varphi] d\varphi$$
$$= \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi) \frac{\sin 2mp\varphi}{\sin p\varphi} d\varphi .$$

Using that

$$\tilde{\Phi}_n = \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos\varphi) \frac{\sin(2m+1)p\varphi}{\sin p\varphi} d\varphi$$

we get

$$\tilde{\eta}_n = \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos\varphi) \frac{\sin(2m+1)p\varphi - \sin(2mp\varphi)}{\sin p\varphi} d\varphi$$

$$= \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos\varphi) \frac{\cos\left(2m + \frac{1}{2}\right)p\varphi}{\cos\frac{p\varphi}{2}} d\varphi .$$

We want to rewrite this limit as a sum containing the pth roots of unity. Making a linear substitution we get the Dirichlet kernel in the integrand and then we can use Lemma V. 4.8 in [1].

Put $\varphi = \pi + 2\theta$. Then we get

$$\tilde{\eta}_n = \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\pi + 2\theta)(1 - \cos 2\theta) \frac{\sin(4m+1)p\theta}{\sin p\theta} d\theta$$

$$= \frac{1}{p} \sum_{\mu=0}^{p-1} g_n\left(\pi + \frac{2\mu\pi}{p}\right)\left(1 - \cos\frac{2\mu\pi}{p}\right)$$

$$= \frac{1}{2p} \sum_{\mu=0}^{p-1} \frac{2 - \frac{e^{i2\mu\pi}}{p} - \frac{e^{-i2\mu\pi}}{p}}{\prod_{\nu=0}^{n}(1 - te^{i(n-2\nu)(\pi+(2\mu\pi/p))})} .$$

To get any further we have to treat the cases n even or odd separately.

Case 1. *n is even.*

Then $e^{i(n-2\nu)\pi} = 1$ and we get with $\alpha = e^{i2\pi/p}$

$$\tilde{\eta}_n = \frac{1}{2p} \sum_{\mu=0}^{p-1} \frac{2 - \alpha^{\mu} - \alpha^{-\mu}}{\prod_{\nu=0}^{n}(1 - t\alpha^{(n-2\nu)\mu})}$$

$$(*) \qquad = \frac{1}{2p} \sum_{\mu=0}^{p-1} (2 - \alpha^{\mu} - \alpha^{-\mu}) \sum_{\nu=0}^{\infty} G_{n+\nu,n}(\alpha^{\mu}, \alpha^{-\mu})t^{\nu}$$

$$= \frac{1}{p} \sum_{\gamma\in H} (1 - \gamma) \sum_{\nu=0}^{\infty} G_{n+\nu,n}(\gamma, \gamma^{-1})t^{\nu}$$

where H is the group of pth roots of unity. $G_{n,r}$ is the *homogeneous Gaussian polynomial* defined in [1] Ch. II. 4

$$G_{n,r}(X, Y) = \frac{(X^n - Y^n)(X^{n-1} - Y^{n-1}) \cdots (X^{n-r+1} - Y^{n-r+1})}{(X^r - Y^r)(X^{r-1} - Y^{r-1}) \cdots (X - Y)} .$$

We also used the formula II. 4.3 in [1]

$$\prod_{j=0}^{r}(1-X^{n-j}Y^{j}t)^{-1}=\sum_{\nu=0}^{\infty}G_{r+\nu,r}(X,\ Y)t^{\nu}\ .$$

From the definition of the Gaussian polynomials we get

$$G_{n+\nu+p,n}(\gamma,\ \gamma^{-1})=G_{n+\nu,n}(\gamma,\ \gamma^{-1})$$

and

$$G_{n+\nu,n}(\gamma,\ \gamma^{-1})=0\quad\text{if}\quad p-n\leqq\nu\leqq p-1\ .$$

Hence

$$\begin{aligned}\tilde{\eta}_n(t)&=\frac{1}{p(1-t^p)}\sum_{\gamma\in H}(1-\gamma)\sum_{\nu=0}^{p-n-1}G_{n+\nu,n}(\gamma,\ \gamma^{-1})t^{\nu}\\&=\frac{1}{1-t^p}\sum_{\nu=0}^{p-n-1}\Big(\frac{1}{p}\sum_{\gamma\in H}(1-\gamma)G_{n+\nu,n}(\gamma,\ \gamma^{-1})\Big)t^{\nu}\ .\end{aligned}$$

It follows that

$$(1-t^p)\tilde{\eta}_n(t)=\sum_{\nu=0}^{p-n-1}\tilde{e}_{n,\nu}t^{\nu}$$

where

$$\tilde{e}_{n,\nu}=\frac{1}{p}\sum_{\gamma\in H}(1-\gamma)G_{n+\nu,n}(\gamma,\ \gamma^{-1})\ .$$

Then $\tilde{e}_{n,\nu}=e_{n,\nu}$ for $\nu=0,1,\cdots,p-1$. But

$$\tilde{e}_{n,\nu+p}-\tilde{e}_{n,\nu}=0$$

and hence $\tilde{e}_{n,\nu}=e_{n,\nu}$ for all $\nu\geqq 0$ and

$$\tilde{\eta}_n(t)=\eta_n(t)\ .$$

From (*) we infer that

$$\eta_n(t^{-1})=-t^{n+1}\eta_n(t)$$

and $\eta_n(t)$ is *symmetric* in the sense of Stanley (see [1] V. 5.1). Using V. 5.6 in [1] we get

$$e_{n,-\nu}=e_{n,\nu-n-1}\qquad\text{for}\quad\nu>n\ .$$

But $e_{n,p-\nu}=e_{n,-\nu}=e_{n,\nu-n-1}$ and replacing ν by $\nu+n+1$ we get

$$e_{n,p-n-\nu+1}=e_{n,\nu}$$

which proves (1).

From (3) $e_{n,\nu}=e_{\nu,n}$ we get (2) from (1)

$$e_{p-n-\nu-1,\nu}=e_{\nu,p-n-\nu-1}=e_{\nu,n}=e_{n,\nu}$$

and we are done in case n is even.

Case 2. *n is odd.*
Then $e^{i(n-2\nu)\pi} = -1$ and

$$\tilde{\eta}_n = \frac{1}{2p}\sum_{\mu=0}^{p-1}\frac{2-\alpha^{\mu}-\alpha^{-\mu}}{\prod_{\nu=0}^{n}(1+t\alpha^{(n-2\nu)\mu})} = (1+t^p)^{-1}\sum_{\nu=0}^{p-n-1}\tilde{e}_{n,\nu}t^{\nu} .$$

We get

$$\eta_n(t) = \frac{1+t^p}{1-t^p}\tilde{\eta}_n = \frac{1+t^p}{1-t^p}\cdot\frac{1}{2p}\sum_{\mu=0}^{p-1}\frac{2-\alpha^{\mu}-\alpha^{-\mu}}{\prod_{\nu=0}^{n}(1+t\alpha^{(n-2\nu)\mu})}$$

and it follows

$$\eta_n(t^{-1}) = -t^{n+1}\eta_n(t) .$$

The proof is then finished as in the even case.

We note that we also have solved problem VI. 3.15 in [1].

THEOREM 2. *$S^r V_{n+1}$ has exactly one nonfree component when $n+r=p-2$. In fact*

$$S^{p-n-2}V_{n+1} = free \oplus \begin{cases} V_{n+1} & if\ \ n\ \ is\ even \\ V_{p-n-1} & if\ \ n\ \ is\ odd\ . \end{cases}$$

Proof. $e_{p-n-2,n} = e_{n,1} = 1$.

For the actual computation of the numbers $e_{n,\nu}$ we can get a formula involving the number of restricted partitions. By II. 4.6 in [1] we have

$$G_{n+\nu,n}(\gamma,\gamma^{-1}) = \sum_{m=0}^{\nu n} A(m,\nu,n)\gamma^{\nu n-2m}$$

where $A(m,\nu,n)$ is the number of partitions of m into at most ν parts all of size $\leqq n$.

PROPOSITION 3. *We have*

$$(-1)^{n\nu}e_{n,\nu} = \sum_{\substack{m=0\\ 2m\equiv\nu n}}^{\nu n} A(m,\nu,n) - \sum_{\substack{m=0\\ 2m\equiv\nu n+1}}^{\nu n} A(m,\nu,n)$$

where the congruences are mod p.

Proof. By the proof of Theorem 1 we get when n is even

$$\begin{aligned} e_{n,\nu} &= \frac{1}{p}\sum_{\gamma\in H}(1-\gamma)G_{n+\nu,n}(\gamma,\gamma)^{-1} \\ &= \frac{1}{p}\sum_{\gamma\in H}(1-\gamma)\sum_{m=0}^{\nu n}A(m,\nu,n)\gamma^{\nu n-2m} \\ &= \frac{1}{p}\sum_{m=0}^{\nu n}A(m,\nu,n)\sum_{\gamma\in H}(\gamma^{\nu n-2m}-\gamma^{\nu n+1-2m}) . \end{aligned}$$

But

$$\sum_{\gamma\in H}\gamma^j = \begin{cases} 0 & \text{if} \quad j \not\equiv 0 \pmod p \\ p & \text{if} \quad j\equiv 0 \pmod p \end{cases}$$

finishes the proof. The case when n is odd is similar.

EXAMPLE 4. Combining Theorem 1 and Proposition 3 we can write down a purely combinatorial identity equivalent to Theorem 1.

Let n be fixed and define $\nu' = p - n - \nu - 1$. Then $e_{n,\nu} = e_{n,\nu'}$ or

$$\sum_{2m\equiv\nu n}A(m,\nu,n) - \sum_{2m\equiv\nu n+1}A(m,\nu,n) = \sum_{2m\equiv\nu' n}A(m,\nu',n) - \sum_{2m\equiv\nu' n+1}A(m,\nu',n)$$

where the sums run over $0\leqq m\leqq \nu n$ or $0\leqq m\leqq \nu' n$ respectively.

REMARK 5. Since $\Lambda^r V_{r+n}\cong S^r V_{n+1}$ we have also computed the number of nonfree components of the exterior powers for which similar symmetry relations are valid.

EXAMPLE 6. Let us show how to compute the central number $e_{6,6} = 18$ in the triangle for $p = 19$ (this is the worst case). By the formula in the proposition

$$e_{6,6} = A(18, 6, 6) - A(9, 6, 6) - A(28, 6, 6) = 58 - 22 - 18 = 18 .$$

As a check we also compute the decomposition

$$S^6V_7 = 2V_1 + 3V_5 + 2V_7 + 4V_9 + V_{11} + 3V_{13} + 2V_{15} + V_{17} + 40V_{19}$$

and we read off $e_{6,6} = 2+3+2+4+1+3+2+1 = 18$.

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Reference

1. G. Almkvist and R. Fossum, *Decompositions of exterior and symmetric powers of indecomposable Z/pZ-modules in characteristic p and relations to invariants*, to appear Séminaire P. Dubreil 1976–77, Springer Lecture Notes of Mathematics, No. 641.

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