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DENTING POINTS IN B^p

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It is shown that in the weighted Bergmann space B^p of analytic functions all points of the unit sphere are denting points of the unit ball.

1. Introduction and definitions. Let Δ be the open unit disk in the complex plane ($\boldsymbol{C}$). For each fixed p in $(0, 1)$ we define a finite positive measure $d\mu(z) \equiv (1 - |z|^2)^{1/p-2}dm(z)$, where $z \in \Delta$ and $dm(z)$ is the usual Lebesgue measure on Δ. We consider the closed subspace $B^p = B^p(d\mu)$ of $L^1(d\mu)$ consisting of all functions in $L^1(d\mu)$ that are analytic on Δ. B^p is the containing Banach space of the Hardy space $H^p(\Delta)$ and indeed B^p is the Mackey completion of H^p. (See Duren, Romberg, and Shields [4] and Shapiro [7].) Let B be the closed unit ball and S the unit sphere in B^p. Although the closed unit ball of $L^1(d\mu)$ has no extreme points we shall show that the ball B has certain smoothness properties. It is not a fortiori clear that B has extreme points. However, several functional analytical properties of the space B^p are known. In particular a result of Shields and Williams [8; p. 295] shows that B^p is complemented in $L^1(d\mu)$. An argument of Lindenstrauss and Pelczynski [5; p. 248] can then be used to prove that B^p is topologically isomorphic to the sequence space l^1. It is known that l^1 (being a separable, dual space) has the Radon-Nikodym property. A good reference on the Radon-Nikodym property is Diestal and Uhl [3]. Hence, if T is a topological isomorphism of B^p onto l^1 then $TB = C$ is a bounded, closed convex subset of l^1 and as such has extreme points. In fact B is the closed, convex hull of its extreme points.

If X is a Banach space and $x \in X$ with $||x|| = 1$ we say that x is a denting point of the unit ball of X if for each $\varepsilon > 0$ the closed convex hull of the set

$$\{y \in X: ||y|| \leqq 1 \text{ and } ||y - x|| \geqq \varepsilon\}$$

does not contain x. The Radon-Nikodym property for l^1 also guarantees that $C(=TB)$ has denting points [3; p. 25, 30] and hence there are points of $S \subseteqq B^p$ which are denting points. Finally we define strong extreme point.

DEFINITION. A point x in a Banach space X, with $||x|| = 1$ is a strong extreme point of the unit ball of X if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\max\{\|x+y\|, \|x-y\|\} \leqq 1+\delta$$

implies $\|y\| \leqq \varepsilon$.

The strong extreme points of the unit balls in H^∞ and H^1 have been characterized (Cima and Thomson [2]). McGuigan [6; p. 116] has shown that any denting point is a strong extreme point.

In the next section we give a straightforward proof that B^p is rotund and we use this to prove that every point on the sphere S is a denting point.

2. The points of S are denting points. We recall an elementary fact from measure theory. The equality (for positive finite measures)

$$\int_\Delta |f(z)+g(z)|\,d\mu(z) = \int_\Delta |f(z)|\,d\mu(z) + \int_\Delta |g(z)|\,d\mu(z)$$

holds if and only if $f(z)/g(z) \equiv P(z)$ almost everywhere, where $P(z)$ is a nonnegative measurable function on the set $\{z \mid f(z)g(z) \neq 0\}$.

THEOREM 1. *Each point of S is an extreme point.*

Proof. Assume f, g, and h are in S and $f = 1/2(g+h)$. The equality

$$\int_\Delta |g(z)+h(z)|\,d\mu(z) = \int_\Delta |g(z)|\,d\mu(z) + \int_\Delta |h(z)|\,d\mu(z)$$

must hold. There is a nonvoid open set $\mathcal{O} \subset \Delta$ such that $g(z)/h(z) = P(z)$ is a real valued, nonnegative holomorphic function. It follows that $g(z) = \lambda h(z)$ in $\mathcal{O}$, where λ is a constant. The fact that g and h are in S shows that they equal on Δ. Hence f is an extreme point of B.

It is not known if B^p is isometrically isomorphic to the dual of a Banach space. However, B^p is equivalent to the dual space $(\lambda_\alpha^{n-1})^*$ (or $(\lambda_*^{n-1})^*$) (see [4; p. 49]). The space λ_α^{n-1} is the Lipschitz space of functions f analytic on Δ whose $(n-1)$st. derivative $f^{(n-1)}$ satisfies

$$\lim_{|h|\to 0} \left| \frac{f^{(n-1)}(e^{i(\theta+h)}) - f^{(n-1)}(e^{i\theta})}{h} \right| = 0\,.$$

The norm is the usual one (a condition on the second differences of $f^{(n-1)}$ is required for f to be a member of λ_*^{n-1}). The duality relationship is given as follows: $\Gamma \in (\lambda^{n-1})^*$ corresponds to $g(z) = \sum_{k=0}^\infty a_k z^k$ in B^p by

$$\Gamma(h) = \langle h, g\rangle = \lim_{r\to 1} \sum_{k=1}^{\infty} b_k a_k r^k ,$$

$h(z) = \sum_{k=0}^{\infty} b_k z^k$ and $C_2\|\Gamma\| \leqq \|g\|_{B^p} \leqq C_1\|\Gamma\|$. The functions

$$h_\zeta(z) = (1-\zeta z)^{-1} = \sum_{k=0}^{\infty} \zeta^k z^k \tag{2}$$

are in λ_α^{n-1} for each fixed ζ, $|\zeta| < 1$. Thus if a sequence $\{f_n\} \subseteqq B$ converges weak $*$ in B^p it also converges pointwise since $f_n(\zeta) = \langle f_n, h_\zeta\rangle$. However, the inequality

$$|f(z)| \leqq C(p)| \, \|f\|(1-|z|)^{-1/p}, f \in B^p \tag{3}$$

implies that norm bounded subsets of B^p are normal families. Then $\{f_n\}$ converges uniformly on compacta of Δ. Thus every sequence in B that converges weak $*$ in B^p also converges uniformly on compacta of Δ.

LEMMA 1. *B is weak $*$ compact. Also the weak $*$ topology, the topology of pointwise convergence and the topology of uniform convergence on compacta all coincide on B.*

Proof. Using the equivalence of B^p and $(\lambda_\alpha^{n-1})^*$ we have a weak $*$ compact ball B_0 of some positive radius such that $B \subset B_0$. Thus we must show B is weak $*$ closed. Since λ_α^{n-1} is separable, B is metrizable. Suppose $\{f_n\} \subset B$ and $\{f_n\}$ tends to f in the weak $*$ topology. By our preceding remarks $\lim_{n\to\infty} f_n(z) = f(z)$ for each $z \in \Delta$. But then

$$\int_\Delta |f(z)|\, d\mu(z) \leqq \lim_{n\to\infty} \int |f_n(z)|\, d\mu(z) \leqq 1$$

by Fatou's lemma. Thus $f \in B$.

By the preceding comments the weak $*$ topology is stronger than the topology of uniform convergence on compacta, which in turn is stronger than the topology of pointwise convergence. Since the topology of pointwise convergence is Hausdorff and the weak $*$ topology is compact all three of these topologies agree on B.

LEMMA 2. *Let $\{f_n\} \subseteqq B$ and $f \in S$. Then f_n converges pointwise to f if and only if $\|f_n - f\| \to 0$ as $n \to \infty$.*

Proof. If f_n converges pointwise to f, then Lemma 1 implies that f_n converges uniformly on compacta to f. Let $\varepsilon > 0$ be given and choose an annulus $A(r) = \{z: r \leqq |z| < 1\} \subseteqq \Delta$ such that

$$\int_{A(r)} |f(z)|\,d\mu(z) < \frac{\varepsilon}{4}\,.$$

There exists a postive integer N such that if $n \geqq N$ and $z \in \tilde{A}(r)$

$$|f_n(z) - f(z)| < \frac{\varepsilon}{4}\,.$$

Thus

$$\int_{\tilde{A}(r)} |f_n(z)|\,d\mu(z) \geqq \int_{\tilde{A}(r)} |f(z)|\,d\mu(z) - \int_{\tilde{A}(r)} |f_n(z) - f(z)|\,d\mu(z)$$
$$> \left(1 - \frac{\varepsilon}{4}\right) - \frac{\varepsilon}{4} = 1 - \frac{\varepsilon}{2}\,.$$

Hence,

$$\int_{A(r)} |f_n(z)|\,d\mu(z) < \frac{\varepsilon}{2}$$

and finally

$$\|f - f_n\| \leqq \int_{\tilde{A}(r)} |f(z) - f_n(z)|\,d\mu(z)$$
$$+ \int_{A(r)} |f(z)|\,d\mu(z) + \int_{A(r)} |f_n(z)|\,d\mu(z)$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon\,.$$

The converse follows directly from (1).

THEOREM 2. *Each point of S is a denting point of B.*

Proof. Let $f \in S$ and let $\varepsilon > 0$. We show that f is not in the closed convex hull of $B \backslash B_\varepsilon(f)(B_\varepsilon(f) = \{g \in B^p : \|f - g\| < \varepsilon\})$. As a consequence of the preceding lemma the weak $*$ neighborhoods of f in B form a neighborhood base for f with the norm topology on B. Thus there exists U, a weak $*$ open set, such that $f \in U$ and $B \cap U \subset B \cap B_\varepsilon(f)$. Then $B \backslash U$ is weak $*$ compact and $f \notin B \backslash U$. By an equivalent formulation of Proposition 25.13 in Choquet [1; p. 107] since f is an extreme point of B, f is not in the weak I closed convex hull of $B \backslash U$. Thus f is not in the norm closed convex hall of $B \backslash B_\varepsilon(f)$.

COROLLARY. *Each point of S is a strong extreme point of B.*

The situation in B^p is similar to that of H^1 where each outer function F of norm one is a strong extreme point. In contrast, a point f in the unit ball of H^∞ is a strong extreme point if and only if it is an inner function.

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