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THE 2-CLASS GROUP OF BIQUADRATIC FIELDS, II

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We describe methods for determining the exact power of 2 dividing the class number of certain cyclic biquadratic number fields. In a recent article, we developed a relative genus theory for cyclic biquadratic fields whose quadratic subfields have odd class number; we considered the case in which the quadratic subfield is $Q(\sqrt{l})$ with $l \equiv 5 \pmod 8$ a prime. Here we shall extend our methods to the cases in which the subfield is $Q(\sqrt{2})$ or $Q(\sqrt{l})$ with $l \equiv 1 \pmod 8$ a prime. We consider all such cases for which the 2-class group of the biquadratic field is of rank at most 3.

2. Notation and preliminaries.

Q: the field of rational numbers.

l: a rational prime satisfying $l = 2$ or $l \equiv 1 \pmod 8$.

p, q, p_i: rational primes.

k: the quadratic field $Q(\sqrt{l})$.

$\varepsilon = (u + v\sqrt{l})/2$, the fundamental unit of k, with $u, v > 0$.

m: a square-free positive rational integer, relatively prime to l.

$d = -m\sqrt{l}\,\varepsilon$.

K: the biquadratic field $k(\sqrt{d})$.

h, h_0: the class numbers of K and k, respectively.

$\left(\frac{x, y}{\pi}\right)$: the quadratic norm residue symbol over k.

$\left[\frac{\alpha}{\beta}\right]$: the quadratic residue symbol for k.

$\left(\frac{a}{b}\right)$: the rational quadratic residue (Legendre) symbol.

$\left(\frac{a}{b}\right)_4$: the rational 4th power residue symbol (defined if and only if $(a/b) = 1$).

$N(\ )$: the relative norm for K/k.

H: the 2-Sylow subgroup of the class group of K.

It is easy to see that K is a cyclic extension of Q of degree 4 which contains k. Recall that ε has (absolute) norm -1, that h_0 is odd and that H has rank $t - 1$, where t is the number of prime ideals of k which ramify in K.

3. Class number divisibility: The case $l \equiv 1 \pmod 8$.

THEOREM 1. *Let* $m = p \equiv 3 \pmod 4$. *Then*

$$h \equiv 2 \pmod 4) \quad if \quad \left(\frac{p}{l}\right) = -1\,;$$
$$\equiv 4 \pmod 8) \quad if \quad \left(\frac{p}{l}\right)_4 = -1\,;$$
$$\equiv 0 \pmod{16}) \quad if \quad \left(\frac{p}{l}\right)_4 = 1\,.$$

Proof. The number t of prime ideals of k which ramify in K is equal to 2 or 3 according as $(p/l) = -1$ or 1. In the first case,

$$\left(\frac{p,\, d}{\sqrt{l}}\right) = \left[\frac{p}{\sqrt{l}}\right] = \left(\frac{p}{l}\right) = -1\,,$$

so that only the principal ambiguous class is in the principal genus. By Theorem 1 of [1] we have $H \simeq Z_2$.

If $(p/l) = 1$, then $p = \pi_1\pi_2$, where π_1 and π_2 are prime ideals of k. The ideals $\pi_1^{h_0}$ and $\pi_2^{h_0}$ are principal ideals, and

$$\pi_1^{h_0} = a + b\sqrt{l} > 0\,,$$
$$\pi_2^{h_0} = a - b\sqrt{l} > 0\,.$$

Thus,
$$\left(\frac{a + b\sqrt{l},\, d}{\sqrt{l}}\right) = \left[\frac{a + b\sqrt{l}}{\sqrt{l}}\right] = \left(\frac{a}{l}\right) = \left(\frac{a^2}{l}\right)_4 = \left(\frac{p}{l}\right)_4\,.$$

Also,
$$\left(\frac{a + b\sqrt{l},\, d}{\pi_2}\right) = \left[\frac{a + b\sqrt{l}}{\pi_2}\right] = \left(\frac{2a}{p}\right)\,.$$

Because $p \equiv 3 \pmod 4$ and h_0 is odd, a is even; if $a = 2^i c$ with c odd, then $i = 1$ if and only if $p \equiv 3 \pmod 8$. Thus,

$$\left(\frac{2a}{p}\right) = \left(\frac{2}{p}\right)^{i+1}\left(\frac{c}{p}\right) = \left(\frac{c}{p}\right) = \left(\frac{-p}{c}\right)$$
$$= \left(\frac{l}{c}\right) = \left(\frac{c}{l}\right) = \left(\frac{c^2}{l}\right)_4 = \left(\frac{a^2}{l}\right)_4 = \left(\frac{p}{l}\right)_4\,.$$

We then have the following table of characters:

Norm\Character	$\sqrt{l}$	π_1	π_2
$\varepsilon\sqrt{l}$	1	$\left(\frac{p}{l}\right)_4$	$\left(\frac{p}{l}\right)_4$
$a + b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	1	$\left(\frac{p}{l}\right)_4$
$a - b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	$\left(\frac{p}{l}\right)_4$	1

If $(p/l)_4 = -1$, then only the principal ambiguous class is in the principal genus; by Theorem 1 of [1], we have $H \simeq Z_2 \times Z_2$, so that $h \equiv 4 \pmod 8$.

If $(p/l)_4 = 1$, then all four ambiguous classes are in the principal genus, so that $h \equiv 0 \pmod{16}$.

THEOREM 2. *Let* $m = p_1 p_2 \cdots p_t \equiv 3 \pmod 4$ *with* $(p_i/l) = -1$ *for all* i. *Then*

$$h \equiv 2^t \pmod{2^{t+1}} .$$

Proof. H has rank t, so we just need to show that the only ambiguous class in the principal genus is the principal class. Now

$$\left(\frac{p_i, d}{\sqrt{l}}\right) = \left[\frac{p_i}{\sqrt{l}}\right] = \left(\frac{p_i}{l}\right) = -1 , \quad \text{and}$$

$$\left(\frac{p_i, d}{p_j}\right) = \left[\frac{p_i}{p_j}\right] = 1 \quad \text{for} \quad i \neq j .$$

It follows that $(p_i, d/p_i) = -1$ and $(\varepsilon\sqrt{l}, d/p_i) = -1$, by the product rule. Thus, no two of the ramified prime ideals belong to the same genus, and so the desired result follows.

THEOREM 3. *Let* $m = pq \equiv 3 \pmod 4$ *with* $(p/l) = 1$ *and* $(q/l) = -1$. *Then*

$$\begin{aligned} h &\equiv 8 \pmod{16} \quad \textit{if} \quad \left(\frac{p}{l}\right)_4 \neq \left(\frac{q}{p}\right) ; \\ &\equiv 16 \pmod{32} \quad \textit{if} \quad p \equiv 1 \pmod 4 \quad \textit{and} \quad \left(\frac{p}{l}\right)_4 = \left(\frac{q}{p}\right) \neq \left(\frac{l}{p}\right)_4 ; \\ &\equiv 0 \pmod{32} \quad \textit{if either} \quad p \equiv 3 \pmod 4 \quad \textit{and} \quad \left(\frac{p}{l}\right)_4 = \left(\frac{q}{p}\right) , \\ &\qquad \textit{or} \quad p \equiv 1 \pmod 4 \quad \textit{and} \quad \left(\frac{p}{l}\right)_4 = \left(\frac{q}{p}\right) = \left(\frac{l}{p}\right)_4 . \end{aligned}$$

Proof. Here H has rank 3. Using the notation of Theorem 1, we have that

$$\left(\frac{a + b\sqrt{l}, d}{\pi_2}\right) = \left[\frac{a + b\sqrt{l}}{\pi_2}\right] = \left[\frac{2a}{\pi_2}\right] = \left(\frac{2a}{p}\right) .$$

If $p \equiv 3 \pmod 4$, then $(2a/p) = (p/l)_4$, as before. However, if $p \equiv 1 \pmod 4$, then

$$\left(\frac{2a}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{a}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{a^2}{p}\right)_4 = \left(\frac{2}{p}\right)\left(\frac{b}{p}\right)\left(\frac{l}{p}\right)_4 .$$

Now $b = 2^i c$ with c odd; furthermore, $i = 1$ if and only if $p \equiv 5 \pmod 8$. Hence,

$$\left(\frac{2}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{2}{p}\right)^{i+1}\left(\frac{c}{p}\right) = \left(\frac{c}{p}\right) = \left(\frac{p}{c}\right) = \left(\frac{a^2}{c}\right) = 1 ;$$

we deduce that $(2a/p) = (l/p)_4$. Furthermore,

$$\left(\frac{a + b\sqrt{l}, d}{q}\right) = \left[\frac{a + b\sqrt{l}}{q}\right] = \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right), \quad \text{and}$$

$$\left(\frac{q, d}{\pi_1}\right) = \left[\frac{q}{\pi_1}\right] = \left(\frac{q}{p}\right).$$

The remaining characters are easily evaluated; if we set $(l/p)_4 = (p/l)_4$ if $p \equiv 3 \pmod 4$, we have the following table of characters:

Norm\Character	$\sqrt{l}$	q	π_1	π_2
$\varepsilon\sqrt{l}$	-1	-1	$\left(\frac{p}{l}\right)_4$	$\left(\frac{p}{l}\right)_4$
q	-1	-1	$\left(\frac{q}{p}\right)$	$\left(\frac{q}{p}\right)$
$a + b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	$\left(\frac{q}{p}\right)$	$\left(\frac{q}{p}\right)\left(\frac{p}{l}\right)_4\left(\frac{l}{p}\right)_4$	$\left(\frac{l}{p}\right)_4$
$a - b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	$\left(\frac{q}{p}\right)$	$\left(\frac{l}{p}\right)_4$	$\left(\frac{q}{p}\right)\left(\frac{p}{l}\right)_4\left(\frac{l}{p}\right)_4$

The theorem follows, as before, from an analysis of the various cases.

THEOREM 4. *Let* $m = p \equiv 1 \pmod 4$ *with* $(p/l) = -1$. *Then*

$$\begin{aligned} h &\equiv 8 \pmod{16} \quad \text{if} \quad \left(\frac{2}{l}\right)_4 \neq \left(\frac{2}{p}\right); \\ &\equiv 16 \pmod{32} \quad \text{if} \quad \left(\frac{2}{l}\right)_4 = \left(\frac{2}{p}\right) = (-1)^{(l+7)/8}; \\ &\equiv 0 \pmod{32} \quad \text{if} \quad \left(\frac{2}{l}\right)_4 = \left(\frac{2}{p}\right) = (-1)^{(l-1)/8}. \end{aligned}$$

Proof. Here, the two prime divisors of 2 in k ramify in K. Put $2 = 2_1 2_2$ in k, with

$$2_1^{h_0} = \alpha = \frac{a + b\sqrt{l}}{2} > 0,$$

and

$$2_1^{h_0} = \bar{\alpha} = \frac{a - b\sqrt{l}}{2} > 0 .$$

Then

$$\left(\frac{\alpha, d}{\sqrt{l}}\right) = \left[\frac{\alpha}{\sqrt{l}}\right] = \left[\frac{a/2}{\sqrt{l}}\right] = \left(\frac{2a}{l}\right)$$
$$= \left(\frac{4a^2}{l}\right)_4 = \left(\frac{2}{l}\right)_4 ,$$
$$\left(\frac{\alpha, d}{p}\right) = \left[\frac{\alpha}{p}\right] = \left(\frac{2}{p}\right) , \quad \text{and}$$
$$\left(\frac{p, d}{2_1}\right) = (-1)^{(p-1)/2} = 1. \quad \text{Now}$$
$$\left[\frac{a + b\sqrt{l}}{2}\right]^2 = \frac{1}{2}(a^2 - 2^{h_0+1} + ab\sqrt{l}) , \quad \text{so that}$$
$$a\bar{\alpha} \equiv \frac{1}{2}(a^2 - ab\sqrt{l}) \equiv a^2 - 2^{h_0} \,(\mathrm{mod}\ 2_1^2) . \quad \text{Thus,}$$
$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = \left(\frac{a, d}{2_1}\right)\left(\frac{a^2 - 2^{h_0}, d}{2_1}\right)$$
$$= (-1)^{(a-1)/2}(-1)^{(a^2-2^{h_0}-1)/2}$$
$$= \left(\frac{-1}{a}\right)(-1)^{2^{h_0-1}} .$$

To evaluate $(-1/a)$, note that

$$\left(\frac{a}{l}\right) = \left(\frac{a^2}{l}\right)_4 = \left(\frac{2}{l}\right)_4$$

and

$$\left(\frac{2}{a}\right) = \left(\frac{-l}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{l}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{a}{l}\right) .$$

Hence,

$$\left(\frac{-1}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{a}{l}\right) = \left(\frac{2}{a}\right)\left(\frac{2}{l}\right)_4 .$$

Since $(2/b) = 1$, we have $b^2 \equiv 1 \,(\mathrm{mod}\ 16)$, so that

$$a^2 - lb^2 \equiv a^2 - l \equiv 2^{h_0+2} \,(\mathrm{mod}\ 16) .$$

If $h_0 = 1$, then $a^2 \equiv l + 8 \,(\mathrm{mod}\ 16)$, so that

$$\left(\frac{2}{a}\right) = 1 \quad \text{if and only if} \quad l \equiv 9 \,(\mathrm{mod}\ 16) ;$$

if $h_0 > 1$, then $a^2 \equiv l \pmod{16}$, so that

$$\left(\frac{2}{a}\right) = 1 \quad \text{if and only if} \quad l \equiv 1 \pmod{16} .$$

In either case,

$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = (-1)^{2^{h_0-1}}\left(\frac{-1}{a}\right) = (-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4 .$$

Finally, we note that

$$\left(\frac{p, d}{\sqrt{l}}\right) = \left(\frac{p, d}{p}\right) = -1 .$$

This yields the following table of generic characters:

Norm\Characters	$\sqrt{l}$	p	2_1	2_2
p	-1	-1	$+1$	$+1$
α	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4$
$\bar{\alpha}$	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)$

If $(2/l)_4 \neq (2/p)$, then all three lines of the table are distinct and only the principal ambiguous class lies in the principal genus; this implies that $h \equiv 8 \pmod{16}$.

If $(2/l)_4 = (2/p) \neq (-1)^{(l-1)/8}$, then the last two lines are identical, but different from the first. Here, exactly two ambiguous classes lie in the principal genus, and so $h \equiv 16 \pmod{32}$.

In the case $(2/l)_4 = (2/p) = (-1)^{(l-1)/8}$, there are 4 ambiguous classes in the principal genus. Thus $h \equiv 0 \pmod{32}$.

COROLLARY. *If* $m = 1$, *then*

$$h \equiv 4 \pmod 8 \quad \textit{if} \quad \left(\frac{2}{l}\right)_4 = -1 ;$$

$$\equiv 8 \pmod{16} \quad \textit{if} \quad l \equiv 9 \pmod{16} \quad \textit{and} \quad \left(\frac{2}{l}\right)_4 = 1 ;$$

$$\equiv 0 \pmod{16} \quad \textit{if} \quad l \equiv 1 \pmod{16} \quad \textit{and} \quad \left(\frac{2}{l}\right)_4 = 1 .$$

Proof. Here $t = 3$ and so H has rank 2. The table of generic characters is obtained by setting $(2/p) = 1$ in the last two lines of the table in Theorem 4. There are 1, 2 or 4 ambiguous classes in the principal genus according as the condition of the first, second or third line of the corollary holds.

THEOREM 5. *If $m = 2$, then*

$$\begin{aligned} h &\equiv 4 \pmod 8, \quad \text{if } \left(\frac{2}{l}\right)_4 = -1; \\ &\equiv 0 \pmod{16}, \quad \text{if } \left(\frac{2}{l}\right)_4 = 1. \end{aligned}$$

Proof. Using the notation of the preceding theorem, we have

$$\begin{aligned} \left(\frac{\bar{\alpha}, d}{2_1}\right) &= \left(\frac{\bar{\alpha}, -2\varepsilon\sqrt{l}}{2_1}\right) = \left(\frac{\bar{\alpha}, 2}{2_1}\right)\left(\frac{\bar{\alpha}, -\varepsilon\sqrt{l}}{2_1}\right) \\ &= \left(\frac{\bar{\alpha}, 2}{2_1}\right)(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4, \end{aligned}$$

the last step following from the calculations of Theorem 4. Now

$$\alpha^3 = \left(\frac{a + b\sqrt{l}}{2}\right)^3 = \left(\frac{1}{2}\right)(a(a^2 - 3\cdot 2^{h_0}) + b(a^2 - 2^{h_0})\sqrt{l}),$$

so that

$$\begin{aligned} \left(\frac{\bar{\alpha}, 2}{2_1}\right) &= \left(\frac{a^2 - 2^{h_0}, 2}{2_1}\right)\left(\frac{a(a^2 - 2^{h_0+1}), 2}{2_1}\right) \\ &= \left(\frac{2}{a^2 - 2^{h_0}}\right)\left(\frac{2}{a}\right)\left(\frac{2}{a^2 - 2^{h_0+1}}\right) \\ &= (-1)^{2^{h_0-1}}\left(\frac{2}{a}\right) = (-1)^{(l-1)/8}. \end{aligned}$$

Hence,

$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = (-1)^{(l-1)/8}(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4 = \left(\frac{2}{l}\right)_4.$$

We obtain the following table of characters and the result follows by considerations similar to those previously mentioned:

Norm\Character	$\sqrt{l}$	2_1	2_2
$\varepsilon\sqrt{l}$	1	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{l}\right)_4$
α	$\left(\frac{2}{l}\right)_4$	1	$\left(\frac{2}{l}\right)_4$
$\bar{\alpha}$	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{l}\right)_4$	1

THEOREM 6. *If $m = 2p$ with $(p/l) = -1$, then*

$$\begin{aligned} h &\equiv 8 \pmod{16} \quad \textit{if} \quad \left(\frac{2}{l}\right)_4 \neq \left(\frac{2}{p}\right); \\ &\equiv 16 \pmod{32} \quad \textit{if} \quad \left(\frac{2}{l}\right)_4 = \left(\frac{2}{p}\right) \neq (-1)^{(l-1)/8}, \\ &\qquad\qquad \textit{and } p \equiv 3 \pmod 8; \\ &\equiv 0 \pmod{32}, \quad \textit{otherwise}. \end{aligned}$$

Proof. First we note that

$$\begin{aligned} \left(\frac{\bar{\alpha}, d}{2_1}\right) &= \left(\frac{\bar{\alpha}, -2p\varepsilon\sqrt{l}}{2_1}\right) = \left(\frac{\bar{\alpha}, 2}{2_1}\right)\left(\frac{\bar{\alpha}, -\varepsilon p\sqrt{l}}{2_1}\right) \\ &= (-1)^{(l-1)/8}\left(\frac{\bar{\alpha}, -\varepsilon p\sqrt{l}}{2_1}\right). \end{aligned}$$

If $p \equiv 1 \pmod 4$, then the last symbol was evaluated in the proof of Theorem 4 and reduces to $(-1)^{(l-1)/8}(2/l)_4$.

If $p \equiv 3 \pmod 4$, then 2_1 is unramified in the extension $Q(\sqrt{d_1})$, where $d_1 = -\varepsilon p\sqrt{l}$. Thus, the last symbol is equal to 1. Hence

$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = \left(\frac{\alpha, d}{2_2}\right) = \left(\frac{2}{l}\right)_4 \quad \text{or} \quad (-1)^{(l-1)/8}$$

according as $p \equiv 1$ or $3 \pmod 4$. Evaluation of the remaining symbols is routine, and we have the following table for $p \equiv 3 \pmod 4$:

Norm\Character	$\sqrt{l}$	p	2_1	2_2
$\varepsilon\sqrt{l}$	-1	-1	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{l}\right)_4$
p	-1	-1	$\left(\frac{2}{p}\right)$	$\left(\frac{2}{p}\right)$
α	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)\left(\frac{2}{l}\right)_4$	$(-1)^{(l-1)/8}$
$\bar{\alpha}$	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)\left(\frac{2}{l}\right)_4$

If $p \equiv 1 \pmod 4$, the four entries in the lower right-hand corner are replaced by

$$\begin{matrix} \left(\frac{2}{p}\right) & \left(\frac{2}{l}\right)_4 \\ \left(\frac{2}{l}\right)_4 & \left(\frac{2}{p}\right) \end{matrix}$$

and the desired results follow as before.

4. Class numbers divisibility: The case $l = 2$.

THEOREM 7. *If $m = p$, then*

$$\begin{aligned} h &\equiv 2 \pmod 4, \quad \text{if } p \equiv \pm 3 \pmod 8; \\ &\equiv 4 \pmod 8, \quad \text{if } p \equiv \pm 7 \pmod{16}; \\ &\equiv 8 \pmod{16}, \quad \text{if } p \equiv 1 \pmod{16} \text{ and } \left(\frac{2}{p}\right)_4 = -1; \\ &\equiv 0 \pmod{16}, \quad \text{if } p \equiv 1 \pmod{16} \text{ and } \left(\frac{2}{p}\right)_4 = 1, \text{ or} \\ &\qquad \text{if } p \equiv 15 \pmod{16}. \end{aligned}$$

Proof. If $p \equiv \pm 3 \pmod 8$ then H is cyclic and

$$\left(\frac{p, d}{\sqrt{2}}\right) = \left(\frac{2}{p}\right) = -1 .$$

Hence, the only ambiguous class in the principal genus is the principal class, and so $H \simeq Z_2$.

If $p \equiv \pm 1 \pmod 8$ then H has rank 2. Let $p = \pi_1\pi_2 = (a + b\sqrt{2})(a - b\sqrt{2})$ with $\pi_1 = a + b\sqrt{2} > 0$. If $p \equiv 7 \pmod 8$, then

$$\begin{aligned} \left(\frac{\pi_1, d}{\pi_2}\right) &= \left[\frac{\pi_1}{\pi_2}\right] = \left[\frac{2a}{\pi_2}\right] = \left(\frac{2a}{p}\right) = \left(\frac{a}{p}\right) \\ &= \left(\frac{-1}{a}\right)\left(\frac{p}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{-2b^2}{a}\right) \\ &= \left(\frac{2}{a}\right) = (-1)^{(a^2-1)/8} = (-1)^{(p+2b^2-1)/8} \\ &= (-1)^{(p+1)/8} , \end{aligned}$$

since b must be odd. Furthermore,

$$b\varepsilon\sqrt{2} = 2b + b\sqrt{2} \equiv 2b - a \pmod{\pi_1} ,$$

so that

$$b^2\varepsilon\sqrt{2} \equiv 2b^2 - ab \equiv a^2 - ab \equiv a(a - b) \pmod{\pi_1} .$$

Thus,

$$\left(\frac{\varepsilon\sqrt{2}, d}{\pi_1}\right) = \left[\frac{\varepsilon\sqrt{2}}{\pi_1}\right] = \left(\frac{a(a-b)}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{a-b}{p}\right) .$$

But $(a - b)(a + b) = a^2 - b^2 = p + b^2$, so if $a - b = 2^t c$ with c odd, we have

$$\left(\frac{a-b}{p}\right)=\left(\frac{2}{p}\right)^i\left(\frac{c}{p}\right)=\left(\frac{c}{p}\right)=\left(\frac{-p}{c}\right)=\left(\frac{b^2}{c}\right)=1\,.$$

Hence,

$$\left(\frac{\varepsilon\sqrt{2}\,,\,d}{\pi_1}\right)=\left(\frac{a}{p}\right)=(-1)^{(p+1)/8}\,.$$

Thus, for $p\equiv 7 \pmod 8$, we have the following table of generic characters:

Norm\Character	$\sqrt{2}$	π_1	π_2
$\varepsilon\sqrt{2}$	1	$(-1)^{(p+1)/8}$	$(-1)^{(p+1)/8}$
π_1	$(-1)^{(p+1)/8}$	1	$(-1)^{(p+1)/8}$
π_2	$(-1)^{(p+1)/8}$	$(-1)^{(p+1)/8}$	1

If $p\equiv 7 \pmod{16}$, then none of the above lines are the same, so that $h\equiv 4 \pmod 8$; if $p\equiv 15 \pmod{16}$, then all of the above lines are the same, so that $h\equiv 0 \pmod{16}$.

Now let $p\equiv 1 \pmod 8$. Then

$$\left(\frac{\pi_1,\,d}{\pi_2}\right)=\left(\frac{a}{p}\right)=\left(\frac{a^2}{p}\right)_4=\left(\frac{2b^2}{p}\right)_4$$
$$=\left(\frac{2}{p}\right)_4\left(\frac{b}{p}\right)\,.$$

Setting $b=2^ic$ with c odd, we have

$$\left(\frac{b}{p}\right)=\left(\frac{2}{p}\right)^i\left(\frac{c}{p}\right)=\left(\frac{c}{p}\right)=\left(\frac{p}{c}\right)=\left(\frac{a^2}{c}\right)=1\,.$$

Hence,

$$\left(\frac{\pi_1,\,d}{\pi_2}\right)=\left(\frac{\pi_2,\,d}{\pi_1}\right)=\left(\frac{2}{p}\right)_4\,.$$

Now

$$\left(\frac{\varepsilon\sqrt{2}\,,\,d}{\pi_2}\right)=\left(\frac{a}{p}\right)\left(\frac{a-b}{p}\right)=\left(\frac{2}{p}\right)_4\left(\frac{a-b}{p}\right)\,.$$

Since $(a-b)(a+b)=p+b^2$, we have

$$\left(\frac{a-b}{p}\right)=\left(\frac{p}{a-b}\right)=\left(\frac{-b^2}{a-b}\right)=\left(\frac{-1}{a-b}\right)\,.$$

A paper of G. Pall [2] contains a table, part of which we reproduce here:

$$p = a^2 - 2b^2 = u^2 + v^2\,,\quad v \text{ even}$$

$p \pmod{16}$	$v \pmod 8$	$a \pmod 8$	$b \pmod 4$
1	4	7	0
1	4	5	2
1	0	3	2
1	0	1	0
9	0	1	2
9	0	3	0
9	4	5	0
9	4	7	2

Thus, if $p \equiv 1 \pmod{16}$, then $(-1/(a-b)) = 1$ if and only if $v \equiv 0 \pmod 8$, and if $p \equiv 9 \pmod{16}$, then $(-1/(a-b)) = 1$ if and only if $v \equiv 4 \pmod 8$, so

$$\left(\frac{-1}{a-b}\right) = (-1)^{v/4}(-1)^{(p-1)/8}\,.$$

Now, Dirichlet's necessary and sufficient condition that $(2/p)_4 = 1$ is that $v \equiv 0 \pmod 8$. Hence, $(2/p)_4 = (-1)^{v/4}$;

$$\begin{aligned}\left(\frac{\varepsilon\sqrt{2}\,,\,d}{\pi_1}\right) &= \left(\frac{a}{p}\right)\left(\frac{a-b}{p}\right) = \left(\frac{2}{p}\right)_4\left(\frac{-1}{a-b}\right)\\ &= \left(\frac{2}{p}\right)_4(-1)^{v/4}(-1)^{(p-1)/8}\\ &= \left(\frac{2}{p}\right)_4\left(\frac{2}{p}\right)_4(-1)^{(p-1)/8} = (-1)^{(p-1)/8}\,.\end{aligned}$$

We thus have the following table:

Norm\Character	$\sqrt{2}$	π_1	π_2
$\varepsilon\sqrt{2}$	1	$(-1)^{(p-1)/8}$	$(-1)^{(p-1)/8}$
π_1	$(-1)^{(p-1)/8}$	$(-1)^{(p-1)/8}\left(\frac{2}{p}\right)_4$	$\left(\frac{2}{p}\right)_4$
π_2	$(-1)^{(p-1)/8}$	$\left(\frac{2}{p}\right)_4$	$(-1)^{(p-1)/8}\left(\frac{2}{p}\right)_4$

If $p \equiv 9 \pmod{16}$, then each line is different; thus, only the principal ambiguous class belongs to the principal genus, and so $H \simeq Z_2 \times Z_2$, $h \equiv 4 \pmod 8$.

If $p \equiv 1 \pmod{16}$, then there are either two or four ambiguous classes in the principal genus, according as $(2/p)_4 = -1$ or 1. In these cases, $h \equiv 8$ or $0 \pmod{16}$, respectively.

THEOREM 8. *If* $m = p_1 \cdots p_t$ *with* $(2/p_i) = -1$ *for all* i, *then*

$$h \equiv 2^t \pmod{2^{t+1}} .$$

Comment. The proof is quite similar to the proof of Theorem 2, so we omit it.

THEOREM 9. *Let* $m = pq$ *with* $(2/p) = 1$ *and* $(2/q) = -1$. *If* $p \equiv 1 \pmod 8$, *then*

$$\begin{aligned} h &\equiv 8 \pmod{16}, \quad \textit{if} \quad \left(\frac{p}{q}\right) \neq (-1)^{(p-1)/8} ; \\ &\equiv 16 \pmod{32}, \quad \textit{if} \quad \left(\frac{2}{p}\right)_4 \neq (-1)^{(p-1)/8} = \left(\frac{p}{q}\right) ; \\ &\equiv 0 \pmod{32}, \quad \textit{otherwise} . \end{aligned}$$

If $p \equiv 7 \pmod 8$, *then*

$$\begin{aligned} h &\equiv 8 \pmod{16}, \quad \textit{if} \quad \left(\frac{p}{q}\right) \neq (-1)^{(p+1)/8} ; \\ &\equiv 16 \pmod{32}, \quad \textit{if} \quad q \equiv 3 \pmod 4 \quad \textit{and} \quad \left(\frac{p}{q}\right) = (-1)^{(p+1)/8} = -1 ; \\ &\equiv 0 \pmod{32}, \quad \textit{otherwise} . \end{aligned}$$

Comment. The proof involves straightforward extensions of the tables, constructed in the proof of Theorem 7, so we will omit it.

5. **Numerical results.** A slight modification of the methods described in [3] allow us to compute the relative class number $h^* = h/h_0$ of K. As $h_0 = 1$ for most small values of l, we have $h^* = h$ for almost all values within the range of our computations. In the tables below we list all fields within the range of our calculations, where the maximum power of dividing h^* exceeds the power predicted in §3. We have only computed values of h^* for the fields discussed in Theorems 1, 4, 5, 6, and 7. The column of the table headed by f gives the prime factorization of h^*.

Table 1 ($d = -\varepsilon\sqrt{l}\, p,\ p \equiv 3 \bmod 4$)

l	p	h^*	f
17	67	160	$2^5\cdot 5$
	103	32	$2^5\cdot$
	251	1088	$2^6\cdot 17$
	463	160	$2^5\cdot 5$
41	23	32	2^5
	59	288	$2^5\cdot 9$
	83	1184	$2^5\cdot 37$
	139	832	$2^6\cdot 13$
	163	1312	$2^5\cdot 41$
	223	256	2^8
	271	160	$2^5\cdot 5$
	283	3328	$2^8\cdot 13$
	379	2080	$2^5\cdot 5\cdot 13$
	491	2592	$2^5\cdot 3^4$

Table 1 (con't) ($d = -\varepsilon\sqrt{l}\, p,\ p \equiv 3 \bmod 4$)

l	p	h^*	f
73	71	640	$2^7\cdot 5$
89	67	128	2^7
97	47	64	2^6
	103	544	$2^5\cdot 17$
113	7	160	$2^5\cdot 5$
193	3	160	$2^5\cdot 5$
	47	576	$2^6\cdot 3^2$
233	71	5696	$2^6\cdot 89$
	107	800	$2^5\cdot 5^2$
257*	11	64	2^6
	23	640	$2^6\cdot 5$
	67	416	$2^5\cdot 13$
281	59	160	$2^5\cdot 5$

(*) $h_0 = 3$ when $l = 257$.

Table 2 ($d = -\varepsilon\sqrt{l}\, p,\ p \equiv 1 \bmod 4$)

l	p	h^*	f
17	149	320	$2^6\cdot 5$
	157	512	2^9
	229	640	$2^7\cdot 5$
	293	640	$2^7\cdot 5$
	353	1024	2^{10}
	389	1600	$2^6\cdot 5^2$
	409	832	$2^6\cdot 13$
41	53	832	$2^6\cdot 13$
	61	320	$2^6\cdot 5$
	109	576	$2^6\cdot 3^2$

Table 2 (con't) ($d = -\varepsilon\sqrt{l}\, p,\ p \equiv 1 \bmod 4$)

l	p	h^*	f
41	173	1856	$2^6\cdot 29$
	181	1088	$2^6\cdot 17$
	197	2048	2^{11}
	229	1600	$2^6\cdot 5^2$
	269	1600	$2^6\cdot 5^2$
	293	3200	$2^7\cdot 5^2$
	373	4096	2^{12}
	389	2176	$2^7\cdot 17$
	433	5248	$2^7\cdot 41$
73	41	320	$2^6\cdot 5$

Table 2 (con't)

($d = -\varepsilon\sqrt{l}\,p,\ p \equiv 1 \bmod 4$)

l	p	h^*	f
78	89	512	2^9
	109	2368	$2^6 \cdot 37$
89	73	2560	$2^9 \cdot 5$
	97	2560	$2^9 \cdot 5$
97	53	512	2^9
	101	832	$2^6 \cdot 13$
	109	3904	$2^6 \cdot 61$
113	17	320	$2^6 \cdot 5$
	41	1088	$2^6 \cdot 17$
	53	832	$2^6 \cdot 13$
	73	1600	$2^6 \cdot 5^2$
	89	3712	$2^7 \cdot 29$
	97	4352	$2^8 \cdot 17$
	109	1664	$2^7 \cdot 13$
137	5	128	2^7
	53	1664	$2^7 \cdot 13$

Table 2 (con't)

($d = -\varepsilon\sqrt{l}\,p,\ p \equiv 1 \bmod 4$)

l	p	h^*	f
137	73	1280	$2^8 \cdot 5$
	109	3136	$2^6 \cdot 7^2$
193	101	10816	$2^6 \cdot 13^2$
233	29	1280	$2^8 \cdot 5$
	37	2304	$2^8 \cdot 3^2$
241	5	128	2^7
	61	4608	$2^9 \cdot 3^2$
	97	16000	$2^7 \cdot 5^3$
257	17	832	$2^6 \cdot 13$
	41	2560	$2^9 \cdot 5$
	73	3200	$2^7 \cdot 5^2$
	89	4672	$2^6 \cdot 73$
281	29	1600	$2^6 \cdot 5^2$
	101	2176	$2^7 \cdot 17$
	109	6400	$2^8 \cdot 5^2$

Note: For tables 1 and 2, $p < 500$ when $l = 17$ or 41 and $p < 110$ otherwise.

Table 3

($d = -m\varepsilon\sqrt{l},\ m = 1$ or 2)

l	m	h^*	f
257	1	32	2^5
337	1	256	2^8
89	2	64	2^6
113	2	32	2^5
233	2	128	2^7

Table 4 ($d = -2\varepsilon\sqrt{l}\,p$)

l	p	h^*	f
17	5	32	2^5
	37	320	$2^6 \cdot 5$
	47	320	$2^6 \cdot 5$
	61	256	2^8
41	3	32	2^5
	11	256	2^8
	13	128	2^7
	19	512	2^9
	23	256	2^8
	31	640	$2^7 \cdot 5$
	53	576	$2^6 \cdot 3^2$
	67	512	2^9
73	17	832	$2^6 \cdot 13$
	37	576	$2^6 \cdot 3^2$
	41	3200	$2^7 \cdot 5^2$
	71	4352	$2^8 \cdot 17$
89	11	512	2^9
	17	320	$2^6 \cdot 5$
	67	1600	$2^6 \cdot 5^2$
	73	1600	$2^6 \cdot 5^2$
97	5	320	$2^6 \cdot 5$
	13	320	$2^6 \cdot 5$
	47	3200	$2^7 \cdot 5^2$

Table 4 (con't) ($d = -2\varepsilon\sqrt{l}\,p$)

l	p	h^*	f
113	7	320	$2^6 \cdot 5$
	23	640	$2^7 \cdot 5$
	31	1152	$2^7 \cdot 3^2$
	41	2368	$2^6 \cdot 3^7$
	53	1600	$2^6 \cdot 5^2$
	71	1664	$2^7 \cdot 13$
	73	3712	$2^7 \cdot 29$
137	13	512	2^9
	43	2624	$2^6 \cdot 41$
	67	3904	$2^6 \cdot 61$
	73	3904	$2^6 \cdot 61$
193	5	320	$2^6 \cdot 5$
	7	1152	$2^7 \cdot 3^2$
	13	3328	$2^8 \cdot 13$
	37	3392	$2^7 \cdot 53$
	53	1664	$2^7 \cdot 13$
	61	11072	$2^6 \cdot 173$
233	19	1280	$2^8 \cdot 5$
	23	3328	$2^8 \cdot 13$
	37	3712	$2^7 \cdot 29$
	71	5248	$2^7 \cdot 41$
	73	3328	$2^8 \cdot 13$

Table 5 ($d = -\varepsilon\sqrt{2}\,p$)

p	h^*	f
47	32	2^5
127	160	$2^5 \cdot 5$
223	160	$2^5 \cdot 5$

Table 5 (con't) ($d = -\varepsilon\sqrt{2}\,p$)

p	h^*	f
239	320	$2^6 \cdot 5$
257	160	$2^5 \cdot 5$
271	160	$2^5 \cdot 5$

Table 5 (con't)			Table 5 (con't)		
($d = -\varepsilon\sqrt{2}\,p$)			($d = -\varepsilon\sqrt{2}\,p$)		
p	h^*	f	p	h^*	f
367	160	$2^5\cdot 5$	1279	640	$2^7\cdot 5$
431	320	$2^6\cdot 5$	1423	1088	$2^6\cdot 17$
463	640	$2^7\cdot 5$	1439	1600	$2^6\cdot 5^2$
479	160	$2^5\cdot 5$	1553	800	$2^5\cdot 5^2$
577	416	$2^5\cdot 13$	1601	640	$2^7\cdot 5$
751	576	$2^6\cdot 3^2$	1663	1088	$2^6\cdot 17$
1039	800	$2^5\cdot 5^2$	1759	1664	$2^7\cdot 13$
1151	640	$2^7\cdot 5$	1823	1184	$2^5\cdot 5\cdot 17$
1153	544	$2^5\cdot 17$	1889	1184	$2^5\cdot 37$
1201	1088	$2^6\cdot 17$	1951	1312	$2^5\cdot 41$
1217	512	2^9			

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