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THE PETERSSON INNER PRODUCT AND THE RESIDUE OF AN EULER PRODUCT

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1. Introduction. Implicit in the work of Rankin [6] and explicit in the work of Petersson [5] is a formula connecting the Petersson inner product of two holomorphic modular forms with a residue of the Dirichlet series formed with the products of the Fourier coefficients of the two modular forms. In view of the modern group theoretic interpretation of the eigenfunctions of the Hecke operators as unitary representations of an adèle group, it appears that the ideas of Rankin and Petersson may have wider applicability; for example they may relate to multiplicity-one problems in the theory of automorphic representations. The purpose of this note is to extend these ideas to real analytic modular forms.

In §2 we recall the results of Petersson and Rankin and show (see Theorem 1) how the residue at $s = 1$ of a certain Euler product can be used to distinguish whether two eigenfunctions of the Hecke operators are orthogonal or not. In §3 we describe the essential nature of the method and derive (see Theorem 2) a formula which expresses the Petersson inner product of two real analytic cusp forms which are eigenfunctions of the Hecke operators as the residue at $s = 1$ of an Euler product.

2. Holomorphic cusp forms. Let H be the upper half plane and Let Γ be the group of linear fractional transformations of $H: z \mapsto \sigma(z) = (az + b)/(cz + d)$, with $a, b, c, d \in \boldsymbol{Z}$ and $ad - bc = 1$. Let $d\Omega = y^{-2}xdy$ be the $\mathrm{SL}(2, \boldsymbol{R})$-invariant measure of H. Let k be a positive integer and let

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n,\ g(z) = \sum_{n=1}^{\infty} b(n)q^n,\ q = \exp(2\pi iz)$$

be two holomorphic cusp forms of weight k on the group Γ; let p be a prime and suppose that $f(z)$ and $g(z)$ are eigenfunctions of the Hecke operator T_p:

$$(f \mid T_p)(z) = \sum_{n=1}^{\infty} a(np)q^n + p^{k-1} \sum_{n=1}^{\infty} a(n)q^{np} = a(p)f(z)\ ;$$

with a similar formula for $g(z)$. A particular case of Petersson's formula (Theorem 6, [4]) is the following.

$$\langle f, g\rangle = \int_{D(\Gamma)} f(z)\overline{g(z)}y^k d\Omega$$
$$= \frac{\pi}{3}\Gamma(k)(4\pi)^{-k}\operatorname*{Res}_{s=1} L_{f,g}(s) ,$$

were

$$L_{f,g}(s) = \sum_{n=1}^{\infty} a(n)b(n)n^{-(s+k-1)},\ \mathrm{Re}\,(s) > 1 .$$

If we use the multiplicativity of the coefficients $a(n)$, $b(n)$ and the duplication formula for the gamma function, then Petersson's equality can be rewritten in the form

$$\langle f, g\rangle = 2^{-k}\operatorname*{Res}_{s=1} L(s, \pi_f \times \pi_g) ,$$

where

$$2(4\pi)^{-(s+k-1)}\pi^{-s}\Gamma(s+k-1)\Gamma(s)\zeta(2s)\sum_{n=1}^{\infty} a(n)b(n)n^{-(s+k-1)}$$
$$= 2^{-k}L(s, \pi_f \times \pi_g)$$
$$= 2^{-k}\Gamma(s, \pi_f \times \pi_g)\zeta(s, \pi_f \times \pi_g)$$

and

$$\Gamma(s, \pi_f \times \pi_g) = G_R(s)G_R(s+1)G_R(s+k-1)G_R(s+k),\ G_R(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) ,$$
$$\zeta(s, \pi_f \times \pi_g) = \prod_p \frac{1}{(1-\lambda_p\eta_p p^{-s})(1-\lambda_p\bar{\eta}_p p^{-s})(1-\bar{\lambda}_p\eta_p p^{-s})(1-\bar{\lambda}_p\bar{\eta}_p p^{-s})} ,$$
$$a(p) = p^{(k-1)/2}(\lambda_p + \bar{\lambda}_p),\ b(p) = p^{(k-1)/2}(\eta_p + \bar{\eta}_p) .$$

REMARK. The Euler product $L(s, \pi_f \times \pi_g)$ is related to those introduced by Langlands ([2], p. 10).

With the above notations we can restate Petersson's result in the following form

THEOREM 1. *Let $f(z)$ and $g(z)$ be cusp forms of weight k on the full modular group Γ which are eigenfunctions of all the Hecke operators. Then $f(z)$ is orthogonal to $g(z)$ if and only if the Euler product*

$$L(s, \pi_f \times \pi_g) = \Gamma(s, \pi_f \times \pi_g)\zeta(s, \pi_f \times \pi_g)$$

is regular at $s = 1$.

3. Real analytic cusp forms. Let $f(z)$ and $g(z)$ be two real analytic cusp forms in the sense of Maass [4]. Let

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

be the Laplace-Beltrami operator for the upper half plane and suppose that

$$\Delta f = \frac{1-\lambda^2}{2} f \,, \quad \Delta g = \frac{1-\eta^2}{2} g \,.$$

Observe that because of the positivity of the Laplace-Beltrami operator the nonzero numbers λ and η must lie in the imaginary axis or in the interval between -1 and 1. Suppose that $f(z)$ and $g(z)$ have Fourier expansions about the cusp at infinity of the form, $z = x + iy$,

$$f(z) = \sum_{m \neq 0} a(m) y^{1/2} K_\lambda(2\pi |m| y) \exp(2\pi i m x) \,, \qquad a(1) = 1 \,,$$
$$g(z) = \sum_{n \neq 0} b(n) y^{1/2} K_\eta(2\pi |n| y) \exp(2\pi i n x) \,, \qquad b(1) = 1 \,,$$

where $K_\lambda(z)$ is the modified Bessel function. If $f(z)$ is an eigenfunction of the Hecke operators then Maass ([4], Theorem 12) has proved that

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \frac{1}{1 - a(p)p^{-s} + p^{-2}} \,;$$

similarly for $g(z)$. To simplify our notation we suppose that $f(-\bar{z}) = f(z)$ and $g(-\bar{z}) = g(z)$. For $\mathrm{Re}(s) > 1$ we have the Euler product identity

$$\begin{aligned} \Gamma(s)^{-1}\pi^{-s}\Gamma\left(\frac{s+\lambda+\eta}{2}\right)&\Gamma\left(\frac{s+\lambda-\eta}{2}\right)\Gamma\left(\frac{s-\lambda-\eta}{2}\right)\Gamma\left(\frac{s-\lambda-\eta}{2}\right) \\ &\times \sum_{n\neq 0} a(n)\overline{b(n)} n^{-s} \\ &= \frac{\Gamma(s, \pi_f \times \pi_g)\zeta(s, \pi_f \times \pi_g)}{\Lambda(2s)} \\ &= \frac{L(s, \pi_f \times \pi_g)}{\Lambda(2s)} \,, \end{aligned}$$

where

$$\Gamma(s, \pi_f \times \pi_g) = G_R(s+\lambda+\eta)G_R(s+\lambda-\eta)G_R(s-\lambda+\eta)G_R(s-\lambda-\eta) \,,$$
$$\zeta(s, \pi_f \times \pi_g) = \prod_p \frac{1}{(1-\lambda_p\eta_p p^{-s})(1-\lambda_p^0\eta_p p^{-s})(1-\lambda_p\eta_p^0 p^{-s})(1-\lambda_p^0\eta_p^0 p^{-s})} \,,$$
$$a(p) = \lambda_p + \lambda_p^0,\ b(p) = \eta_p + \eta_p^0,\ \lambda_p\lambda_p^0 = \eta_p\eta_p^0 = 1 \,,$$

and $\Lambda(s) = G_R(s)\zeta(s)$. This equality follows from the formal power series identity

$$\sum_{v=0}^{\infty} a(p^v)\overline{b(p^v)}T^v = \frac{1 - T^2}{(1-\lambda_p\eta_p T)(1-\lambda_p^0\eta_p T)(1-\lambda_p\eta_p^0 T)(1-\lambda_p^0\eta_p^0 T)}$$

With these notations we have the following result.

THEOREM 2. *Let $f(z)$ and $g(z)$ be two real analytic cusp forms on the full modular group Γ which are eigenfunctions of all Hecke operators. Let $L(s, \pi_f \times \pi_g)$ be the Euler product introduced above. Then we have*

$$\langle f, g\rangle = \frac{1}{2} \operatorname*{Res}_{s=1} L(s, \pi_f \times \pi_g) .$$

In particular f and g are orthogonal if and only if $L(s, \pi_f \times \pi_g)$ is regular at $s = 1$.

Proof. Let

$$S_\Gamma = \left\{z = x + iy \colon |x| \leqq \frac{1}{2}, \; y \geqq 0\right\} .$$

Let $\Gamma_\infty = \left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \colon n \in \boldsymbol{Z}\right\}$ and let $D(\Gamma)$ be a fundamental domain for Γ in H. We then have the formal congruence identity

$$(\,*\,) \qquad S_\Gamma \equiv \sum_{\sigma \in \Gamma/\Gamma_\infty} \sigma D(\Gamma) .$$

Now for $\operatorname{Re}(s)$ sufficiently large we have

$$\begin{aligned}
&\int_{S_\Gamma} f(z)\overline{g(z)}y^s d\Omega \\
&\quad = \int_0^\infty \left(\int_{-1/2}^{1/2} f(z)\overline{g(z)}dx\right) y^{s-2}dy \\
&\quad = \int_0^\infty \left(\sum_m \sum_n a(m)\overline{b(n)}K_\lambda(2\pi\,|\,m\,|\,y)K_\eta(2\pi\,|\,n\,y) \int_{-1/2}^{1/2} e^{2\pi i x(m-n)}dx\right) y^{s-1}dy \\
&\quad = \sum_m \sum_n a(m)\overline{b(m)}\left(\int_0^\infty K_\lambda(2\pi\,|\,m\,|\,y)K_\eta(2\pi\,|\,m\,|\,y)y^{s-1}dy\right) .
\end{aligned}$$

The interchange of the order of summation and integration is justified by the fact that uniformly in x both $f(z)$ and $g(z)$ are $O(y^A)$ (resp. $O(y^{-B})$) when $y \to \infty$ (resp. $y \to \infty$) for some positive constants A and B.

We now use the well known identity ([3], p 102)

$$\begin{aligned}
&\int_0^\infty K_\mu(\alpha t)K_\nu(\alpha t)t^{-\zeta}dt \\
&\quad = \frac{\alpha^{\zeta-1}}{2^{\zeta+2}\Gamma(1-\zeta)}
\end{aligned}$$

$$\times \Gamma\Big(\frac{1-\zeta+\mu+\nu}{2}\Big)\Gamma\Big(\frac{1-\zeta+\mu-\nu}{2}\Big)\Gamma\Big(\frac{1-\zeta-\mu+\nu}{2}\Big)$$
$$\times \Gamma\Big(\frac{1-\zeta-\mu-\nu}{2}\Big)$$

which is valid for $\mathrm{Re}\,(\alpha) > 0$ and $\mathrm{Re}\,(1-\zeta \pm \mu \pm \nu) > 0$. We apply this identity with $\alpha = 2\pi n y$ and $\zeta = 1 - s$ to obtain

$$\begin{aligned}\int_{S_\Gamma} & f(z)\overline{g(z)}y^s d\Omega \\ &= \frac{\pi^{-s}}{8\Gamma(s)}\Gamma\Big(\frac{s+\lambda+\eta}{2}\Big)\Gamma\Big(\frac{s+\lambda-\eta}{2}\Big)\Gamma\Big(\frac{s-\lambda+\eta}{2}\Big) \\ &\quad \times \Gamma\Big(\frac{s-\lambda-\eta}{2}\Big)\sum_{m\neq 0} a(m)\overline{b(m)}m^{-s} \\ &= \frac{\pi^{-s}}{4\Gamma(s)}\Gamma\Big(\frac{s+\lambda+\eta}{2}\Big)\Gamma\Big(\frac{s+\lambda-\eta}{2}\Big)\Gamma\Big(\frac{s-\lambda+\eta}{2}\Big) \\ &\quad \times \Gamma\Big(\frac{s-\lambda-\eta}{2}\Big)\sum_{m=1}^{\infty} a(m)b(m)m^{-s} \\ &= \frac{L(s, \pi_f \times \pi_g)}{4\Lambda(2s)} .\end{aligned}$$

We now use the congruence identity (*) for the region S_Γ to obtain

$$\begin{aligned}\int_{S_\Gamma} & f(z)\overline{g(z)}y^s d\Omega \\ &= \sum_{\sigma\in\Gamma/\Gamma_\infty}\int_{D(\Gamma)} f(z)\overline{g(z)}y^s d\Omega\circ\sigma .\end{aligned}$$

By the automorphy property of $f(z)$ and $g(z)$ we get

$$f(z)\overline{g(z)}y^s d\Omega\circ\sigma = f(z)\overline{g(z)}(\mathrm{Im}\,\sigma(z))^s d\Omega ,$$

and therefore

$$\frac{L(s, \pi_f \times \pi_g)}{4\Lambda(2s)} = \int_{D(\Gamma)} f(z)\overline{g(z)}E(z, 2s-1)d\Omega ,$$

where

$$E(z, s) = \sum_{\sigma\in\Gamma/\Gamma_\infty}(\mathrm{Im}\,\sigma(z))^{(1+s)/2}$$

is the well known Eisenstein series associated to Γ. The Fourier expansion of $E(z, s)$ is ([1], p. 46)

$$\begin{aligned}E(z, s) &= y^{(1+s)/2} + \frac{\Lambda(s)}{\Lambda(s+1)}y^{(1-s)/2} \\ &\quad + \sum_{m\neq 0}\frac{2}{\Lambda(s+1)}\,\frac{\sigma_{s(|m|)}}{|m|^{s/2}}y^{1/2}K_{s/2}(2\pi|m|y)e^{2\pi i x m x}\end{aligned}$$

and the only poles of the summation term are those arising from the zeros of Riemann's Euler product $\Lambda(s+1)$. Hence the only pole of $E(z, s)$ at $s = 1$ comes from the numerator in the expression $\Lambda(s)/\Lambda(s+1)y^{(1+s)/2}$. In fact a well known application of the Kronnecker limit formula gives the Laurent expansion in a small neighborhood of $s = 1$

$$E(z, 2s-1) = \frac{3}{\pi}\cdot\frac{1}{s-1} + \frac{6}{\pi}(c - \log 2) - \frac{1}{2\pi}\log(y^6|\Delta(z)|) + \sum_{n=1}^{\infty} c(n)(s-1)^n ,$$

were

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} .$$

In particular it should be observed that the residue of $E(z, s)$ at $s = 1$ is independent of z. Therefore we have

$$\begin{aligned}\operatorname*{Res}_{s=1}\int_{D(\Gamma)} f(z)\overline{g(z)}E(z, 2s-1)d\Omega &= \left(\int_{D(\Gamma)} f(z)\overline{g(z)}d\Omega\right). \quad \operatorname*{Res}_{s=1} E(z, 2s-1) \\ &= \frac{3}{\pi}\langle f, g\rangle .\end{aligned}$$

On the other hand

$$\begin{aligned}\operatorname*{Res}_{s=1}\frac{L(s, \pi_f\times\pi_g)}{4\Lambda(2s)} &= \frac{3}{2\pi}\operatorname*{Res}_{s=1} L(s, \pi_f\times\pi_g) .\end{aligned}$$

This completes the proof of Theorem 2.

REMARK 1. The functional equation for the Eisenstein series

$$E(z, s) = \frac{\Lambda(s)}{\Lambda(s+1)}E(z, -s)$$

leads to the functional equation

$$L\left(\frac{1+s}{2}, \pi_f\times\pi_g\right) = L\left(\frac{1-s}{2}, \pi_f\times\pi_g\right) .$$

2. The residue of $L(s, \pi_f\times\pi_g)$ at $s = 1$ can be thought of as an intertwining operator.

3. In [7], Shimura has used the connection between the Petersson inner product and L-series to study the arithmetic properties of the periods of Eichler differentials.

REFERENCES

1. T. Kubota, *Elementary Theory of Eisenstein Series*, Kodansha Ltd., Tokyo 1973.
2. R. P. Langlands, *Euler Products*, Yale Univ. Press, 1967.
3. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Function of Mathematical Physics*, Springer-Verlag, New York 1966.
4. H. Maass, *Über eine meue Art von nichtanalytischen automorphen Funktionen und die Besti mmung Dirichletscher Reihen durch Funktional gleichungen*, Math. Ann., **121** (1944), 141–182.
5. H. Petersson, *Über die Berechnung der Skalarprodukte ganzer Modulf ormen*, Comment, Math. Helv., **22** (1949), 168–199.
6. R. A. Rankin, *Contributions to the Theory of Ramanujan's function $\tau(n)$ and similar arithmetic function*, II, Proc. Camb. Phil. Soc., **35** (1939), 357–372.
7. G. Shimura, *The special values of the zeta function associated with cusp forms*, Comm. Pure App. Math., **29** (1976), 783-804.

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