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CAPACITIES OF COMPACT SETS IN LINEAR SUBSPACES OF R^n

TORD SJÖDIN

We consider two types of spaces, the Bessel potential spaces $L_\alpha^p(R^n)$ and the Besov spaces $\Lambda_\alpha^p(R^n)$, $\alpha > 0$, $1 < p < \infty$. Associated in a natural way with these spaces are classes of exceptional sets. We characterize the exceptional sets for $\Lambda_\alpha^p(R^n)$ by an extension property for continuous functions and prove an inequality between Bessel and Besov capacities.

The classes of exceptional sets for the spaces $L_\alpha^p(R^n)$ have been studied by the concept of capacity [5]. Capacity definitions of these classes are given in § 2.

Bessel potential spaces and Besov spaces in R^n and R^{n+1} are connected by restriction theorems. A short statement of these results is the following:

$$L_\beta^p(R^{n+1})|_{R^n} = \Lambda_\alpha^p(R^n) \tag{1.1}$$

$$\Lambda_\beta^p(R^{n+1})|_{R^n} = \Lambda_\alpha^p(R^n) , \tag{1.2}$$

where $\alpha>0$, $1<p<\infty$, and $\beta = \alpha + 1/p$. (O. V. Besov [4] and E.M. Stein [7].)

The restriction theorem above gives relations between exceptional classes of different spaces L_α^p and Λ_α^p in R^n and R^{n+1}.

This enables us to prove an extension theorem for continuous functions on a compact set $K \subset R^n$ into $\Lambda_\alpha^p(R^n)$ (Theorem 1) analogous to the $L_\alpha^p(R^n)$ — case contained in [6, Theorem 1]. Finally we prove an inequality between the capacities defining the classes of exceptional sets for $\Lambda_\alpha^p(R^n)$ and $L_\alpha^p(R^n)$ (Theorem 2).

2. Preliminaries and statements of the theorems.

We consider the n-dimensional space R^n of n-tuples $x = (x_1, x_2, \cdots, x_n)$. Points in R^{n+1} are written (x, x_{n+1}), where $x \in R^n$ and $x_{n+1} \in R^1$. Then R^n is identified as the subspace $\{(x, 0); x \in R^n\}$ of R^{n+1}. Compact sets are denoted by K. If $K \subset R^n$ then K is a compact subset of R^{n+1} as well. As usual, the space of p-summable functions is denoted by $L^p(R^n)$ with norm $||\cdot||_p$. The Bessel kernel G_α^n in R^n is the $L^1(R^n)$-function whose Fourier transform equals $(1 + |x|^2)^{-\alpha/2}$, $\alpha > 0$.

The space of convolutions $U = G_\alpha^n * f$, where $f \in L^p(R^n)$, with the norm $||U||_{\alpha,p} = ||f||_p$, is denoted by $L_\alpha^p(R^n)$, $\alpha > 0$, $1 \leqq p < \infty$. A function $U \in \Lambda_\alpha^p(R^n)$, $1 \leqq p \leqq \infty$, $0 < \alpha < 1$ if

$$|U|_{\alpha,p} = ||U||_p + \left(\iint \frac{|U(x) - U(y)|^p}{|x - y|^{\alpha p+n}} dxdy\right)^{1/p}$$

is finite. (When no limits of integration are indicated it is understood that the integration is over the whole space.)

When $1 \leqq \alpha < 2$ we replace the first difference by the second difference. Finally, for $\alpha \geqq 2$, $U \in \Lambda_\alpha^p(R^n)$ if and only if $U \in L^p$ and $\partial U/\partial x_i \in \Lambda_{\alpha-1}^p(R^n)$, $1 \leqq i \leqq n$, with the norm

$$|U|_{\alpha,p} = \|U\|_p + \sum_{i=1}^{n} \left|\frac{\partial U}{\partial x_i}\right|_{\alpha-1,p} .$$

We consider the following two capacities for compact sets $K \subset R^n$, $\alpha > 0$, $1 < p < \infty$.

$$A_{\alpha,p}^n(K) = \inf |\varphi|_{\alpha,p}^p ,$$
$$B_{\alpha,p}^n(K) = \inf \|\varphi\|_{\alpha,p}^p ,$$

where, in both cases, the infimum is taken over all $\varphi \in C_0^\infty(R^n)$ such that $\varphi(x) \geqq 1$ for every $x \in K$. $C_0^\infty(R^n)$ is the infinitely differentiable functions on R^n with compact support.

The $B_{\alpha,p}^n$-capacity has several equivalent definitions [2, 5]. We mention that

$$B_{\alpha,p}^n(K) = \inf \|f\|_p^p$$

where infimum is over $f \in L_+^p$ such that $G_\alpha^n * f(x) \geqq 1$ on K. (A lower superscript + indicates positive elements.) The sign $\sim$ means that the ratio is bounded from below and above by positive real numbers. Further, $B_{\alpha,p}^n(K) = (\sup \|\mu\|_1)^p$ where supremum is over positive Borel measures μ concentrated on K with total variation $\|\mu\|_1 < \infty$ and $\|G_\alpha^n * \mu\|_q \leqq 1$.

Here $q = p/p-1$. See [5] where this capacity is denoted by $b_{\alpha,p}$. Let K be a compact subset of R^n. We have proved that $B_{\alpha,p}^n(K) = 0$ if and only if every continuous function on K is the restriction to K of a continuous function in $L_\alpha^p(R^n)$ [6, Theorem 1]. We prove here the analogue for $\Lambda_\alpha^p(R^n)$. Let $C(E)$ denote the space of continuous functions on a set E in R^n.

THEOREM 1. *Let $1 < p < \infty$, $0 < \alpha \cdot p \leqq n$ and let K be a compact subset of R^n. Then $A_{\alpha,p}^n(K) = 0$ if and only if every function $f_0 \in C(K)$ has an extension $f \in \Lambda_\alpha^p(R^n) \cap C(R^n)$.*

When $\alpha p > n$, the capacities $A_{\alpha,p}^n$ and $B_{\alpha,p}^n$ are positive unless K is empty [3].

We denote the exceptional classes for $L_\alpha^p(R^n)$ and $\Lambda_\alpha^p(R^n)$, $1<p<\infty$, $\alpha \cdot p \leqq n$, by $\mathfrak{B}_{\alpha,p}^n$ and $\mathfrak{U}_{\alpha,p}^n$ respectively [3]. It is well known that for $K \subset R^n$:

$$K \in \mathfrak{A}^n_{\alpha,p} \quad \text{if and only if} \quad A^n_{\alpha,p}(K) = 0$$
$$K \in \mathfrak{B}^n_{\alpha,p} \quad \text{if and only if} \quad B^n_{\alpha,p}(K) = 0\,.$$

See [3].

It is interesting to note that $\mathfrak{A}^n_{\alpha,p}$ and $\mathfrak{B}^n_{\alpha,p}$ can be proved to be identical for $2 - \alpha/n < p < \infty$ [1, Theorem 1] inspite of the fact that $L^p_\alpha(R^n) \neq \Lambda^p_\alpha(R^n)$ when $\alpha > 0$ and $p \neq 2$ [3].

THEOREM 2. *Let $\alpha > 0$, $1 < p < \infty$, and let K be a compact subset of R^n. Then*

$$B^n_{\alpha,p}(K) \leqq c \cdot A^n_{\alpha,p}(K)\,.$$

Constants depending on n, p, and α only, not necessarily the same at each occurance, are denoted by c.

REMARK. David R. Adams [1, p. 3] has proved that $A^n_{\alpha,p}(K) = 0$ implies $B^n_{\alpha,p}(K) = 0$ for $\alpha > 0, 1 < p < \infty$. Theorem 2 makes it possible to compare the capacities $B^n_{\alpha,p}$ and $A^n_{\alpha,p}$ for all sets.

It will become clear from the proofs of Theorem 1 and Theorem 2 that the restriction theorem described in (1, 1) and (1, 2) is an essential tool. (An exact formulation is given in Theorem A is § 3.)

At this point we just note that Theorem 2 has an alternative formulation. Under the assumptions of Theorem 2,

$$B^n_{\alpha,p}(K) \leqq c \cdot B^{n+1}_{\beta,p}(K)\,,\ K \subset R^n\,, \qquad \beta = \alpha + \frac{1}{p}\,.$$

The inclusions $L^p_\alpha(R^n) \subset \Lambda^p_\alpha(R^n)$ for $2 \leqq p < \infty$ and $\Lambda^p_\alpha(R^n) \subset L^p_\alpha(R^n)$, for $1 < p \leqq 2$, are well known [3]. They give immediately the inequalities

$$B^n_{\alpha,p}(K) \leqq c \cdot A^n_{\alpha,p}(K)\,, \qquad 1 < p \leqq 2\,,$$

and

$$A^n_{\alpha,p}(K) \leqq c \cdot B^n_{\alpha,p}(K)\,, \qquad 2 \leqq p < \infty\,. \tag{2.1}$$

Combining Theorem 2 with (2.1) gives,

$$A^n_{\alpha,p}(K) \sim B^n_{\alpha,p}(K)\,, \qquad 2 \leqq p < \infty\,.$$

3. Proof of Theorem 1.

We first define two operators E and R in the following way. Let $\varphi \in C^\infty_0(R^{n+1})$, then

$$R\varphi(x) = \varphi(x, 0),\ x \in R^n\,.$$

Let $f \in C^\infty_0(R^1)$ and $g \in C^\infty_0(R^n)$ be such that $f(0) = 1$ and $\int g(x)dx = 1$.

When $\psi \in C_0^\infty(R^n)$ we put

$$E\psi(x, x_{n+1}) = f(x_{n+1}) \cdot \int \psi(x - x_{n+1} \cdot y) \cdot g(y) dy ,$$

$x \in R^n$, $x_{n+1} \in R^1$. See for example, E. M. Stein [7].

THEOREM A. *Let* $\alpha > 0, 1 < p < \infty$, *and* $\beta = \alpha + 1/p$. *Then*

(a) *the map* R *is a continuous map from* $L_\beta^p(R^{n+1})(\Lambda_\beta^p(R^{n+1}))$ *to* $\Lambda_\alpha^p(R^n)$;

(b) *the map* E *is a continuous map from* $\Lambda_\alpha^p(R^n)$ *to* $L_\beta^p(R^{n+1})(\Lambda_\beta^p(R^{n+1}))$.

This theorem is due to E.M. Stein [7] and O.V. Besov [4]. Let $K \subset R^n$, $\alpha > 0$, $1 < p < \infty$, then

$$B_{\beta,p}^{n+1}(K) \sim A_{\alpha,p}^n \sim A_{\beta,p}^{n+1}(K) \tag{3.1}$$

where $\beta = \alpha + 1/p$.

This is an immediate consequence of Theorem A and the definitions of the capacities.

Proof of Theorem 1. Let K be a compact subset of R^n such that $A_{\alpha,p}^n(K) = 0$. Let $f_0 \in C(K)$. Since $B_{\beta,p}^{n+1}(K) = 0$, $\beta = \alpha + 1/p$, by (3.1), there is a function $f \in L_\beta^p(R^{n+1}) \cap C(R^{n+1})$ such that $f(x) = f_0(x)$ when $x \in K$ [6, Theorem 1]. Taking the restriction Rf we have $Rf \in \Lambda_\alpha^p(R^n) \cap C(R^n)$ by Theorem A.

Conversely suppose that every $f_0 \in C(K)$ has an extension $f \in \Lambda_\alpha^p(R^n) \cap C(R^n)$. Let $f_0 \in C(K)$ then $Ef \in L_\beta^p(R^{n+1}) \cap C(R^{n+1})$, $\beta = \alpha + 1/p$. By [6, Theorem 1] we must have $B_{\beta,p}^{n+1}(K) = 0$, which implies $A_{\alpha,p}^n(K) = 0$. The proof is complete.

4. **Proof of Theorem 2.** We begin with a lemma. Let $f \in L_+^p(R^{n+1})$ then we define $g(y) = \left(\int f(y, t)^p dt\right)^{1/p}$, $y \in R^n$.

The function g belongs to $L_+^p(R^n)$ and

$$\|g\|_p = \|f\|_p .$$

(The notation $\|\cdot\|_p$ means that the integral defining the norm is taken over all the variables and over the whole space.)

LEMMA 1. *Let* $\alpha > 0, 1 < p < \infty$, $\beta = \alpha + 1/p$. *Then for* $f \in L_+^p(R^{n+1})$,

$$G_\beta^{n+1} * f(x, 0) \leqq c \cdot G_\alpha^n * g(x) , \qquad x \in R^n .$$

In the proof of Lemma 1 we use some well known properties of the Bessel kernel $G_\alpha^n(r)$ (see for example [3]):

$$G_\alpha^n(r) \sim r^{\alpha-n},\ r \longrightarrow 0\ , \quad \text{for} \quad 0 < \alpha < n$$
$$G_\alpha^n(r) \sim r^{(\alpha-n-1)/2}\cdot e^{-r}\ , \quad r \longrightarrow \infty\ , \quad \text{for} \quad \alpha > 0\ .$$

Proof of Lemma 1. Suppose $\alpha\cdot p \leqq n$ and let $f \in L_+^p(R^{n+1})$ and $g(y) = \left(\int f(y, t)^p dt\right)^{1/p}$. We have

$$G_\beta^{n+1} * f(x, 0) = \iint G_\beta^{n+1}(\sqrt{|x-y|^2 + t^2})\cdot f(y, t)dydt\ .$$

For $|y - x| \leqq 1$ we get the estimate:

$$\begin{aligned}\int G_\beta^{n+1}(\sqrt{|x-y|^2+t^2})\cdot f(y,t)dt &\leqq c\cdot\int(\sqrt{|x-y|^2+t^2})^{\beta-n-1}\\ &\quad\cdot f(y,t)dt\\ &= c\cdot|x-y|^{\beta-n}\cdot\int(\sqrt{1+t^2})^{\beta-n-1}\cdot f(y,|x-y|\cdot t)dt\\ &\leqq c\cdot|x-y|^{\beta-n}\cdot\left(\int f(y,|x-y|\cdot t)^p dt\right)^{1/p}\\ &= c\cdot|x-y|^{\beta-n-1/p}\cdot\left(\int f(y,t)^p dt\right)^{1/p}\\ &\leqq c\cdot G_\alpha^n(x-y)\cdot g(y)\ .\end{aligned}$$

For $|y - x| \geqq 1$ we get

$$\begin{aligned}\int G_\beta^{n+1}(\sqrt{|x-y|^2+t^2})f(y,t)dt &\leqq c\cdot\int(\sqrt{|x-y|^2+t^2})^{(\beta-n-2)/2}\\ &\quad\cdot e^{-\sqrt{|x-y|^2+t^2}}\cdot f(y,t)dt\\ &= c\cdot|x-y|^{(\beta-n)/2}\cdot\int(\sqrt{1+t^2})^{(\beta-n-2)/2}\cdot e^{-|x-y|\cdot\sqrt{1+t^2}}\\ &\quad\cdot f(y,|x-y|\cdot t)dt\ .\end{aligned}$$

We divide the last integral in two parts

$$I = \int_{-1}^{1} \text{ and } II = \int_{|t|\geqq 1}\ .$$

Then using the inequality $\sqrt{1+x} \geqq 1 + x/3$, $0 \leqq x \leqq 1$ we get

$$\begin{aligned}I &\leqq e^{-|x-y|}\cdot\int_{-1}^{1} e^{-|x-y|\cdot t^2/3}\cdot f(y,|x-y|\cdot t)dt\\ &\leqq |x-y|^{-1/2}\cdot e^{-|x-y|}\cdot\int e^{-t^2/3}\cdot f(y,\sqrt{|x-y|}\,t)dt\\ &\leqq c\cdot|x-y|^{(-1-1/p)\cdot/2}\cdot e^{-|x-y|}\cdot g(y)\ .\end{aligned}$$

Further we have

$$II \leqq e^{-\sqrt{2}\cdot|x-y|}\cdot\int(\sqrt{1+t^2})^{(\beta-n-2)/2}\cdot f(y, |x-y|\cdot t)dt$$
$$\leqq c\cdot e^{-\sqrt{2}\cdot|x-y|}\cdot|x-y|^{-1/p}\cdot g(y) .$$

Collecting our results we have

$$\int G_\beta^{n+1}(\sqrt{|x-y|^2+t^2})\cdot f(y, t)dt \leqq c\cdot G_\alpha^n(x-y)\cdot g(y)$$

which gives

$$G_\beta^{n+1} * f(x, 0) \leqq c\cdot G_\alpha^n * g(x) ,$$

where $\beta = \alpha + 1/p$.

The case $\alpha\cdot p > n$ is much simpler and the proof is omitted.

Proof of Theorem 2. According to the relation (3.1) it suffices to prove that for every $f \in L_+^p(R^{n+1})$ such that $G_\beta^{n+1} * f(x, 0) \geqq 1$ for $x \in K$, there exists $g \in L_+^p(R^n)$ such that $G_\alpha^n * g(x) \geqq 1$ for $x \in K$ and

$$||g||_p \leqq c\cdot ||f||_p .$$

But this follows immediately from Lemma 1. This proves the theorem.

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UNIVERSITY OF UMEA
SWEDEN