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EQUIDISTRIBUTION THEORY IN HIGHER DIMENSIONS

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Let X, Y be complex spaces, and $f: X \rightharpoonup Y$ a meromorphic map. Assume in Y an admissible family $\mathfrak{A} = \{S_b\}_{b \in N}$ of analytic subsets S_b is given. Assume f is almost adapted to $\mathfrak{A}$. The purpose of this paper is to prove that, if f satisfies certain growth conditions, the valence of S_b (for almost all $S_b \in \mathfrak{A}$) grows to infinity at the same rate as the characteristic of f. Here X is assumed to carry an exhaustion function which is, e.g., g-concave, centrally g-convex or g-quasiparabolic.

The results obtained generalize the Casorati-Weierstrass type theorems of Chern [4] [6], Cowen [7], Griffiths-King [12], Stoll [23] [26], Wu [31, II-III] (see also Griffiths [10]).

Introduction. It is well-known that the classical Casorati-Weierstrass theorem is not true in higher dimensions. In fact, the standard example of Fatou-Bierberbach [2, p. 45] gives a holomorphic imbedding of $\boldsymbol{C}^2$ into $\boldsymbol{P}_2$ with a nondense image. Chern [4] first showed that a holomorphic map $f: \boldsymbol{C}^n \to \boldsymbol{P}_n$ whose characteristic grows sufficiently rapidly assumes almost every point in $\boldsymbol{P}_n$. This result was generalized to subvarieties of a general codimension in a complex manifold by Hirshfelder [13] and Stoll [21]-[23]. In Wu [31] certain geometric conditions were given which ensure the Casorati-Weierstrass property. For instance, if $\boldsymbol{C}^n$ is given the Fubini-Study metric, then a nondegenerate quasi-conformal holomorphic map $f: \boldsymbol{C}^n \to \boldsymbol{P}_n$ assumes almost every point in $\boldsymbol{P}_n$. This in fact carries over to a balanced holomorphic map of $\boldsymbol{C}^m$ into $\boldsymbol{P}_n$ (see [10, p. 54]), whose image intersects almost every $(n-p)$-dimensional linear subspace of $\boldsymbol{P}_n$ (where $0 < p \leqq \min(m, n)$).

Let $f: X \rightharpoonup Y$ be a meromorphic map between complex spaces X, Y. Assume in Y an admissible family $\mathfrak{A} = \{S_b\}_{b \in N}$ is given. This means $\mathfrak{A}$ is defined by two holomorphic maps $Y \overset{h}{\leftarrow} M \overset{\pi}{\rightarrow} N$ (where M is a complex space, N a compact complex manifold) such that (i) π is open, surjective; (ii) h is proper, locally trivial at every point of M; (iii) each S_b is the topological image of $\pi^{-1}(b)$ under h, and S_b contains no branch of Y. Then S_b is analytic of pure codimension s in Y for all b. The main purpose of this paper is to establish the equidistribution property that, for almost every $S_b \in \mathfrak{A}$, the valence of S_b grows (over suitable sequence of domains) at the same rate as the characteristic of f. The admissible family defined here is more general than the one given in Stoll [23]. This makes it possible to include, for instance, the Schubert varieties as special cases. Moreover, the Kählerian assumption on the index manifold is no longer required in view of the results of Dektyarev [8] and Stoll [23] (see Theorem 2.5).

The equidistribution theorems are proved in §4 for different types of spaces. Here only the centrally g-convex type will be set out. Let φ be a C^∞ exhaustion function of X. Thus $\varphi: X \to \boldsymbol{R}$ is a C^∞ map such that the sets $X[r] = \{x \in X \mid \varphi(x) \leqq r\}$ are compact for all $r \geqq 0$. Let $L(\varphi) = dd^c\varphi$ be the Levi form of φ. Let $g: \boldsymbol{R}(0, \infty) \dashrightarrow \boldsymbol{R}$ be an increasing function of class C^1 with $\|e^{-g}\|_1^r = \int_1^r e^{-g(t)}dt \longrightarrow \infty$ as $r \to \infty$. Then φ is said to be *centrally g-convex* (*c.* g-convex) if $X[0]$ has measure zero and if

$L(\varphi) \geqq (g' \circ \varphi) d\varphi \wedge d^c\varphi$ off a closed nowhere dense set.

It follows that $L(\varphi) \geqq 0$ on X (Lemma 2.1) and, setting $u = e^{-g}$,

$$\omega_u = (u \circ \varphi)[L(\varphi) - (g' \circ \varphi) d\varphi \wedge d^c\varphi] \geqq 0 \qquad \text{on} \quad X - X[0] .$$

If further

$$(\omega_u)^m \equiv 0 \quad \text{off a compact set } (m = \dim X) ,$$

then φ is called *g-semiparabolic*. A logarithmic pseudoconvex exhaustion function (in the sense of Stoll [25]) is g-convex (with $g = \log$). It is not clear to what extent the g-convexity generalizes logarithmic pseudoconvexity, except in the trivial case where $g =$ constant (see §4 for an example).

Assume φ is $c.g$-convex. Define $X(r) = \{x \in X \mid \varphi(x) < r\}$, $\chi_p = L(\varphi)^p$. Let $k = \dim N$. Let $\omega_{N,1}$ be the fundamental form on N of a Hermitian metric normalized so that $\int_N (\omega_{N,1})^k = 1$. The fiber integration operator induced by h is denoted by h_*. For $\xi \in A_0^{p',p'}(N)$, $p' \geqq k - s$, define $\xi_Y = h_* \pi^* \xi$. Also set $\Omega_p = ((\omega_{N,1})^{k-s+p})_Y$, $0 \leqq p \leqq s$. If $r > r' > 0$, $p' = k - s + p$, and $\xi \geqq 0$, define

$$D^u_{f,p}(r, \xi_Y) = \int_{X(r)} f^*\xi_Y \wedge (\omega_u)^{m-p}$$

$$A^u_{f,p}(r, \xi_Y) = u(r)^{m-p} \int_{X(r)} f^*\xi_Y \wedge \chi_{m-p}$$

$$T^u_{f,p}(r, r', \xi_Y) = \int_{r'}^r A^u_{f,p}(t, \xi_Y) u(t) dt .$$

(The existence of the integrals will be established in §4.)

THEOREM. *Assume $f: X \rightharpoonup Y$ is almost adapted to $\mathfrak{A}$. Assume either $\chi_q(q = m - s \geqq 0)$ is semi-positive on an effective open set or*

$\chi_q \wedge f^*\Omega_s \not\equiv 0$. (1) *Assume there exists a positive form* $\xi \in A_0^{k-1,k-1}(N)$ *such that over some* φ*-admissible sequence* $\sigma = \{r_j\}$, *one of the following conditions holds*:

(a) $A^u_{f,s-1}(r, \xi_Y) = o'(T^u_{f,s}(r, 1, \Omega_s))$.

(b) $D^u_{f,s-1}(r, \xi_Y) = o'\left(\int_1^r D^u_{f,s}(t, \Omega_s)u(t)dt + A^u_{f,s}(o, \Omega_s)\|u\|_1^r\right)$.

Then there is a set $N_\sigma \subseteqq N$ of measure zero such that for every $b \in N - N_\sigma$, the valence

$$N^u_{f,s}(r, r_0, S_b) = \int_{r_0}^r N_f(X(t), S_b, \chi_q)u(t)^{q+1}dt$$

grows at the same rate as the characteristic $T^u_{f,s}(r, r_0, \Omega_s)$ over a subsequence (of σ) $\to \infty$ depending on S_b. (2) If φ is g-semiparabolic and if $s = 1$, then taking σ to be any φ-admissible sequence, the same conclusion holds. (3) Assume there exists a positive form $\xi \in A_0^{k-1,k-1}(N)$ and a positive continuous $\gamma: \boldsymbol{R}[a_0, \infty) \to \boldsymbol{R}$ with $\|\gamma u\|_{a_0}^r \to \infty$ such that for some constants $\alpha > 1$, $B \geqq 0$,

$$\gamma(r)|D^u_{f,s-1}(r, \xi_Y) - B|^\alpha = O(D^u_{f,s}(r, \Omega_s)) \qquad (r \longrightarrow \infty) .$$

Then there is a φ-admissible sequence σ for which the conclusion in (1) holds.

A preliminary report of this paper appeared in [29].

1. Adaptation to admissible families. In the following, all complex spaces are reduced, pure dimensional and have a countable basis. A family $\mathfrak{A} = \{S_b\}_{b \in N}$ is said to be *admissible* in a complex space Y iff:

(A_1) The index set N is a locally irreducible complex space.

(A_2) There exists a complex space M and holomorphic maps $\pi: M \to N$, $h: M \to Y$, such that π is open, surjective, and h is proper, locally trivial at every point of M.

(A_3) For each $b \in N$ the restriction $h: \pi^{-1}(b) \to Y$ is injective and $S_b = h(\pi^{-1}(b))$.

(A_4) No S_b contains a branch of Y.

It follows that each S_b is an analytic set in Y of pure (constant) codimension $s > 0$. If in addition, $h: M \to Y$ is surjective, then $\mathfrak{A}$ is called *strictly admissible* (st. adm.).

To give some examples, take integers p, q, n with $0 \leqq p \leqq q \leqq n - 1$. Let V be a complex vector space of dimension $n + 1$. Let $G_q(V)$ be the Grassmann manifold of projective q-planes in $\boldsymbol{P}(V)$. If $y \in G_q(V)$, the affine $(q + 1)$-plane spanned by y is denoted by $E(y)$. Then the flag manifold

$$F_{p,q} = \{(x, y) \in G_p(V) \times G_q(y) \mid E(x) \subseteq E(y))\}$$

together with the projections $G_p(V) \overset{h}{\leftarrow} F_{p,q} \overset{\pi}{\rightarrow} G_q(V)$ defines a st. adm. family $\mathfrak{A}_{p,q}$ in $G_p(V)$ ([22]).

The family $\mathfrak{A}_{0,q}$ belongs to the class of Schubert varieties. Let $A = (a_0, a_1, \cdots, a_p) \in \boldsymbol{Z}^{p+1}$ with $0 \leqq a_0 \leqq a_1 \leqq \cdots \leqq a_p \leqq n - p$. The flag manifold of A is the set $\boldsymbol{F}(A)$ of all $v = (v_0, \cdots, v_p)$ with $v_j \in G_{a_j+j}(V)$ such that $E(v_0) \subseteq E(v_1) \subseteq \cdots \subseteq E(v_p)$. For each $v \in \boldsymbol{F}(A)$ the Schubert variety

$$S_v(A) = \bigcap_{j=0}^{p} \{x \in G_p(V) \mid \dim E(x) \cap E(v_j) \geqq j + 1\}$$

has dimension $\sum_{j=0}^{p} a_j$. It was shown in Cowen [7] that the Schubert family $\{S_v(A)\}_{v \in \boldsymbol{F}(A)}$ is st. adm. in $G_p(V)$. Here the total space M is given by the irreducible complex space

$$S(A) = \bigcup_{v \in \boldsymbol{F}(A)} S_v(A) \times \{v\} .$$

For $b \in G_{n-p-1}(V)$, define

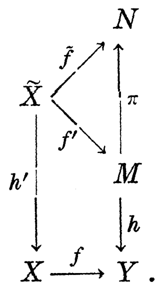
$$\Sigma_b = \{x \in G_p(V) \mid E(x) \cap E(b) \neq \{0\}\}$$

(see Chern [5, p. 79]). The collection $\mathscr{D}_{p,n} = \{\Sigma_b\}_{b \in G_{n-p-1}(V)}$ is a Schubert family (see below); it is in fact also admissible relative to $G_{n-p-1}(V)$:

LEMMA 1.1. *$\mathscr{D}_{p,n}$ is st. adm. in $G_p(V)$ of codimension* 1.

Proof. Let $A = (n - p - 1, n - p, \cdots, n - p) \in \boldsymbol{Z}^{p+1}$. If $v = (v_0, \cdots, v_p) \in \boldsymbol{F}(A)$, the Schubert variety $S_v(A)$ has pure codimension 1 in $G_p(V)$ and $\Sigma_{v_0} = S_v(A)$. The unitary group (of a Hermitian metric on V) acts transitively and biholomorphically on $\boldsymbol{F}(A)$. It follows that the projection $\rho\colon \boldsymbol{F}(A) \rightarrow G_{n-p-1}(V)$ is open. Hence the triplet $G_p(V) \overset{h}{\longleftarrow} S(A) \overset{\rho\circ\pi}{\longrightarrow} G_{n-p-1}(V)$ is admissible and defines $\mathscr{D}_{p,n}$.

Let $\mathfrak{A} = \{S_b\}_{b \in N}$ be admissible in Y, $s = \operatorname{codim} S_b$, and $k = \dim N$. Assume X is a complex space and $f\colon X \rightarrow Y$ is holomorphic. To obtain the equidistribution property of f rel. to $\mathfrak{A}$, it is necessary to impose a general position requirement on the image set of f. Consider the relative fiber product (f', h') of (f, h):



Then f is said to be *almost adapted to* $\mathfrak{A}$ iff $\tilde{f} = \pi \circ f'$ has strict maximal rank, i.e., the restriction of $\tilde{f}$ to every branch of $\tilde{X}$ has rank k. (Observe that if $\dim X < s$, f is not almost adapted to $\mathfrak{A}$.) Take $(a, x) \in N \times X$. The map f is said to be adapted to a at x iff there exist open neighborhoods U, V of a, resp. x, such that the set $f^{-1}(S_b) \cap V$ either is empty for all $b \in U$ or has pure codimension s in V for all $b \in U$; in the latter case, f is called *truly adapted to* a *at* x.

Assume now $f\colon X \rightharpoonup Y$ is a meromorphic map ([1] [18]). Then f may be thought of as a holomorphic correspondence $[f]\colon X \rightharpoonup Y$ (see [20]). Let $'X \subseteqq X \times Y$ be the graph of $[f]$ and $P\colon {'X} \to X$, $F\colon {'X} \to Y$ be the projections. There is a largest open set $X^0 \subseteqq X$ such that $P\colon P^{-1}(X^0) \to X^0$ is biholomorphic. Define $f_0 = F \circ P^{-1}\colon X^0 \to Y$. Then $f_0 = [f] | X^0$ is holomorphic. The indeterminacy $I_f = X - X^0$ is thin analytic in X. The map f is said to be: (1) *nondegenerate* if F has strict maximal rank; (2) *almost adapted* to $\mathfrak{A}$ if so is F.

Define $D = D(\tilde{F}) = \{z \in {'\tilde{X}} \mid \operatorname{rank}_z \tilde{F} < k\}$. Let $G \subseteqq X$. If $[f](G) \cap S_b \neq \varnothing$ for some $b \in N$, then G is called *effective* (for $\mathfrak{A}$). The set of all $b \in N$ to which F is adapted at every point of $'G = P^{-1}(G)$ is denoted by $N_{G,f}$. Let $\tilde{G} = h'^{-1}('G)$. Then $N_{G,f} = N - \tilde{F}(\tilde{G} \cap D)$. Hence if G is compact, $N_{G,f}$ is open. Let $G_f^{(0)}$ be the set of all $x \in G$ such that F is truly adapted to some $b \in N$ at some $w \in P^{-1}(x)$.

LEMMA 1.2. *Assume f is almost adapted to $\mathfrak{A}$. Then* (i) *for all $G \subseteqq X$, $N - N_{G,f}$ has zero (Hausdorff) $2k$-measure*; (ii) *an open set $G \subseteqq X$ is effective iff $G_f^{(0)} \neq \varnothing$*; (iii) *for every branch X_j of X, there exists $x_j \in X_j \cap X^0$ such that f_0 is truly adapted to some point of N at x_j.*

Proof. By [1, 1.24], $\tilde{F}(D)$ is almost thin of dimension $k - 2$. Since $N - N_{G,f} \subseteqq \tilde{F}(D)$, assertion (i) follows. Assume $G \subseteqq X$ is open. If G is effective there exists $w \in {'G}$ such that $F(w) \in h(M)$. Then $G_f^{(0)} \supseteqq P(h'(\tilde{G} - D)) \neq \varnothing$, since D is thin analytic in $'\tilde{X}$ ([1, 1.16]). The converse is trivial. Now (iii) is a consequence of (ii).

LEMMA 1.3. *Assume every effective branch X_j of X contains a point $x_j \in X^0$ such that for every branch B of $W = h^{-1}(f(x_j))$, f_0 is truly adapted to some point of $\pi(B)$ at x_j. Then f is almost adapted to $\mathfrak{A}$.*

Proof. Let (f_0', h_0') be the fiber product of (f_0, h). Take an arbitrary branch A of $\tilde{X}^0$. The map $h_0'|A: A \to X^0$ is proper of pure maximal rank. Hence the image $A' = h_0'(A)$ is a branch of X^0 ([1, 1.27]). Let $\{B_i\}$ be the family of branches of W. Since A' is the intersection of X^0 and an effective branch X_j of X, it contains a point x_j at which f_0 is truly adapted to a point of $\pi(B_i)$ for every B_i. Also, $(h_0')^{-1}(x_j) \cap A \neq \emptyset$ and contains a branch of the form $(f_0')^{-1}(B_i) \cap (h_0')^{-1}(x_j)$. Hence there exists $z \in A$ with $\operatorname{rank}_z \tilde{f}_0|A = k$. It follows that $\tilde{F}$ has strict rank k.

COROLLARY 1.4. *Let $\mathfrak{A}$ be a st. adm. family defined by $Y \overset{h}{\leftarrow} M \overset{\pi}{\rightarrow} N$. Assume Y is nonsingular, connected and simply connected, and M is irreducible. Let $f: X \rightharpoonup Y$ be a meromorphic map such that for every branch X_j of X, there is a point $x_j \in X_j - I_f$ at which f_0 is truly adapted to a point of N. Then f is almost adapted to $\mathfrak{A}$.*

Proof. By [27, 1.3], $h^{-1}(y)$ is irreducible for all $y \in Y$. Apply Lemma 1.3.

2. **The Crofton formula and the F.M.T.** Let X be a complex space of dimension $m > 0$. Let $A_k^p(X)$, resp. $A_k^{q,r}(X)$, denote the set of all differential forms of class C^k and degree p, resp. bidegree (q, r) on X. A form $\zeta \in A_0^{p,p}(X)$ is said to be *nonnegative* ($\geqq 0$) iff for any holomorphic map α of a nonvoid open set $U \subseteq C^p$ into X, $\alpha^*\zeta \geqq 0$ on U; ζ is said to be *strictly nonnegative* ($\gg 0$) iff $\zeta \wedge \eta \geqq 0$ for all nonnegative forms η on X. The form ζ is said to be *positive* at $a \in X$ iff ζ has a positive extension into a local embedding space of X at a. Also, ζ is said to be *simi-positive* on X iff it is positive outside a thin analytic subset of X.

LEMMA 2.1. *If $\zeta \in A_0^{p,p}(X)$ is nonnegative on an open, dense subset of X, then ζ is nonnegative on X.*

Proof. Take arbitrary $\xi_j \in A_0^{1,0}(X)$, $j = 1, \cdots, m - p$. Then the form $\eta = i^{m-p}\zeta \wedge \xi_1 \wedge \bar{\xi}_1 \wedge \cdots \wedge \xi_{m-p} \wedge \bar{\xi}_{m-p}$ is $\geqq 0$ on an open, dense subset of X. By continuity, $\eta \geqq 0$ on X_{reg} (the manifold of regular points of X). It follows that $\eta \geqq 0$ (hence also $\zeta \geqq 0$) on X ([28, §4.2]).

Let M, M' be complex spaces and $h: M \to M'$ a holomorphic map. If h is proper and locally trivial at every point of M, then the fiber integration operator h_* exists, which associates to each $\zeta \in A_l^{p,p}(M)$, $p \geqq r =$ fiber dim. h, a form $h_*\zeta \in A_l^{p-r,p-r}(M')$ ([28, §8.2]). Moreover, if ζ is $\geqq 0$, resp. $\gg 0$, then so is $h_*\zeta$ ([ibid.]). If h has strict rank $n = \dim M'$ and if ζ' is $\gg 0$ on M', then $h^*\zeta' \gg 0$ on M ([ibid.]).

Assume $Y \overset{h}{\leftarrow} M \overset{\pi}{\rightarrow} N$ defines an admissible family $\mathfrak{A}$ in Y. Then the linear map $\Psi = h_*\pi^*: A_l^{p,p}(N) \rightarrow A_l^{p-k+s,p-k+s}(Y)$ exists if $p \geqq k - s$. Assume $f: X \rightharpoonup Y$ is a meromorphic map almost adapted to $\mathfrak{A}$.

LEMMA 2.2. *Let ξ, $\xi' \in A_0^{p,p}(N)$ be real forms where $p=k$ or $k-1$. Let $\xi_Y = \Psi(\xi)$. (1) If $\xi \geqq 0$, then $f^*\xi_Y \gg 0$ on any open subset of X^0. (2) Assume N is compact and $\xi > 0$. Then there is a constant $C > 0$ such that if $\zeta \in A_0^{r,r}(W)$ is $\geqq 0$ on an open subset W of X, $\zeta \wedge f^*\xi'_Y \leqq C\zeta \wedge f^*\xi_Y$ on W.*

Proof. Observe that $\xi \gg 0$ on N. Since $\tilde{F}$ has strict rank k, fiber integration yields $F^*\xi_Y \gg 0$ on $'X$([ibid.]). This implies $f^*\xi_Y \gg 0$ on any open subset of X^0 ([28, 4.2.5.]). To prove (2), let $C > 0$ be a constant such that $\xi' \leqq C\xi$. Then $C\xi - \xi' \gg 0$. Hence $\zeta \wedge f^*\xi'_Y \leqq C\zeta \wedge f^*\xi_Y$ on W.

THEOREM 2.3 (*Crofton Formula*). *Let $\zeta \in A_0^{2(m-s)}(X)$ and $\omega \in A_0^{2k}(N)$. Assume $G \subseteqq X$ is open and $K = \bar{G} \cap \operatorname{supp} \zeta$ is compact. Define $'\zeta = P^*\zeta$, $'G = P^{-1}(G)$, and*

$$N_f(G, S_b, \zeta) = \int_{F^{-1}(S_b) \cap 'G} \nu_F^b{}'\zeta \qquad (b \in N_{K,f}) .$$

Then $N_f(G, S_b, \zeta)$ is measurable on N and

$$\int_N N_f(G, S_b, \zeta)\omega = \int_G \zeta \wedge f^*\omega_Y .$$

REMARKS 1. The intersection multiplicity ν_F^b of F with S_b is included because of its appearance in the F. M. T. For the definition and properties of the multiplicity, see [28].

2. It can be shown that for almost all $S_b \in \mathfrak{A}$,

$$N_f(G, S_b, \zeta) = \int_{f_0^{-1}(S_b) \cap G} \zeta .$$

Proof. Since

$$\int_{G \cap X^0} \zeta \wedge f_0^*\omega_Y = \int_{'G} {}'\zeta \wedge F^*\omega_Y$$

exists, the measurable form $\zeta \wedge f^*\omega_Y$ is integrable over G. Moreover, it suffices to consider the case where f is holomorphic.

Let $\tilde{K} = h'^{-1}(K)$, $\tilde{\zeta} = h'^{*}\zeta$, $D = D(\tilde{f})$ and $N_0 = \{b \in N \mid \omega(b) \neq 0\}$. Then

$$\begin{aligned}\int_G \zeta \wedge f^*\omega_Y &= \int_K \zeta \wedge h'_*(f'^*\pi^*\omega) \qquad ([28,\ \S 8.2])\\ &= \int_{\tilde{K}-D} \tilde{\zeta} \wedge \tilde{f}^*\omega\\ &= \int_{N_0}\Big(\int_{\tilde{f}^{-1}(b)\cap(\tilde{K}-D)} \nu_{\tilde{f}}\tilde{\zeta}\Big)\omega \qquad ([28,\ 5.2.1])\\ &= \int_{N_{K,f}}\Big(\int_{\tilde{f}^{-1}(b)\cap\tilde{K}} \nu_{\tilde{f}}\tilde{\zeta}\Big)\omega\\ &= \int_N N_f(G, S_b, \zeta)\omega\ .\end{aligned}$$

LEMMA 2.4. *Let $\omega \in A_0^{2k}(N)$ be semi-positive. Then* (1) $\omega_Y \not\equiv 0$ *on* Y. (2) *Let f be as above and $\zeta \in A_0^{m-s,m-s}(X)$. If ζ is semi-positive on an effective open set $G \subseteqq X$, then $\zeta \wedge f^*\omega_Y \not\equiv 0$ on $G^0 = G \cap X^0$.*

Proof. Observe that the identity map of Y is almost adapted to $\mathfrak{A}$ and positive form $\zeta' \in A_0^{n-s,n-s}(Y)$ ($n = \dim Y$) exists. Hence (2) implies (1).

To prove (2), let R be a thin analytic subset of G such that $\zeta > 0$ on $G - R$. The open set $('G)_f^{(0)}$ contains a point $x \in P^{-1}(G^0)$. Therefore it may be assumed w.l.o.g. that f is holomorphic. Let $w \in \tilde{X}$ such that f is adapted to $a = \tilde{f}(w)$ at $x = h'(w) \in G$. Let $G_1 \Subset G$, $Q \subseteqq N$ be neighborhoods of x, resp. a, such that $f^{-1}(S_b) \cap G_1$ has pure codimension s for all $b \in Q$. Then $\tilde{f}(\tilde{G}_1)$ constains a nonvoid open subset Q_1 of Q. Define $V_b = \tilde{f}^{-1}(b) \cap \tilde{G}_1$ for $b \in Q$, and $\tilde{R} = h'^{-1}(R)$. According to [1, 1.26], the set $T = \{b \in Q \mid V_b \cap \tilde{R}$ is not thin in $V_b\}$ is almost thin in Q. Since h' maps $\tilde{f}^{-1}(b)$ homeomorphically onto $f^{-1}(S_b)$, there is an open set $H_b \subseteqq G_1$ such that $H_b \cap f^{-1}(S_b) = h'(V_b)$ (for each $b \in Q$). Let $Z_b = (H_b - R) \cap f^{-1}(S_b)$. Then

$$\begin{aligned}\int_{G_1} \zeta \wedge f^*\omega_Y &= \int_N N_f(G_1, S_b, \zeta)\omega\\ &\geqq \int_{Q-T}\Big(\int_{V_b-\tilde{R}} \nu_{\tilde{f}}\tilde{\zeta}\Big)\omega\\ &\geqq \int_{Q_1-T}\Big(\int_{Z_b} \nu_f^b\zeta\Big)\omega > 0\ .\end{aligned}$$

Let $f\colon X \rightharpoonup Y$ be a meromorphic map. Relative to a family of subvarieties in Y, say $\mathfrak{A} = \{S_b\}_{b\in N}$, the so called First Main Theorem for f measures the difference between the valence of S_b and its mean value on N. The theorem requires the existence of certain differential forms $\{\Lambda_b\}_{b\in N}$ (where Λ_b is singular on S_b) with special properties.

For instance, if $\mathfrak{A} = \mathfrak{A}_{0q}$ (the family of projective q-planes in $\boldsymbol{P}(V)$), the explicitly constructed Chern-Levine forms ([4] [22]) suffice for this purpose. In general, if $\mathfrak{A}$ satisfies (A_1)-(A_4), the forms Λ_b may be obtained by fiber integration from a singular potential λ on N. The latter means that λ is a set of forms $\{\lambda_b\}_{b\in N}$ depending infinitely smoothly on b such that (i) λ_b is $\geqq 0, C^\infty$ on $N - \{b\}$ of bidegree $(k-1, k-1)$; (ii) λ_b is singular at b as described in [23, p. 55, (2)-(3)] ([28, 7.2.1] if N is singular); (iii) $dd^c\lambda_b$ extends to a nonnegative, C^∞ form on N independent of b. For a compact Kähler manifold the existence of such forms was proved by Wu [31, I, II], Hirschfolder [14] (see also [13]), and Stoll [23]. By the method of elliptic operators, Dektyarev [8] [9] constructed similar forms without requiring the Kähler condition. Actually, combining [8] and [23], the following can be proved:

THEOREM 2.5. *Assume N is a connected, compact complex manifold of dimension $k > 0$. Let ω be a volume form on N normalized so that $\int_N \omega = 1$. Then there exists a singular potential $\lambda = \{\lambda_b\}_{b\in N}$ such that $dd^c\lambda_b = \omega$ on $N - \{b\}$.*

Proof. By [23, 5.3], there exist differential forms $\{\lambda_b^*\}_{b\in N}$ depending infinitely smoothly on b such that λ_b^* satisfies the above conditions (i)-(ii), and for some C^∞ $g: N \times N \to \boldsymbol{R}$,

$$dd^c\lambda_b^* = g_b\omega \quad \text{on} \quad N - \{b\} .$$

(Here $g_b(x) = g(x, b)$ if $x \in N$.) Take a positive form $\xi \in A_\infty^{k-1,k-1}(N)$. Consider the linear operator E on the space $\mathscr{C}^\infty(N)$ of real-valued C^∞-functions such that for $u \in \mathscr{C}^\infty(N)$,

$$(Eu)\omega = dd^c(u\xi) .$$

The adjoint operator E^* is given by

$$(E^*u)\omega = dd^cu \wedge \xi .$$

Then E and E^* are elliptic. By the maximum principle, the kernel of E^* consists of constant functions. The residue theorem ([23, 6.4]) applied to the identity map of N gives $\int_N g_b\omega = 1$ for $b \in N$. Hence the equation

$$Eu = 1 - g_b$$

has a solution $u_b \in \mathscr{C}^\infty(N)$ depending infinitely smoothly on b (see, for example, [15, 10.5.3] [19]). According to [8, p. 961], the form

ξ can be chosen so that $dd^c\xi = 0$. Hence for some positive constant C, $\lambda_b = (C + u_b)\xi + \lambda_b^*$ defines a set of forms with all required properties.

If G is an open subset of the complex space X, there exists a maximal open subset dG of $\partial G \cap X_{\text{reg}}$ such that dG is a smooth, (oriented) C^∞-boundary manifold of G in X_{reg}. A relatively compact open set $G \subseteqq X$ is called a Stokes domain iff ∂G has locally finite (Hausdorff) $(2m-1)$-measure and $\partial G - dG$ has zero $(2m-1)$-measure. A bump (g, G, ψ) in X is given by Stokes domains G and g in X with $\emptyset \subseteqq g \Subset G$ and a continuous function $\psi: X \to \boldsymbol{R}[0, \infty)$ such that (i) supp $\psi \subseteqq \bar{G}$; (ii) $\psi | \bar{G} - g$ is of class C^2; (iii) $\psi | \bar{g} = \operatorname{Max} \psi | \bar{G} > 0$ (if $g \neq \emptyset$).

THEOREM 2.6 (F.M.T.). *Let $f: X \rightharpoonup Y$ be a meromorphic map of a complex space X of dimension $m > 0$ into a complex space Y. Assume $\mathfrak{A} = \{S_b\}_{b \in N}$ is admissible in Y with $q = m - \text{codim } \mathfrak{A} \geqq 0$. Assume $\omega \in A_\infty^{2k}(N)$ is nonnegative and $\lambda = \{\lambda_b\}_{b \in N}$ is a singular potential with $dd^c\lambda_b = \omega$. Define $\Lambda_b = \Psi(\lambda_b)$ on $Y - S_b$ and $\Omega = \Psi(\omega)$ on Y. Assume $\chi \in A_1^{q,q}(X)$ is closed, strictly nonnegative. Let (g, G, ψ) be a bump in X, and $K = \bar{G} \cap \text{supp}\, \chi$. Then for every $b \in N_{K,f}$,*

$$T_f(G) - N_f(G, b) = m_f(\Gamma, b) - m_f(\gamma, b) - D_f(G, b)\ .$$

Here

$$N_f(G, b) = \int_{F^{-1}(S_b) \cap 'G} \nu_F^b{}' \psi' \chi \qquad (\textit{valence})$$

$$m_f(\Gamma, b) = \int_\Gamma f^*\Lambda_b \wedge d^\perp \psi \wedge \psi \geqq 0 \quad (\textit{exterior proximity})$$

$$m_f(\gamma, b) = \int_\gamma f^*\Lambda_b \wedge d^\perp \psi \wedge \chi \geqq 0 \quad (\textit{interior proximity})$$

$$T_f(G) = \int_G \psi f^*\Omega \wedge \chi \qquad (\textit{characteristic})$$

$$D_f(G, b) = \int_{G - \bar{g}} f^*\Lambda_b \wedge dd^\perp \psi \wedge \chi \quad (\textit{deficit})$$

are continuous functions of b on $N_{K,f}$; $'G$, $'\psi$, resp. $'\chi$, denotes its lifting to $'X$; $\Gamma = \partial G$, $\gamma = \partial g$, and $d^\perp = i(\bar{\partial} - \partial) = -d^c$.

REMARKS 1. If, in the F.M.T., f is almost adapted to $\mathfrak{A}$, the hypothesis "$\chi \gg 0$" can be weakened to "$\chi \geqq 0$ in a neighborhood of $\bar{G} - g$". 2. The theorem was proved in [28] for a holomorphic map; the case of a meromorphic map is an easy consequence.

3. Integral averages. Some general assumptions shall be stated here for later reference.

(I) X is a complex space of dimension $m > 0$ with at least one noncompact branch.

(II) $\mathfrak{A} = \{S_b\}_{b \in N}$ is admissible in a complex space Y, where N is compact, connected and nonsingular. Let $k = \dim N$, $s = \operatorname{codim} S_b$, and $q = m - s$.

(III) $f: X \rightharpoonup Y$ is a meromorphic map almost adapted to $\mathfrak{A}$.

(IV-a) $\omega \in A_\infty^{2k}(N)$ is semi-positive, normalized so that $\int_N \omega = 1$; $\{\lambda_b\}_{b \in N}$ is a singular potential with $dd^c \lambda_b = \omega$. Define $\Lambda_b = \Psi(\lambda_b)$, $\Omega = \omega_Y$.

(IV-b) $\omega_{N,1}$ is the fundamental form on N of a Hermitian metric normalized so that $\int_N \omega_{N,1}^k = 1$. Define $\Omega_l = (\omega_{N,1}^{k-s+l})_Y$, $0 \leqq l \leqq s$.

(V) If $q = 0$, let $\chi = 1$, if $q > 0$, assume $\chi \in A_2^{p,q}(X)$ is closed, nonnegative.

(VI) Either χ is semi-positive on an effective open subset of X or $\chi \wedge f^*\Omega \not\equiv 0$ on X^0.

Assume (I)-(IV-a). For a measurable function u on N, define $I(u) = \int_N u\omega$ (if the integral exists). By [23, 6.3], the integral average $\Lambda(y) = \int_N \omega(b) \otimes \lambda_b(y)$, $y \in N$, defines a nonnegative form $\Lambda \in A_0^{k-1,k-1}(N)$.

LEMMA 3.1. *Let $G \subseteqq X$ be a relatively compact open set. Assume $\zeta \in A_0^{q+1,q+1}(X)$. Then the integral*

$$D_f(G; \zeta) = \int_G f^*\Lambda_Y \wedge \zeta$$

exists and

$$I\left(\int_G f^*\Lambda_b \wedge \zeta\right) = D_f(G; \zeta) . \tag{3.1}$$

Proof. The existence of $D_f(G; \zeta)$ follows from the continuity of $F^*\Lambda_Y \wedge {}'\zeta$ on ${}'G$. By [28, 7.2.2 and §8.2], the integral $\int_G f^*\Lambda_b \wedge \zeta$ is a continuous function of b on $N_{\bar{G},f}$. Also by [28, §9.1], $F^*\Lambda_b \gg 0$ on ${}'X - F^{-1}(S_b)$. Let $\tilde{\zeta} = h'^*({}'\zeta)$. If $\zeta \geqq 0$ on G, then

$$\begin{aligned} I\left(\int_G f^*\Lambda_b \wedge \zeta\right) &= \int_N \left(\int_{\tilde{G}} \tilde{F}^*\lambda_b \wedge \tilde{\zeta}\right)\omega \\ &= \int_{'G} F^*\Lambda_Y \wedge {}'\zeta . \end{aligned}$$

If ζ is real, there exists a positive form $\eta \in A_0^{q+1,q+1}(X)$ such that $\eta + \zeta > 0$ on $\bar{G}$. Then (3.1) holds for η and $\eta + \zeta$, hence also for ζ.

If ζ is complex-valued, a splitting into real and imaginary parts yields the result.

LEMMA 3.2. *Let $\chi \in A_2^{q,q}(X)$ and $G \subseteqq X$ be a Stokes domain. Assume $\psi: X \to \boldsymbol{R}$ is of class C^2 with $\psi|\partial G = 0$, and, for some neighborhood W of ∂G, either $\psi|W \cap G \geqq 0$ or $\psi|W - G \geqq 0$. Then*

$$I\Big(\int_{\partial G} f^*\Lambda_b \wedge d^c\psi \wedge \chi\Big) = \int_{\partial G} f^*\Lambda_Y \wedge d^c\psi \wedge \chi$$
$$= D_f(G; dd^c(\psi\chi)) .$$

Proof. For each $b \in N_{\bar{G},f}$, it was proved in [30] that the following residue formula holds:

$$\int_G \psi f^*\Omega \wedge \chi - \int_G f^*\Lambda_b \wedge dd^c(\psi\chi) = N_f(G, S_b, \psi\chi) - \int_{\partial G} f^*\Lambda_b \wedge d^c\psi \wedge \chi .$$

Therefore (3.1) and the Crofton Formula yield

$$I\Big(\int_{\partial G} f^*\Lambda_b \wedge d^c\psi \wedge \chi\Big) = D_f(G; dd^c(\psi\chi)) .$$

Now assume $\chi \geqq 0$ in a neighborhood of $\bar{G}$. Let $\Gamma = \partial G$, $'\Gamma = P^{-1}(\Gamma)$, $\tilde{\Gamma} = h'^{-1}('\Gamma)$, etc. By [23, 3.2 and AII, 4.11, 4.6],

$$I\Big(\int_{\partial G} f^*\Lambda_b \wedge d^c\psi \wedge \chi\Big) = \int_N\Big(\int_{'\Gamma} F^*\Lambda_b \wedge d^{c'}\psi \wedge {}'\chi\Big)\omega$$
$$= \int_N\Big(\int_{\tilde{\Gamma}} \tilde{F}^*\lambda_b \wedge d^c\tilde{\psi} \wedge \tilde{\chi}\Big)\omega$$
$$= \int_{'\Gamma} F^*\Lambda_Y \wedge d^{c'}\psi \wedge {}'\chi$$
$$= \int_{\Gamma} f^*\Lambda_Y \wedge d^c\psi \wedge \chi .$$

The general case follows the same way as in Lemma 3.1.

Assume (I)-(IV-a) and (V). Assume (g, G, ψ) is a bump in X. Let T_f, N_f, m_f, and D_f denote the associated value distribution functions. Let $\hat{\psi}$ be a C^2-extension of ψ on X. Then Lemma 3.2 applied to $\hat{\psi}$, resp. $(\hat{\psi}|g) - \hat{\psi}$, shows that the mean proximities $m_f(\Gamma) = I(m_f(\Gamma, b))$ and $m_f(\gamma) = I(m_f(\gamma, b))$ exist. Moreover, with $G_\mu = G$ or g,

$$m_f(\partial G_\mu) = \int_{\partial G_\mu} f\Lambda_Y \wedge d^\perp\psi \wedge \chi = D_f(G_\mu; d^c d\hat{\psi} \wedge \chi) . \tag{3.2}$$

Hence

$$m_f(\Gamma) - m_f(\gamma) = I(D_f(G, b)) . \tag{3.3}$$

4. Equidistribution theorems. Let X be a complex space of dimension $m > 0$. A *semi-exhaustion function* of X is an upper semi-continuous map $\varphi: X \to \boldsymbol{R}_{-\infty} = \boldsymbol{R} \cup \{-\infty\}$ such that the half spaces $X[r] = \{x \in X \mid \varphi(x) \leqq r\}$ are compact for all $r \geqq 0$. An *exhaustion function of* X is a semi-exhaustion $\varphi: X \to \boldsymbol{R}_{-\infty}$ which is C^∞ outside a compact set.

Let φ be a semi-exhaustion function of X. If $r > r' \geqq -\infty$, define $X(r) = \{x \in X \mid \varphi(x) < r\}$, $X[r', r) = X(r) - X(r')$, $X(r', r) = X(r) - X[r']$, etc. Assume $U: \boldsymbol{R}[r_0, r) \to \boldsymbol{R}$ is absolutely continuous and let $u = U'$. For $r > r' \geqq r_0$, define

$$\varphi_{rr'u}(x) = \begin{cases} 0 & , \text{ if } x \in X[r, \infty) \\ U(r) - U(\varphi_{[r']}(x)) , & \text{ if } x \in X(r) . \end{cases}$$

Here $\varphi_{[r']}(x) = \operatorname{Max}(r', \varphi(x))$. Define $U_\varphi = U \circ \varphi$.

LEMMA 4.1. (Cf. [23, 8.3] [25, 10.6].) *Assume ζ is a locally integrable $2m$-form on X. Take $a_0 \in \boldsymbol{R}[-\infty, r)$. Define*

$$v(t) = \int_{X[a_0, t)} \zeta , \quad v[t] = \int_{X[a_0, t]} \zeta \quad (t > a_0) .$$

Then if $r > r' \geqq \operatorname{Max}(a_0, r_0)$,

$$\begin{aligned} \int_{X[a_0,\infty)} \varphi_{rr'u}\zeta &= \int_{r'}^{r} v(t)u(t)dt \\ &= U(r)v(r) - U(r')v[r'] - \int_{X(r', r)} U_\varphi \zeta . \end{aligned} \tag{4.1}$$

Proof. Observe that $\varphi_{rr'u}$ is bounded, measurable on X. W.l.o.g. assume ζ is real. There exist nonnegative, integrable forms $\zeta_j (j = 1, 2)$ on $X_{\text{reg}}[r]$ such that $\zeta_1 - \zeta_2 = \zeta$. Hence it may be assumed that $\zeta \geqq 0$ on $X_{\text{reg}}[r]$. Likewise assume $u \geqq 0$. Let $C(x, t) = 1$ if $\varphi(x) < t$, and $C(x, t) = 0$ if $\varphi(x) \geqq t$. Then

$$\begin{aligned} \int_{X[a_0,\infty)} \varphi_{rr'u}\zeta &= \int_{X[a_0, r)} \left(\int_{\varphi_{[r']}(x)}^{r} u(t)dt\right)\zeta(x) \\ &= \int_{r'}^{r} \left(\int_{X[a_0, r)} C(x, t)\zeta(x)\right)u(t)dt \\ &= \int_{r'}^{r} v(t)u(t)dt . \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{X[a_0,\infty)} \varphi_{rr'u}\zeta &= \int_{X[a_0, r)} [U(r) - U(\varphi_{[r']}(x))]\zeta(x) \\ &= U(r)v(r) - U(r')v[r'] - \int_{X(r', r)} U_\varphi \zeta . \end{aligned}$$

Assume (I) and let $\varphi: X \to \boldsymbol{R}_{-\infty}$ be an exhaustion function. Call $r > -\infty$ φ-admissible if $X(r)$ is a Stokes domain. A strictly increasing sequence $\{r_j\}_{j=1}^{\infty}$ is said to be φ-admissible if each r_j is φ-admissible and $r_j \to \infty$. The set of non-φ-admissible values $>r'$ has measure zero in $\boldsymbol{R}[r', \infty)$ for large r' (cf. [28, 7.1.6]). If φ is C^∞ on X, this is true for arbitrary r'.

LEMMA 4.2. *Let $L(\varphi) = dd^c\varphi$ be the Levi form of φ on $X - X[r_0]$ (where φ is C^∞). Assume $U: \boldsymbol{R}[r_0, \infty) \to \boldsymbol{R}$ is of class C^2 and $u = U'$. Let $\zeta \in A_2^{m-1,m-1}(X)$. (1) If r, r' are φ-admissible with $r > r' > r_0$, then*

$$
\begin{aligned}
&u(r)\int_{dX(r)} d^c\varphi \wedge \zeta - u(r')\int_{dX(r')} d^c\varphi \wedge \zeta \\
&\quad = \int_{X(r',r)} L(U_\varphi) \wedge \zeta + \int_{r'}^{r}\left(\int_{X(t)} dd^c\zeta\right)u(t)dt\,.
\end{aligned} \tag{4.2}
$$

(2) Assume $\varphi: X \to \boldsymbol{R}$ is C^∞ and ζ is d^cd-closed. Then for all $r > r' > r_0$,

$$
u(r)\int_{X(r)} L(\varphi) \wedge \zeta - u(r')\int_{X(r')} L(\varphi) \wedge \zeta = \int_{X[r',r)} L(U_\varphi) \wedge \zeta\,. \tag{4.3}
$$

Proof. By Lemma 4.1, if r, r' are φ-admissible with $r > r' > r_0$, then

$$
\begin{aligned}
\int_{r'}^{r}\left(\int_{X(t)} dd^c\zeta\right)u(t)dt &= U(r)\int_{X(r)} dd^c\zeta - U(r')\int_{X[r']} dd^c\zeta - \int_{X(r',r)} U_\varphi dd^c\zeta \\
&= \int_{X(r',r)} d(U_\varphi) \wedge d^c\zeta \\
&= \int_{X(r',r)} d\zeta \wedge d^c(U_\varphi) \\
&= u(r)\int_{dX(r)} d^c\varphi \wedge \zeta - u(r')\int_{dX(r')} d^c\varphi \wedge \zeta \\
&\quad - \int_{X(r',r)} \zeta \wedge L(U_\varphi)\,.
\end{aligned}
$$

Now assertion (2) follows from (1) by repeated application of the Stokes theorem ([28, §7.1]) and the left-continuity of $v(t) = \int_{X(t)} \chi$ (where $\chi \in A_{\mathrm{loc}}^{2m}(X)$).

Let $\varphi: X \to \boldsymbol{R}_{-\infty}$ be an exhaustion function. If there exists an increasing $g: \boldsymbol{R}(c_0, \infty) \to \boldsymbol{R}$ ($c_0 \geqq 0$) of class C^1 such that

(i) $L(\varphi) \geqq g'_\varphi d\varphi \wedge d^c\varphi$ on $X_{\mathrm{reg}} - X[c_0]$,

(ii) $\|e^{-g}\|_{c_0}^{r} \to \infty$ as $r \to \infty$,

then φ is called *g-convex*. If instead of (i),

(i)′ $L(\varphi) \leqq g'_\varphi d\varphi \wedge d^c\varphi$ on $X_{\mathrm{reg}} - X[c_0]$, and (ii) hold, then φ is called *g-concave.*

A g-convex exhaustion function may not be logarithmic pseudoconvex. As an example, consider the variety $A = \{z \in \boldsymbol{C}^n \mid z_1 + \cdots + z_{n-1} = 0\}$, $n > 1$. Let $\|z\|^2 = \Sigma z_j \bar{z}_j$, $z \in \boldsymbol{C}^n$. Then $\varphi = \log \|z\|^2 \colon A \to \boldsymbol{R}_{-\infty}$ is pseudoconvex (i.e., g-convex with $g =$ constant). Let $Z = \{z \in \boldsymbol{C}^n \mid z_1 = \cdots = z_{n-1} = 0\}$ and $\iota\colon Z \to A$ be the inclusion. Then

$$\iota^*\Big(L(\varphi) - \frac{d\varphi \wedge d^c\varphi}{\varphi}\Big) = \frac{-2dx_n \wedge dy_n}{|z_n|^2} < 0$$

on $Z - A[0]$ (where $z_n = x_n + iy_n$). Hence φ is not logarithmic psc. ($=\log -$ convex). Similarly, one can construct a g-concave exhaustion function which is not pseudoconcave.

Assume $g\colon \boldsymbol{R}(c_0, \infty) \to \boldsymbol{R}$ $(c_0 \geqq 0)$ is of class C^1 and $u = e^{-g}$. Let U_p be a primitive of u^p on $\boldsymbol{R}(c_0, \infty)$ for $p \in \boldsymbol{Z}[1, m]$. Define

$$\omega_u = u_\varphi[L(\varphi) - g'_\varphi d\varphi \wedge d^c\varphi] \tag{4.4}$$

off a compact set, say $X[c]$, $c \geqq c_0$. Setting $\chi_l = L(\varphi)^l$, (4.4) yields

$$(\omega_u)^p = dd^c(U_p \circ \varphi) \wedge \chi_{p-1} \qquad \text{on} \quad X - X[c]\,. \tag{4.5}$$

Now assume (I) and the exhaustion $\varphi\colon X \to \boldsymbol{R}$ is C^∞. Let $\zeta \in A_2^{p,p}(X)$ be dd^c-closed; if $p = 0$, set $\zeta = 1$. Define

$$A_p^u(r) = u(r)^{m-p}\int_{X(r)} \zeta \wedge \chi_{m-p} \qquad (r > c_0)\,.$$

For $r > r' > c$, (4.3) yields

$$A_p^u(r) - A_p^u(r') = \int_{X[r',r)} \zeta \wedge (\omega_u)^{m-p}\,.$$

Hence if φ is $c.g$-convex $(c_0 = 0)$ and if $\zeta \geqq 0$, $A_p^u(0) = \lim_{r\to 0+} A_p^u(r)$ exists and

$$A_p^u(r) = A_p^u(0) + \int_{X(r)} \zeta \wedge (\omega_u)^{m-p} \qquad (r > 0)\,. \tag{4.6}$$

A $c.g$-convex exhaustion φ of a complex space X is called *g-quasiparabolic* if

$$\int_{X(r)} (\omega_u)^m = o(\|u\|_1^r) \qquad (r \longrightarrow \infty)\,.$$

Consider the following example. Let $M \subseteqq \boldsymbol{P}_n$ be a projective variety of dimension $m - 1 > 0$. Let U be the restriction of the universal line bundle (over $\boldsymbol{P}_n$) to M. There is a proper, holomorphic map σ of U onto an algebraic set A in $\boldsymbol{C}^{n+1}$ such that $\sigma\colon U - \sigma^{-1}(0) \to A - \{0\}$ is biholomorphic. Let $\psi = 1 + \|z\|^2\colon A \to \boldsymbol{R}$ and $\varphi = \sigma^*\psi\colon U \to \boldsymbol{R}$.

With $g = \log r$ in (4.4), the (1, 1)-form ω_u is $\geqq 0$ on U and >0 on $U - \sigma^{-1}(0)$. Moreover,

$$\int_{U(r)} (\omega_u)^m = \int_{A_\psi(r)} (L(\log \psi))^m = O(1) \qquad (r \longrightarrow \infty) .$$

Thus the exhaustion φ of U is log-quasiparabolic but not parabolic in the sense of [25].

LEMMA 4.3. *Assume u, A: $\boldsymbol{R}[r_0, \infty) \to \boldsymbol{R}$ where u is positive, continuous, and $A \geqq 0$. Assume*

(4.7) *There exists a continuous α: $\boldsymbol{R}[r_0, \infty) \to \boldsymbol{R}(0, \infty)$ such that $\|u\alpha\|_{r_0}^r \to \infty$ and A/α is increasing in r.*

Then

$$\frac{\|uA\|_{r_0}^r}{\|u\alpha\|_{r_0}^r} \longrightarrow \lim_{r\to\infty} \frac{A(r)}{\alpha(r)} .$$

Proof. Let $G(r) = \|u\alpha\|_{r_0}^r$. For all $r > r' > r_0$,

$$\frac{A(r')}{\alpha(r')}\left[1 - \frac{G(r')}{G(r)}\right] \leqq \frac{\|uA\|_{r_0}^r}{G(r)} \leqq \frac{A(r)}{\alpha(r)} .$$

From this the lemma follows.

THEOREM 4.4. *Assume* (I)–(IV-a), (V)–(VI). *Let φ: $X \to \boldsymbol{R}_{-\infty}$ be an exhaustion function. Let U: $\boldsymbol{R}[r_0, \infty) \to \boldsymbol{R}$ be of class C^2 with $U' = u > 0$. For $r > r' \geqq r_0$, define*

$$N_f^u(r, r', S_b) = \int_{r'}^r N_f(X(t), S_b, \chi)u(t)dt , \quad (b \in N_{X[r],f})$$

$$T_f^u(r, r', \Omega) = \int_X \varphi_{rr'u} f^*\Omega \wedge \chi ,$$

and for φ-admissible $r > r_0$,

$$m_f^u(r) = u(r) \int_{dX(r)} f^*\Lambda_\Upsilon \wedge d^c\varphi \wedge \chi .$$

Assume one of the following holds:

(1) *$L(U_\varphi) \wedge \chi \geqq 0$ off a compact set, and*

(4.8) $$m_f^u(a) = o'(T_f^u(a, r_0, \Omega))$$

over some φ-admissible sequence $\sigma = \{a_j\}_{j=1}^\infty$ (this fact is denoted by "o'").

(2) *$L(U_\varphi) \wedge \chi \leqq 0$ off a compact set and $T_f^u(r, r_0, \Omega) \to \infty$. Let*

$\sigma = \{a_j\}$ *be any* φ*-admissible sequence.*

Then there exists a set $N_\sigma \subseteqq N$ *of measure zero such that for every* $b \in N - N_\sigma$, *there is a subsequence* $\{r_\mu\} \subseteqq \sigma$, $r_\mu \to \infty$, *for which*

$$\lim_{\mu\to\infty} \frac{N_f^u(r_\mu, r_1, S_b)}{T_f^u(r_\mu, r_1, \Omega)} = 1 .$$

Proof. There exists $a_{l-1} \geqq r_0$ such that that on $X - X[a_{l-1}]$ φ is C^∞ and $L(U_\varphi) \wedge \chi \geqq 0$(or $\leqq 0$). Define $N^{[\sigma]} = \bigcap_{j=l}^\infty N_{X[a_j],f}$. Let $G_j = X(a_j)$, $\psi_j = \varphi_{a_j a_l u}$; associated to the bump (G_l, G_j, ψ_j), $j > l$, there are the deficit $D_f(a_j, b) = D_f(G_j, b)$, proximity $m_f(a_j, b) = m_f(\partial G_j, b)$, etc., for all $b \in N^{[\sigma]}$. Observe that

$$m_f(a_j, b) = u(a_j) \int_{\partial X(a_j)} f^* \Lambda_b \wedge d^c\varphi \wedge \chi \qquad (j \geqq l) .$$

For $b \in N^{[\sigma]}$, define

$$\Delta_f(a_j, b) = |D_f(a_j, b)| + m_f(a_j, b) + m_f(a_l, b) \qquad (j > l) .$$

Then it follows from (3.3) and (4.8) that

$$I(\Delta_f(a_j, b)) = o(T_f(a_j)) \qquad (j \longrightarrow \infty) .$$

Define

$$N_{j\mu} = \{b \in N^{[\sigma]} \mid \Delta_f(a_j, b) \geqq 2^{-\mu} T_f(a_j)\} \qquad (j \geqq \mu > l) ,$$
$$N^{(\mu)} = \bigcap_{j \geqq \mu} N_{j\mu} .$$

Then $N^{(\mu)}$ is measurable, and since

$$I(\Delta_f(a_j, b)) \geqq 2^{-\mu} T_f(a_j) \int_{N^{(\mu)}} \omega ,$$

each $N^{(\mu)}$ has measure zero. Define

$$N_\sigma = (N - N^{[\sigma]}) \cup \bigcup_{\mu=l+1}^\infty N^{(\mu)} .$$

Then N_σ has measure zero. For each $b \in N - N_\sigma$, there exists a subsequence $\{r_\mu\}_{\mu=1}^\infty \subseteqq \sigma$ with $a_l \leqq r_1 < r_u \to \infty$ such that

$$\Delta_f(r_\mu, b) = o(T_f(r_\mu)) \qquad (\mu \longrightarrow \infty) .$$

Hence from this and the F. M. T. the theorem follows.

THEOREM 4.5. *Assume* (I)-(III), (IV-b), (V). *Let* φ, U, u *be the same as in Theorem* 4.4. *Assume* $A(r) = \int_{X(r)} f^*\Omega_s \wedge \chi \not\equiv 0$ *and* (4.7) *holds for* (u, A). *Assume* $\zeta = L(U_\varphi) \wedge \chi \geqq 0$ *off a compact set.* (1)

If

$$\int_{X(r_0,r)} f^*\Lambda_Y \wedge \zeta = o'(T_f^u(r, r_0, \Omega_s)) \tag{4.9}$$

(*over some* φ*-admissible sequence* $\sigma = \{r_j\}$), *then* Im $[f]$ *intersects almost every* $S_b \in \mathfrak{A}$ *with*

$$\lim_{r\to\infty}{}' \frac{N_f^u(r, r_0, S_b)}{T_f^u(r, r_0, \Omega_s)} = 1 . \tag{4.10}$$

Here r *runs over a subsequence* (*of* σ) $\to \infty$ *depending on* S_b. (2) (*Cf. Griffiths-King* [12, 5.3].) *If* $\omega_{N,1}$, $\omega'_{N,1}$, *are cohomologous Kähler forms and if there exists a positive form* $\xi \in A_0^{k-1,k-1}(N)$ *such that*

$$\int_{X(r_0,r)} f^*\xi_Y \wedge \zeta = o'(T_f^u(r, r_0, \Omega_s)) \tag{4.11}$$

(*over* σ *as above*), *then*

$$\lim_{j\to\infty} \frac{T_f^u(r_j, r_0, \Omega'_s)}{T_f^u(r_j, r_0, \Omega_s)} = 1 . \tag{4.12}$$

Proof. By (4.1) and Lemma 4.3, $T_f^u(r, r_0, \Omega_s) \to \infty$. (1) Let r, r' be φ-admissible with $r > r' > r_0$. By (3.1) and (3.3),

$$\begin{aligned}\int_{X(r_0,r)} f^*\Lambda_Y \wedge \zeta &= O(1) + \int_{X(r',r)} f^*\Lambda_Y \wedge d^c d\varphi_{rr'u} \wedge \chi \\ &= O(1) + m_f^u(r) - m_f^u(r') .\end{aligned}$$

Thus (4.8) follows from (4.9) and therefore (4.10) holds. (2) Lemma 2.2 and (4.11) yield

$$\int_{X(r_0,r)} f^*\Omega_{s-1} \wedge \zeta = o'(T_f^u(r, r_0, \Omega_s)) . \tag{4.13}$$

There exists $\eta \in A_\infty^{k-1,k-1}(N)$ such that $dd^c\eta = \omega'^k_{N,1} - \omega^k_{N,1}$. Assume r, r' are φ-admissible, $r > r' > r_0$, and $\zeta \geqq 0$ on $X - X[r']$. By (4.2),

$$u(r)\int_{dX(r)} d^c\varphi \wedge f^*\Omega_{s-1} \wedge \chi = O(1) + \int_{X(r_0,r)} L(U_\varphi) \wedge f^*\Omega_{s-1} \wedge \chi .$$

Observe that on $dX^0(r)$,

$$d^c\varphi \wedge f^*\eta_Y \wedge \chi \leqq \text{const.}\ d^c\varphi \wedge f^*\Omega_{s-1} \wedge \chi$$

([23, 3.2]). Therefore (4.2) yields

$$\begin{aligned}
&T_f^u(r, r', \Omega_s') - T_f^u(r, r', \Omega_s) \\
&= O(1) + u(r)\int_{dX(r)} d^c\varphi \wedge f^*\eta_Y \wedge \chi - \int_{X(r',r)} f^*\eta_Y \wedge \zeta \\
&\leqq O(1) + \text{const.}\int_{X(r_0,r)} f^*\Omega_{s-1} \wedge \zeta .
\end{aligned}$$

Now (4.12) follows from (4.13).

COROLLARY 4.6. *Assume* (I)-(IV-a), (V)-(VI). *Assume φ is a C^∞, g-convex exhaustion function of X. If (with $u=e^{-g}$ defined on $\boldsymbol{R}[r_0, \infty)$)*

$$u(r)\int_{X(r)} f^*\Lambda_Y \wedge \chi_1 \wedge \chi = o'(T_f^u(r, r_0, \Omega)) ,$$

then for almost every $S_b \in \mathfrak{A}$,

$$\lim_{r\to\infty}{}' \frac{N_f^u(r, r_0, S_b)}{T_f^u(r, r_0, \Omega)} = 1 . \tag{4.14}$$

Proof. If $r > r_0$ is φ-admissible, Lemma 3.2 yields

$$D_f(X(r); \chi_1 \wedge \chi) = \int_{dX(r)} f^*\Lambda_Y \wedge d^c\varphi \wedge \chi .$$

Hence Theorem 4.4 *concludes the proof.*

COROLLARY 4.7. *Assume* (I)-(IV-a), (V)-(VI). *Assume $\varphi: X \to \boldsymbol{R}_{-\infty}$ is a g-concave exhaustion function. Let $\sigma = \{r_j\}$ be an arbitrary φ-admissible sequence. Then there is a set $N_\sigma \subseteqq N$ of measure zero such that* (4.14) *holds for every S_b with $b \in N - N_\sigma$.*

Proof. Apply Theorem 4.4.

Let $W \to Y$ be a holomorphic vector bundle of fiber dimension $p \geqq 1$ over a complex space Y. Assume $\Gamma(Y, W)$ contains an ample linear subspace V (see [17]) of dimension $n + 1 \geqq 2$. Take $q \in \boldsymbol{Z}[0, n - p]$. For $b \in G_q(V)$, define $Z_b = \bigcap \{\text{Zero}\,(\sigma) \mid \sigma \in E(b)\}$. Let $d(q, n) = \dim G_q(V)$. Let $\omega_{q,1}$ be a normalized Kähler form on $G_q(V)$ such that, setting $\omega_{[q]} = \omega_{q,1}^{d(q,n)}$, $\int_{G_q(V)} \omega_{[q]} = 1$. The classifying map $c_V: Y \to G_{n-p}(V)$ is given by

$$E(c_V(y)) = \{\sigma \in V \mid \sigma(y) = 0\} \qquad (y \in Y) .$$

It can be easily shown that $Z_b = c_V^{-1}(S_b)$ for every $S_b \in \mathfrak{A}_{n-p,q}$. Here S_b has codimension $s = p(q + 1)$ in $G_{n-p}(V)$ (see [27]). Define

$$\Omega_{p,q} = c_V^*\Psi(\omega_{[q]}) \in A_\infty^{s,s}(Y) .$$

Then $\Omega_{p,q} \gg 0$ and $d\Omega_{p,q} = 0$.

COROLLARY 4.8. *Assume X satisfies* (I) *and $\varphi: X \to \boldsymbol{R}_{-\infty}$ is a g-concave exhaustion function. Let W, V, q be as above, and $f: X \rightharpoonup Y$ be a meromorphic map. Assume for every branch X_j of X there is a point $(x_j, b_j) \in (X_j - I_f) \times G_q(V)$ such that $codim_{x_j} f_0^{-1}(Z_{b_j}) = s$. Assume $\chi \in A_2^{m-s,m-s}(X)$ is closed, nonnegative, and $\chi > 0$ at some point of X, (if $m = s$, take $\chi = 1$). Then for almost all $b \in G_q(V)$, $f_0^{-1}(Z_b)$ has pure codimension s and*

$$\lim_{r\to\infty}{}' \frac{N_f^u(r, r_0, Z_b)}{T_f^u(r, r_0, \Omega_{p,q})} = 1 .$$

(Cf. the definitions in Theorem 4.4.)

Proof. The meromorphic map $c_V \circ f: X \rightharpoonup G_{n-p}(V)$ is almost adapted to $\mathfrak{A}_{n-p,q}$ by Corollary 1.4. Hence Corollary 4.7 yields the result.

THEOREM 4.9. *Assume* (I)-(III) *and* (IV-b). *Assume φ is a c.g-convex exhaustion function of X such that* (VI) *holds with $\chi = \chi_q$ and $\Omega = \Omega_s$.* (1) *Assume for some positive form $\zeta \in A_0^{k-1,k-1}(N)$, one of the following conditions holds (over a φ-admissible sequence σ):*

(a) $A_{f,s-1}^u(r, \xi_Y) = o'(T_{f,s}^u(r, 1, \Omega_s))$

(b) $D_{f,s-1}^u(r, \xi_Y) = o'\left(\int_1^r D_{f,s}^u(t, \Omega_s)u(t)dt + A_{f,s}^u(0, \Omega_s)\|u\|_1^r\right)$.

Then there is a set $N_\sigma \subset N$ of measure zero such that for every $b \in N - N_\sigma$,

$$\lim_{r\to\infty}{}' \frac{N_{f,s}^u(r, r_0, S_b)}{T_{f,s}^u(r, r_0, \Omega_s)} = 1 . \tag{4.15}$$

(Here $r_0 > 0$ is an arbitrary constant.) (2) *If φ is g-semiparabolic and if $s = 1$, the above conclusion holds for every φ-admissible sequence σ.*

Proof. For fixed $r' > 0$, $T_{f,s}^u(r, r', \Omega_s) \to \infty$ by (4.7), Lemmas 2.4 and 4.3. Let $\xi' \in A_\infty^{k-1,k-1}(N)$ be a $d^c d$-closed positive form ([8, p. 961]). Then (4.6) and Lemma 2.2 imply

$$D_{f,s-1}^u(r, \xi_Y') \leqq \text{const. } A_{f,s-1}^u(r, \xi_Y) .$$

Therefore (a) $\Rightarrow$ (b). (Similarly (b) $\Rightarrow$ (a).) Now (4.5) and Theorem 4.5 yield (4.15). Clearly (2) is a consequence of (1).

COROLLARY 4.10. *Assume* (I)-(III), (IV-b) *and $\mathfrak{A}$ is strictly adm. of codimension* 1. *Assume φ is an exhaustion function of X such that one of the following holds:*

(a) *φ is c.g-convex and*

$$A_0^u(r) = o'(T_{f,1}^u(r, 1, \Omega_1)) .$$

(b) *φ is g-quasiparabolic.*
If $m > 1$, assume $\chi_m \not\equiv 0$. Assume either Y is compact, or $d\omega_{N,1} = 0$ and Y has a finite number of connectivity components. Then (4.15) *holds for all S_b with $b \in N - N_\sigma$. (Here N_σ has measure zero, and in the case of* (b), *σ is an arbitrary φ-admissible sequence.)*

Proof. Since $\chi_1 \geqq 0$, $\chi_m \not\equiv 0$ implies $\chi_1 > 0$ at some point of X_{reg}. Also, by hypotheses, Ω_0 is a bounded function on Y. Hence the corollary follows from Theorem 4.9 and Lemma 4.3.

LEMMA 4.11. *Let $V, W: \boldsymbol{R}[c_0, \infty) \to \boldsymbol{R}[0, \infty)$, where W is increasing $\not\equiv 0$, and V is measurable, locally bounded. Let $u, \gamma: \boldsymbol{R}[c_0, \infty) \to \boldsymbol{R}(0, \infty)$ be continuous functions with $\|\gamma u\|_{c_0}^r \to \infty$. Assume for some constants $\alpha > 1$, $B \geqq 0$,*

$$\gamma(r)|V(r) - B|^\alpha = O(W(r)) \qquad (r \longrightarrow \infty) .$$

Let $E \subseteqq \boldsymbol{R}[c_0, \infty)$ be a set of measure zero. Then there exists a sequence $\{r_j\}$ in $\boldsymbol{R}[c_0, \infty) - E$ tending to infinity such that

$$V(r_j) = O(\|uW\|_{c_0}^{r_j}) \qquad (j \longrightarrow \infty) .$$

Proof. Put $H(r) = \|\gamma u\|_{c_0}^r$ and let J be the inverse function of H on $\boldsymbol{R}[0, \infty)$. There exist constants $K > 0$, $r_1 \geqq c_0$ such that

$$\gamma(r)|V(r) - B|^\alpha \leqq KW(r) \qquad (r \geqq r_1) .$$

For $a > a_1 = H(r_1)$ define $Q(a) = |V(J(a)) - B|^\alpha$, and $P(a) = \|Q\|_{a_1}^a$. The case $P(a) \equiv 0$ is trivial, hence assume $P(a) > 0$ for $a > a_1$. Then with $a = H(r) > a_1$,

$$|V(r) - B| \leqq K[(P^{-\alpha}Q)(a)]^{1/\alpha}\|uW\|_{r_1}^r .$$

Since $P^{-\alpha}Q \in L^1([a', \infty))$ for large a', there exists a sequence $\{r_j\}$ in $\boldsymbol{R}[c_0, \infty) - E$ tending to infinity such that $(P^{-\alpha}Q)(H(r_j)) < 2^{-j}$ for every j. From this the conclusion follows.

COROLLARY 4.12. *Assume* (I)-(III) *and* (IV-b). *Assume φ is a c.g-convex exhaustion function of X such that* (VI) *holds with $\chi = \chi_q$ and $\Omega = \Omega_s$. Assume there exists a positive form $\xi \in A_0^{k-1,k-1}(N)$ and a positive continuous $\gamma: \boldsymbol{R}[a_0, \infty) \to \boldsymbol{R}$ with $\|\gamma u\|_{a_0}^r \to \infty$ such that for some constants $\alpha > 1$, $B \geqq 0$, one of the following holds*:

(a) $\gamma(r)|D_{f,s-1}^u(r, \xi_Y) - B|^\alpha = O(D_{f,s}^u(r, \Omega_s))$.

(b) $\gamma(r)|A_{f,s-1}^u(r, \xi_Y) - B|^\alpha = O(A_{f,s}^u(r, \Omega_s))$.

Then there exists a φ-admissible sequence σ for which (4.15) *holds for almost every $S_b \in \mathfrak{A}$.*

Proof. Apply Lemma 4.11 and Theorem 4.9.

To give some applications of the preceding results, consider the following:

1. Let $f: X \to G_p(V)$ be a meromorphic map (where V has dim $n+1$) and $\mathfrak{A} = \mathscr{D}_{p,n}$. If $n = 1$, assume f is nondegenerate; if $n > 1$, assume every branch of X contains a point $x \notin I_f$ for which there is a $\Sigma_b \in \mathscr{D}_{p,n}$ with $\dim_x f_0^{-1}(\Sigma_b) = m - 1$. Assume $\varphi: X \to \boldsymbol{R}$ is an exhaustion function such that either 4.10 (a) or 4.10 (b) (with $\Omega_1 = \Psi(\omega_{[n-p-1]})$) holds. If $m > 1$, assume $\chi_m \not\equiv 0$. Then (4.15) holds for almost every $\Sigma_b \in \mathscr{D}_{p,n}$.

2. Theorem 4.6 of Stoll [26] for a family $\{S_v\}_{v \in E(A)}$ of Schubert zeros holds for a $c.g$-convex space (X, φ) (under conditions similar to (4.3), ibid.) in the stronger sense that the valence of almost all S_v grows at the same rate as the characteristics of f. Especially, the theorem holds if φ is g-quasiparabolic and q (see [26, assump. (13)]) $= m - 1$. This can be proved using Theorem 4.4 and Lemma 3.2.

3. (*Cf.* Stoll [23, 9.5].) Assume (I)-(IV-a) with X nonsingular, connected, and $\mathfrak{A}$ strictly admissible of codimension 1. Assume $\chi \in A_\infty^{m-1,m-1}(X)$ is closed and positive. Let $\sigma = \{G_j\}_{j=0}^\infty$ be a sequence of domains in X such that $\varnothing \neq G_j \Subset G_{j+1}$, $X - G_0$ has no compact component, $dG_j = \partial G_j$, and $\cup G_j = X$. Then there exist functions $\psi_j: X \to \boldsymbol{R}$ $(j \geqq 1)$ solving the Dirichlet problem

$$\chi \wedge dd^c\psi_j = 0 \qquad \text{on} \quad G_j - \bar{G}_0 ,$$

with $\psi_j | \bar{G}_0 = 1$, $\psi_j | X - G_j = 0$. The capacity of G_j (relative to χ) is defined by

$$C(G_j) = \int_{G_j - \bar{G}_0} \chi \wedge d\psi_j \wedge d^c\psi_j \qquad (j \geqq 1) .$$

It follows that $\psi_j \leqq \psi_{j+1}$, and $0 < C(G_{j+1}) \leqq C(G_j)$ ([23, 9.3]). Assume either (a) $C(G_j) \to 0$ or (b) $T_f(G_j) = \int_{G_j} \psi_j \chi \wedge f^*\Omega \to \infty$. Then for almost every $S_b \in \mathfrak{A}$,

$$\lim_{j \to \infty}{}' \frac{N_f(G_j, S_b, \psi_j \chi)}{T_f(G_j)} = 1 .$$

This follows from (3.3), Lemma 2.4 and the proof of Theorem 4.4, observing that for $j \geqq 1$,

$$\int_{dG_0} f^*\Lambda_Y \wedge d^\perp\psi_j \wedge \chi \leqq \text{const.}\ C(G_j) .$$

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