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COMMUTATIVE NON-ARCHIMEDEAN C^*-ALGEBRAS

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Commutative non-archimedean C^*-algebras are defined, their properties established, and a representation theory is developed for them. Their closed ideals are completely analyzed in terms of the closed subsets of the spectrum where they 'vanish.' A large class of C^*-algebras is exhibited. A Stone-Weierstrass theorem generalizing a result of Kaplansky is proved.

Introduction. In this paper F denotes a complete non-archimedean valued field, and it is assumed that the valuation is non-trivial. *A non-archimedean normed vector space* over F is a vector space X with a norm satisfying the *strong triangle inequality* $||x + y|| \leqq \max(||x||, ||y||)$ for all $x, y \in X$. If X is complete, X is called a *Banach space* over F.

Let A be an associative algebra over F, and suppose that $||\cdot||$ is a norm on A making A a non-archimedean normed space. If for all $x, y \in A$, $||xy|| \leqq ||x||, ||y||$ (and if A is unital, $||1|| = 1$), then we call A a *non-archimedean algebra*. If, further, A is a Banach space, then we call A a *Banach algebra*. In this paper a Banach algebra will be understood to be commutative and unital unless the contrary is explicitly assumed in a particular context.

If A is a unital commutative C^*-algebra over the complex numbers $\boldsymbol{C}$, then the Gelfand-Naimark theorem shows that if T is the spectrum of A, then A is isometrically isomorphic to $C(T, \boldsymbol{C})$, the algebra of continuous functions on T with values in $\boldsymbol{C}$. In this paper we define a class of algebras, called L-algebras, which play an analogous role in the non-archimedean theory to that played by the algebras $C(T, \boldsymbol{C})$ in the theory over $\boldsymbol{C}$. We prove a Stone-Weierstrass theorem concerning these algebras, and we establish their properties. In the second section we give an abstract definition of a non-archimedean commutative C^*-algebra. Such a definition has been sought for a number of years. We show that every C^*-algebra can be represented by an L-algebra. We derive a number of interesting properties of these C^*-algebras, and in the third section we give some examples of C^*-algebras.

1. The Stone-Weierstrass theorem.

DEFINITION 1.1. A *bundle* is a family $(X_t)_{(t \in T)}$ of Banach algebras over F indexed by a topological space T. $\bigoplus_{t \in T} X_t$ denotes the set of all elements x of the Cartesian product of the X_t which have $\|x\| =: \sup\{\|x(t)\|: t \in T\} < \infty$. Under the pointwise operations and this norm, $\bigoplus_{t \in T} X_t$ is a Banach algebra. If A is a subalgebra of $\bigoplus_{t \in T} X_t$ with $1 \in A$, and if for all $x \in A$ the maps $\psi_x: T \to \boldsymbol{R}\ t \to \|x(t)\|$ are upper semi-continuous (USC), then we call A an *algebra on the bundle*. ($\boldsymbol{R}$ denotes the set of real numbers.)

If A is an algebra on the bundle, $x \in \bigoplus_{t \in T} X_t$, and $t_0 \in T$, we say that *x is in A locally at t_0* if for all $\delta > 0$, there is an open set U in T with $t_0 \in U$, and there is an element y in A, such that for all $t \in U$, $\|x(t) - y(t)\| \leqq \delta$. We call A an *L-algebra on the bundle* if A contains all the elements of $\bigoplus_{t \in T} X_t$ which are in A locally at all points of T.

A simple example of an L-algebra is the following: Let $\beta = (X_t)_{(t \in T)}$, where T is any topological space, and $X_t = F$ for all $t \in T$. Let $C_b(T, F)$ denote the algebra of bounded continuous functions on T with values in F. Then $C_b(T, F)$ is an L-algebra on the bundle β. (See the observations following Corollary 1.5.)

THEOREM 1.1. *If A is an L-algebra on the bundle $(X_t)_{(t \in T)}$, then A is a Banach algebra.*

Proof. If x_n is a Cauchy sequence in A, then for each $t \in T$, $x_n(t)$ is a Cauchy sequence in the Banach algebra X_t, so there is an element $x(t)$ in X_t to which $x_n(t)$ converges. Let $x = (x(t))_{(t \in T)}$ and $\delta > 0$. There is an integer N such that for all $n, m > N$, and all $t \in T$, $\|x_n(t) - x_m(t)\| \leqq \delta/2$. Letting $m \to \infty$, we get $\|x_n(t) - x(t)\| \leqq \delta/2$, so $\|x_n - x\| < \delta$, for $n > N$. Thus we see that $x \in \bigoplus_{t \in T} X_t$, and x is in A locally, so $x \in A$. Hence A is complete.

THEOREM 1.2. *If A is any algebra on the bundle $\beta =: (X_t)_{(t \in T)}$, then $\gamma[\beta, A] =: \{x \in \bigoplus_{t \in T} X_t: x$ is in A locally$\}$ is the smallest L-algebra containing A.*

Proof. Suppose $x \in \gamma[\beta, A]$, and $\delta > 0$. If $\|x(t_0)\| < \delta$, there is $y \in A$ such that $\|x(t) - y(t)\| < \delta$ near t_0 (i.e., in a neighborhood of t_0 in T). But $\|x(t)\| \leqq \max(\|x(t) - y(t)\|, \|y(t)\|)$, so $\|x(t)\| < \delta$ near t_0. This shows the map $\psi_x: T \to \boldsymbol{R}\ t \to \|x(t)\|$ is USC for all $x \in \gamma[\beta, A]$.

Now suppose $x, y \in \gamma[\beta, A]$, $\alpha \in F$, $\delta > 0$, and $t_0 \in T$. Then for some $x', y' \in A$, we have

$$\begin{aligned} \|xy(t) - x'y'(t)\| &\leqq \|x(t)y(t) - x(t)y'(t) + x(t)y'(t) - x'(t)y'(t)\| \\ &\leqq \max(\|x(t)\| \cdot \|y(t) - y'(t)\|, \|x(t) - x'(t)\| \cdot \|y'(t)\|) \\ &\leqq \max((1 + \|x(t_0)\|)\|y(t) - y'(t)\|, \|x(t) - x'(t)\|(1 + \|y'(t_0)\|)) . \end{aligned}$$

These inequalities hold for t near t_0, because by the USC property, $||x(t)|| \leqq 1 + ||x(t_0)||$ near t_0, and $||y'(t)|| \leqq 1 + ||y'(t_0)||$ near t_0. Now we can choose y' to have $||y(t) - y'(t)|| \leqq \delta(1 + ||x(t_0)||)^{-1}$ near t_0, and then x' so that $||x(t) - x'(t)|| \leqq \delta(1 + ||y'(t_0)||)^{-1}$ near t_0. This gives us $||xy(t) - x'y'(t)|| \leqq \delta$ near t_0. So as $x'y' \in A$, xy is in A locally at each point t_0 of T. Hence $xy \in \gamma[\beta, A]$. It is easy to see that $x + y$ and αx are also in $\gamma[\beta, A]$. Thus $\gamma[\beta, A]$ is an algebra on β, and it clearly contains A.

If x is in $\gamma[\beta, A]$ locally, then for each $t_0 \in T$, and $\delta > 0$, there is $x' \in \gamma[\beta, A]$ with $||x'(t) - x(t)|| \leqq \delta$ for t near t_0. But then there is $y \in A$ with $||x'(t) - y(t)|| \leqq \delta$ for t near t_0. So $||x(t) - y(t)|| \leqq \delta$ for t near t_0. Hence $x \in \gamma[\beta, A]$. Thus $\gamma[\beta, A]$ is an L-algebra.

If γ' is any other L-algebra containing A, then any element $x \in \gamma[\beta, A]$ is in A locally, so x is in γ' locally, as A is contained in γ', and so $x \in \gamma'$, as γ' is an L-algebra. Hence $\gamma[\beta, A]$ is contained in γ'.

DEFINITION 1.2. If A is an algebra on a bundle $\beta = (X_t)_{(t \in T)}$, and if for all distinct points s, t of T there is $x \in A$ with $||x|| \leqq 1$, $x(s) = 0(s)$, and $x(t) = 1(t)$, then we say A is *separating on* β.

If E is any clopen set of T, define φ_E by $\varphi_E(t) = 1(t)$ if $t \in E$, and $\varphi_E(t) = 0(t)$ if $t \in T - E$. If A is any L-algebra on β, then clearly $\varphi_E \in A$. Hence if T is a *Boolean* space (i.e., a compact, Hausdorff, totally disconnected space)—in this case we call β a *Boolean* bundle—then every L-algebra on β is a separating algebra. The converse of this statement is our generalization of the Stone-Weierstrass theorem. First we need a lemma whose proof is a simple induction.

LEMMA 1.3. *If A is a normed (non-archimedean) algebra, $x_1, \cdots, x_n \in A$, and $0 < \delta < 1$, and $||x_i|| \leqq 1$, $||1 - x_i|| < \delta$, $(i = 1, \cdots, n)$, then $||1 - x_1 \cdots x_n|| < \delta$ and $||x_i|| = 1$.*

THEOREM 1.4 (*Stone-Weierstrass*). *Let A be a separating Banach algebra on a Boolean bundle. Then A is an L-algebra on the bundle.*

Proof. Let $\beta = (X_t)_{(t \in T)}$ be the bundle, and $\gamma = \gamma[\beta, A]$. First we show that if E is a clopen set in T, then $\varphi_E \in A$. For let $s \in E^c = T - E$, and $t \in E$. As $s \neq t$, there is a $y^t \in A$ such that $||y^t|| \leqq 1$, $y^t(s) = 1(s)$, and $y^t(t) = 0(t)$. If $0 < \delta < 1$, then by the USC property, there is a clopen set V_t in T with $t \in V_t$ such that for all $t' \in V_t$, $||y^t(t')|| \leqq \delta$. Thus E is contained in $\bigcup_{t \in E} V_t$, and so as E is compact, there is a finite number $V_{t_1}, \cdots, V_{t_n}$ covering E.

Define $y_s = y^{t_1} \cdots y^{t_n}$. Then $y_s \in A$, and $\|y_s\| \leqq 1$, $y_s(s) = 1(s)$ and $\|y_s(t)\| < \delta$ for all $t \in E$. But $h_s = 1 - y_s$. Then $h_s \in A$ and $\|h_s\| \leqq 1$, $h_s(s) = 0(s)$. Moreover, for all $t \in E$, $\|1(t) - h_s(t)\| < \delta$. Once again, by the USC property, there is a clopen set W_s in T with $s \in W_s$ such that for all $s' \in W_s$, $\|h_s(s')\| < \delta$. So E^c is covered by the sets W_s, $s \in E^c$, and as E^c is closed in T and so compact, a finite number $W_{s_1}, \cdots, W_{s_m}$ cover E^c. Define $h = h_{s_1} \cdots h_{s_m}$. Then $h \in A$, $\|h\| \leqq 1$, and for all $s' \in E^c$, $\|h(s')\| < \delta$. Now by the lemma, for any $t \in E$, $\|1(t) - h(t)\| = \|1(t) - h_{s_1}(t) \cdots h_{s_m}(t)\| < \delta$. Thus for all $t \in T$, $\|\varphi_E(t) - h(t)\| < \delta$, so $\|\varphi_E - h\| \leqq \delta$. But A is closed in $\bigoplus_{t \in T} X_t$, and $h \in A$. Therefore as δ was arbitrarily small, $\varphi_E \in A$.

Now suppose that $x \in \gamma$, $\delta > 0$, and $t_0 \in T$. Then there is $z_{t_0} \in A$ such that $\|x(t) - z_{t_0}(t)\| \leqq \delta$ for all t near t_0, i.e., for all t in some clopen set U_{t_0} with $t_0 \in U_{t_0}$. Thus T is a union of such sets, and so by compactness there is a finite number $U_{t_1}, \cdots, U_{t_p}$ covering T. Put $E_1 = U_{t_1}$, and for $i = 2, \cdots, p$, $E_i = U_{t_i} - (\bigcup_{j<i} U_{t_j})$. Then the the E_i form a pairwise disjoint family of clopen sets covering T. The element $y = \varphi_{E_1} z_{t_1} + \cdots + \varphi_{E_p} z_{t_p}$ is in A. Also $\|x(t) - y(t)\| = \|x(t) - z_{t_i}(t)\|$ if $t \in E_i$, and this is less than or equal to δ, so $\|x(t) - y(t)\| \leqq \delta$, for all t in T, i.e., $\|x - y\| \leqq \delta$. Thus $x \in A$, as A is closed in $\bigoplus_{t \in T} X_t$. Therefore γ is contained in A, and so $\gamma = A$. Thus A is an L-algebra on β.

COROLLARY 1.5. *Let $\beta = (X_t)_{(t \in T)}$ be a Boolean bundle, and $\beta' = \{x \in \bigoplus_{t \in T} X_t \colon \psi_x \colon T \to \boldsymbol{R}\ t \to \|x(t)\|$ is USC$\}$. If I is any subset of $\bigoplus_{t \in T} X_t$ let $I_t = \{x(t) \colon x \in I\}$ for each $t \in T$. If A is a separating Banach algebra on β, then*

$$A = \{x \in \beta' \colon x(t) \in A_t \text{ for all } t \in T,\ x - y \in \beta' \text{ for all } y \in A\}\,.$$

Proof. Let the set on the R.H.S. of the equation be denoted by B. Then clearly A is contained in B. So suppose that $x \in B$ and $t_0 \in T$. Then there is an element $x_{t_0} \in A$ such that $x(t_0) = x_{t_0}(t_0)$. Let $\delta > 0$. Since the map $\psi_{x - x_{t_0}}$ is USC, there is a clopen set U_{t_0} with $t_0 \in U_{t_0}$ such that for all $t \in U_{t_0}$, $\|x(t) - x_{t_0}(t)\| < \delta$. These sets cover T, so by the compactness of T there is a finite number of them $U_{t_1}, \cdots, U_{t_n}$ covering T. As in the proof of the Stone-Weierstrass theorem we can replace these sets by a pairwise disjoint family $(E_i)_{(i=1,\cdots,n)}$ of clopen sets covering T and such that E_i is contained in U_{t_i} for $i = 1, \cdots, n$. Now $\varphi_{E_i} \in A$ for each i, from Theorem 1.4, so $y = \colon \varphi_{E_1} x_{t_1} + \cdots + \varphi_{E_n} x_{t_n}$ is in A, and $\|x - y\| \leqq \delta$. But as A is closed, this implies $x \in A$. Thus $A = B$.

Suppose X is a Banach algebra over F, and T is any topological space. For each $t \in T$, let $X_t = X$. Let K denote the algebra of constant functions from T to X. Then K is clearly an algebra on the bundle $\beta = (X_t)_{(t \in T)}$. So $\gamma[\beta, K]$ is an L-algebra on β. Suppose $x \in \bigoplus_{t \in T} X_t$, and $t_0 \in T$. Then x is in K locally at t_0 iff for all $\delta > 0$, for all t near t_0, $||x(t) - x(t_0)|| < \delta$. Thus x is in K locally at t_0 iff x is continuous at t_0. Hence $\gamma[\beta, K] = C_b(T, X)$, the algebra of all bounded continuous functions defined on T with values in X. When T is compact this is of course $C(T, X)$, the algebra of continuous functions on T with values in X. We can now state the Stone-Weierstrass theorem for these algebras.

COROLLARY 1.6. *Let X be a Banach algebra, and T a compact space. If A is a closed separating subalgebra of $C(T, X)$ and A contains the constants X, then $A = C(T, X)$.*

Proof. This follows immediately from Corollary 1.5 if we show T is a Boolean space.

Suppose s, t are distinct points of T. Then there is an element $x \in A$ such that $x(s) = 0$ and $x(t) = 1$. Hence s is an element of the clopen set $\{u \in T: ||x(u)|| < 1\}$, and t is not. Thus T is Hausdorff. Moreover the connected component of s is contained in the above clopen set, and that of t is contained in its complement. So s and t are disconnected. Thus T is totally disconnected. Hence T is a Boolean space.

COROLLARY 1.7. *Let T be a compact space and A a closed separating subalgebra of $C(T, F)$ containing the constants. Then $A = C(T, F)$.*

Proof. Trivial. Just take $X = F$ in Corollary 1.6.

This is Kaplansky's non-archimedean Stone-Weierstrass theorem.

We now investigate the closed ideals of L-algebras. For this the following theorem is fundamental.

THEOREM 1.8. *Let A be an L-algebra on a Boolean bundle $(X_t)_{(t \in T)}$, and I be a closed ideal in A. Then if $x \in A$, $x \in I$ if and only if $x(t) \in I_t$ for all $t \in T$.*

Proof. The "only if" part of the equivalence is obvious. Suppose then $x(t) \in I_t$ for all $t \in T$. Then for each $t \in T$, there is some $y_t \in I$ such that $x(t) = y_t(t)$. If $\delta > 0$, then by the USC property there is a clopen set U_t with $t \in U_t$ such that for all $t' \in U_t$, $||x(t') - y_t(t')|| < \delta$.

By a familiar argument we can replace the covering $(U_t)_{(t\in T)}$ of T by a finite covering of pairwise disjoint clopen sets E_i contained in U_{t_i}, say, $(i = 1, \cdots, n)$ and $U_{t_1} \cup \cdots \cup U_{t_n} = T$. Let $y = \varphi_{E_1} y_{t_1} + \cdots + \varphi_{E_n} y_{t_n}$. Then as all the $\varphi_{E_i} \in A$, and the $y_{t_i} \in I$, so $y \in I$. Also $\|x(t) - y(t)\| = \|x(t) - y_{t_i}(t)\| < \delta$ if $t \in E_i$. Thus $\|x - y\| \leqq \delta$. But as I is closed, this implies $x \in I$.

COROLLARY 1.9. *If I, J are closed ideals in A, the $I = J$ if and only if $I_t = J_t$ for all $t \in T$.*

Proof. This is obvious from Theorem 1.8.

DEFINITION 1.3. Let A be an algebra on a bundle $(X_t)_{(t\in T)}$. We say A is *full* if $A_t = X_t$ for all $t \in T$.

If all the X_t are fields, we call the bundle a *field* bundle.

THEOREM 1.10. *Let A be a full separating Banach algebra on a Boolean field bundle $\beta = (X_t)_{(t\in T)}$. For each $t \in T$, let $M^t = \{x \in A: x(t) = 0(t)\}$. Then M^t is a maximal ideal in A, and the map $T \to T(A)$ $t \to M^t$ is a homeomorphism. (Here $T(A)$ is the maximal ideal space of A endowed with the Hull-Kernel topology.)*

Proof. If $s, t \in T$, then $(M^t)_s = X_s$ if $s \neq t$, and $(M^t)_s = 0$ if $s = t$. The second equation is obvious, so let us prove the first. If $a \in X_s$, then there is $x \in A$ such that $x(s) = a$, since A is full. Also there is a $y \in A$ such that $y(t) = 0(t)$ and $y(s) = 1(s)$. Let $z = xy$. Then $z \in M^t$, and $z(s) = a$. Hence $a \in (M^t)_s$. Thus $(M^t)_s = X_s$.

Suppose now that I is a closed ideal in A containing M^t. Then if $s \neq t$, $(M^t)_s = I_s = X_s$. Also $I_t = 0$ or X_t. Hence $I_s = X_s$ for all $s \in T$, and so $I = A$, or $I_s = (M^t)_s$ for all $s \in T$, and so $I = M^t$. Thus M^t is a maximal ideal in A.

Now suppose that M is any maximal ideal in A. Then M is closed and $M \neq A$, so there is $t \in T$ such that $M_t \neq X_t$. Therefore $M_t = 0$, and so M is contained in M^t, and hence $M = M^t$. Thus $T(A) = \{M^t: t \in T\}$.

Let φ denote the map $t \to M^t$. It has just been shown that φ is surjective, and if $M^t = M^s$, and $s \neq t$, there is $y \in A$ such that $y(s) = 0(s)$ and $y(t) = 1(t)$. Hence $y \in M^s$ and $y \notin M^t$. But this is impossible, so $s = t$. Hence φ is injective. To prove φ is a homeomorphism it is sufficient to show φ^{-1} is continuous, because $T(A)$ is compact and T is Hausdorff. Let E be a clopen set in T. Then as $\varphi_E \in A$, $\varphi(E) = \{M^t: (1 - \varphi_E)(t) = 0(t)\} = \{M^t: \varphi_E \notin M^t\}$. Thus $\varphi(E)$ is the complement in $T(A)$ of the closed set $V(A\varphi_E) = \{M \in T(A)$: M contains $A\varphi_E\}$. This shows φ^{-1} is continuous.

LEMMA 1.11. *Let A be a full separating Banach algebra on a Boolean field bundle $(X_t)_{(t\in T)}$. If S is any subset of T, let $id(S) = \{x \in A: x(s) = 0(s)$ for all $s \in S\}$.*

(a) *$id(S)$ is a closed ideal in A.*

(b) *If S_1, S_2 are subsets of T with S_1 contained in S_2, then $id(S_1)$ contains $id(S_2)$.*

(c) *For all S contained in T, $id(S) = id(\mathrm{cl}\,(S))$.*

(d) *If S_1, S_2 are closed subsets of T, then $id(S_1) = id(S_2)$ if and only if $S_1 = S_2$.*

(e) *If S is any subset of T, then $id(S)$ is a maximal ideal in A if and only if S is a singleton.*

Proof. (a) and (b) are obvious, so consider (c). Clearly $id(\mathrm{cl}\,(S)) \subseteqq id(S)$, so suppose $x \in id(S)$ and $x \notin id(\mathrm{cl}\,(S))$. Then there is an element s of $\mathrm{cl}\,(S)$ with $x(s) \neq 0(s)$. Now $V(Ax) = \{M \in T(A): M \supseteqq Ax\} = \{M^t: x(t) = 0(t)\}$ is closed in $T(A)$, so using the homeomorphism of Theorem 1.10, $\{t \in T: \|x(t)\| = 0\}$ is a closed set in T, and so $U = \{t \in T: \|x(t)\| > 0\}$ is open in T. Therefore as $s \in U$, the intersection of S and U is nonempty. But this is clearly a contradiction. So $id(S) = id(\mathrm{cl}\,(S))$.

To prove (d), suppose that $id(S_1) = id(S_2)$, and S_1 is not contained in S_2. Then there is $s \in S_1$, $s \notin S_2$. But as $T - S_2$ is open in T, there is a clopen set E contained in $T - S_2$ such that $s \in E$. So $\varphi_E \in A$, and $\varphi_E(t) = 0(t)$ for all $t \in S_2$. Hence $\varphi_E \in id(S_2) = id(S_1)$. So $\varphi_E(s) = 0(s)$, implying $s \notin E$. This contradiction shows that $S_1 = S_2$.

Finally consider (e). Clearly $id(\{t\}) = M^t$, which is a maximal ideal. Suppose now that $id(S)$ is a maximal ideal, and $s, t \in S$. Then $id(S) \subseteqq M^s, M^t$, so $id(S) = M^s = M^t$. Hence $s = t$, and $S = \{t\}$ (if S were empty, then $id(S) = A$).

LEMMA 1.12. *Let T be a Boolean space, U an open subset, and C a closed subset, with C contained in U. Then there is a clopen set E in T such that $C \subseteqq E \subseteqq U$.*

Proof. For each $x \in C$ there is a clopen set U_x with $x \in U_x \subseteqq U$. Hence the family U_x cover the compact set C, so there is a finite number so that C is contained in their union E, say. Clearly E is clopen, contains C, and is contained in U.

The following theorem is a structure theorem for the closed ideals of certain L-algebras.

THEOREM 1.13. *Let A be a full separating Banach algebra on a Boolean field bundle $(X_t)_{(t\in T)}$. If I is a closed ideal in A, let*

$k(I) = \{t \in T$: for all $x \in I$, $x(t) = 0(t)\}$. Then $k(I)$ is closed in T, and $I = id(k(I))$.

Proof. Suppose that $t \in \mathrm{cl}\,(k(I))$. Then there is a net $(t_\alpha)_\alpha$ in $k(I)$ converging to t. Hence if $x \in I$, then t_α is in the closed set $E = \{s \in T: x(s) = 0(s)\}$ for all indices α. So $t \in E$. Therefore $x(t) = 0(t)$. This implies that $t \in k(I)$. Thus $k(I)$ is closed in T.

Suppose now $x \in A$, and G is an open set in T containing $k(I)$, and that $x = 0$ on G. Then $t \in T - G$ implies $t \notin k(I)$, so $I_t \neq 0$. Hence there is $x_t \in I$ with $x_t(t) \neq 0(t)$. There is therefore a clopen set U_t with $t \in U_t$ such that x_t is nonzero on U_t. Now $T - G \subseteq \bigcup_{t \in G} U_t$. But as $T - G$ is closed in T, it is compact, and so we can cover $T - G$ by a finite number $U_{t_1}, \cdots, U_{t_n}$, say. Let $E_1 = U_{t_1}$, and for $i = 2, \cdots, n$ let $E_i = U_{t_i} - (\bigcup_{j<i} U_{t_j})$. Then the family of sets $(E_i)_i$ form a pairwise disjoint covering by clopen sets of $T - G$. Let $P = \varphi_{(T-(E_1 \cup \cdots \cup E_n))}$. Then $\varphi_{E_1}, \cdots, \varphi_{E_n}$, P are all in A. Define $y = \varphi_{E_1} x_{t_1} + \cdots + \varphi_{E_n} x_{t_n} + P$. Then $y \in A$, and for all $t \in T$, $y(t) \neq 0(t)$. Hence y is invertible in A. Let $z = (1 - P)(\varphi_{E_1} x_{t_1} + \cdots + \varphi_{E_n} x_{t_n}) y^{-1}$. Again $z \in A$; also $z(t) = 1(t)$ if $t \in E_1 \cup \cdots \cup E_n$, and $z(t) = 0(t)$ otherwise. So $x(t)z(t) = x(t)$ for all $t \in T$. I.e., $xz = x$. But because all the $x_{t_i} \in I$, so $z \in I$. Hence $x \in I$.

Suppose now $x \in id(k(I))$, and $\delta > 0$. Then as x is zero on $k(I)$, so $k(I) \subseteq \{t \in T: \|x(t)\| < \delta\}$. Hence by Lemma 1.12, there is a clopen set E containing $k(I)$ and contained in $\{t \in T: \|x(t)\| < \delta\}$. Then $\varphi_{T-E} \in I$, from the above argument, because $\varphi_{T-E} = 0$ on E. Also $\|x - x\varphi_{E^c}\| = \sup_{t \in T} \|x(t) - x(t)\varphi_{T-E}(t)\| \leqq \delta$, and since I is closed, this gives $x \in I$. Hence $id(k(I)) \subseteq I$.

The reverse inclusion is trivial, so these ideals are equal.

2. C^*-Algebras.

DEFINITION 2.1. Let A be a Banach algebra satisfying the following two conditions:

(a) If $t \in T(A)$, $x \in t$, and $\delta > 0$, then there is an idempotent $p \in t$ such that $\|x - xp\| < \delta$.

(b) For all idempotents $p \in A$, $\|p\| \leqq 1$.

Then we call A a *C^*-algebra.*

For example, if T is a compact space, then $C(T, F)$ is a C^*-algebra, the idempotents being characteristic functions of clopen sets in T.

The above conditions on a Banach algebra will be seen to be necessary and sufficient conditions to ensure that the algebra is an isometric isomorph of an L-algebra on a Boolean field bundle. This is precisely the class of algebras we want the term "C^*-algebra" to cover.

If A is any Banach algebra we define $||\cdot||_{\sup}$ by $||x||_{\sup} = \sup\{||x + t||: t \in T(A)\}$ for all $x \in A$. Here $||x + t||$ is the quotient norm of $x + t$ in A/t, $||x + t|| = \inf\{||x + y||: y \in t\}$. Thus $||\cdot||_{\sup}$ is a norm on A if A is semisimple.

Before proving the next theorem, let us just make some remarks here relating the C^* concept to the V^*-algebras defined in [3]. Using Theorem 2.1 below, and Theorem 4, p. 149 of [3], we easily see that a C^*-algebra is a V^*-algebra. Conversely, from [3] p. 165, Cor. 2, a V^*-Gelfand algebra with compact maximal ideal space (in the Gelfand topology) is a C^*-algebra.

THEOREM 2.1. *If A is a C^*-algebra, then* $||\cdot||_{\sup} = ||\cdot||$.

Proof. Suppose $x \in A$, $t \in T(A)$, and $||x + t|| < ||x||$. Now it is easy to see that because of condition (a) in Definition 2.1, $t = \mathrm{cl}(\{pa: p = p^2 \text{ and } p \in t, a \in A\})$. Hence $||x + t|| = \inf\{||x - xp||: p = p^2$ and $p \in t\}$. So there is an idempotent $p \in t$ such that $||x - xp|| < ||x||$, as $||x + t|| < ||x||$. Let $I = \cup\{pA: ||px|| < ||x||, p \in A, \text{ and } p = p^2\}$. Then I is a proper ideal in A. For suppose that p, q are idempotents in A such that $||px||, ||qx|| < ||x||$. Then $r = p + q - pq$ is also an idempotent in A, and $pA, qA \subseteqq rA$, because $pr = p$, $qr = q$. Moreover $||rx|| \leqq \max(||px||, ||qx||, ||pqx||) < ||x||$. This shows that I is an ideal, and if I contained 1, then there would be an idempotent p of A such that $1 \in pA$, and $||px|| < ||x||$. Then $1 = pa$ for some $a \in A$, hence $1 = p$, so $||x|| < ||x||$. This contradiction shows that I is proper. Hence there is a maximal ideal s in A containing I. If p is any idempotent in s, then $1 - p \notin I$, so $||(1 - p)x|| = ||x||$. Hence $\inf\{||x - xp||: p \in s \text{ and } p = p^2\} = ||x||$, or $||x + s|| = ||x||$. Thus $||x||_{\sup} = ||x||$ for all $x \in A$.

If A is a Banach algebra, and $x \in A$, define $\bar{x} = (x + t)_{(t \in T(A))}$. Define $\bar{A} = \{\bar{x} \in \bigoplus_{t \in T(A)} A/t: x \in A\}$. Then $\bar{A}$ is a normed subalgebra of $\bigoplus_t A/t$, and the map $\mathfrak{G}: A \to \bar{A}$, $x \to \bar{x}$ is an algebra homomorphism, and is clearly surjective.

THEOREM 2.2. *If A is a C^*-algebra then $\bar{A}$ is a Banach full separating algebra on the Boolean field bundle $(A/t)_{(t \in T(A))}$. Moreover the map $\mathfrak{G}: A \to \bar{A}$ $x \to \bar{x}$ is an isometric isomorphism.*

Proof. Suppose $x \in A$, $\delta > 0$, and $E = \{t \in T(A): ||x + t|| < \delta\}$. Then if $t \in E$, there is an idempotent $p \in t$ such that $||x - px|| < \delta$. Hence if s is a maximal ideal with $p \in s$, then $||x + s|| = \inf\{||x - qx||:$

$q = q^2 \in s\} \leqq ||x - xp|| < \delta$, and so $s \in E$. Hence $V(Ap) = \{t' \in T(A)$: Ap is contained in $t'\}$ satisfies $t \in V(Ap) \subseteqq E$, and $V(Ap)$ is open. (In fact, $V(Ap)$ is clopen, as its complement in $T(A)$ is $V(A(1 - p))$, which is closed. Recall that every maximal ideal is prime, and for all $p = p^2$, $p(1 - p) = 0$, so for any maximal ideal M, p or $1 - p \in M$.) Thus E is a neighborhood of all its points, and so E is open. Hence the map $\psi_{\bar{x}}$: $T(A) \to \boldsymbol{R}$ $t \to ||x + t||$ is USC, for all $x \in A$. Thus $\bar{A}$ is an algebra on the field bundle $(A/t)_t$. To show that $T(A)$ is a Boolean space, suppose that s, t are distinct points of $T(A)$. Then from the condition (a) of Definition 2.1, we see there is an idempotent $p \in s$, $p \notin t$. Thus $s \in V(Ap)$, and $t \notin V(Ap)$. As $V(Ap)$ is clopen, this shows that $T(A)$ is Hausdorff. Also the connected component of t is contained in $V(A(1 - p))$, and the connected component of s is contained in its complement $V(Ap)$. Hence $T(A)$ is totally disconnected. Thus $T(A)$ is a Boolean space.

It is clear from Theorem 2.1 that the map $\mathfrak{G}$ is an isometric isomorphism, so $\bar{A}$ is a Banach algebra, as A is. That $\bar{A}$ is full is obvious, so we have only now to show that it is separating. But we have seen above that if s, t are distinct points of $T(A)$ there is an idempotent $p \in s$, $p \notin t$. Hence, as t is a maximal ideal, $1 - p \in t$. However $||p|| \leqq 1$. Thus $||\bar{p}|| \leqq 1$, $\bar{p}(s) = 0(s)$, and $\bar{p}(t) = 1(t)$. Also $\bar{p} \in \bar{A}$. Thus $\bar{A}$ is separating.

THEOREM 2.3. *Let A be a full separating Banach algebra on a Boolean field bundle. Then A is a C^*-algebra.*

Proof. Let $(X_t)_{(t \in T)}$ be the bundle. We know from Theorem 1.10 that the map φ: $T \to T(A)$ $t \to M^t$ is a homeomorphism. So suppose $x \in M^t$, and $\delta > 0$. Then $||x(t)|| = 0 < \delta$, so there is a clopen set E, say, with $t \in E$, such that for all $s \in E$, $||x(s)|| < \delta$. Hence the idempotent $p = 1 - \varphi_E \in A$, and $p(t) = 0(t)$. Hence $p \in M^t$. Also $||x - px|| = \sup\{||x(s) - p(s)x(s)||: s \in T\} = \sup\{||x(s)||: s \in E\} \leqq \delta$. Finally it is clear that if q is any idempotent of A, then $||q|| \leqq 1$, because for all $t \in T$, $q(t) = 0(t)$ or $1(t)$, giving $||q(t)|| \leqq 1$. Hence A is a C^*-algebra.

THEOREM 2.4. *Let I be a closed ideal in a C^*-algebra A. Then $I = \cap V(I) = \mathrm{cl}(\cup \{pA: p \in I$ and $p = p^2\}) = \mathrm{cl}(\cup \{pI: p = p^2 \in I\})$. (For every ideal I in A, $V(I)$ is the set of maximal ideals containing I.)*

Proof. We know from Theorem 1.13 that $\bar{I} = id(k(\bar{I}))$. Now if $x \in \cap V(I)$, then $\bar{x} = 0$ on $k(\bar{I})$, for if $\bar{I}_t = 0$, then $I \subseteqq t$. So $\bar{x} \in id(k)(\bar{I})) = \bar{I}$, whence $x \in I$. Thus $\cap V(I)$ is contained in I, and the reverse inclusion is trivial, so $\cap V(I) = I$.

Now suppose that $x \in I$. The set $G_n = \{t \in T(A): ||x + t|| < 1/n\}$ is an open set containing the closed set $k(\bar{I})$, hence there is a clopen set E_n containing $k(\bar{I})$ and contained in G_n (using Lemma 1.12). Let $p_n = 1 - \varphi_{E_n}$. Then p_n is an idempotent in $\bar{A}$, and $p_n = 0$ on E_n. Hence $p_n \in \bar{I}$. Also $||\bar{x} - \bar{x}p_n|| = \sup\{||\bar{x}(t) - \bar{x}p_n(t)||: p_n(t) = 0(t)\} = \sup\{||x + t||: t \in E_n\} \leqq 1/n$. Now there are idempotents q_n in A such that $\bar{q}_n = p_n$ $n = 1, 2, \cdots$. Hence these q_n must be in I, and $||x - xq_n|| \to 0$ $(n \to \infty)$. So $x \in \mathrm{cl}(\cup\{pA: p = p^2 \in I\})$. So I is equal to this set.

Let A be a Banach algebra, and I be a closed ideal in A. Then the map $V(I) \to T(A/I)$ $t \to t/I$ is well known to be a homeomorphism. Also the maximal modular ideals of I are precisely the ideals of the form $t \cap I$, where t is a maximal ideal of A not containing I. Another useful remark which it is easy to verify is the following: If $x \in A$, and $t \in V(I)$, then $||x + I + t/I|| = ||x + t||$.

DEFINITION 2.2. If I is a nonunital Banach algebra we say that I is a C^*-algebra if the following three conditions hold:

(a) If t is a maximal modular ideal of I, $x \in t$, and $\delta > 0$, then there is an idempotent p of I such that $||x - px|| < \delta$ and $p \in t$, or there is an idempotent q of I such that $||qx|| < \delta$ and $q \notin t$.

(b) For all idempotents p of I, $||p|| \leqq 1$.

(c) $I = \mathrm{cl}(\cup\{pI: p = p^2 \in I\})$.

The following interesting lemma is used in our next theorem.

LEMMA 2.5. *If A is a C^*-algebra, and I is a closed ideal in A, then for all $x \in A$, $||x + I|| = \sup\{||x + t||: t \in V(I)\}$.*

Proof. We know that $\bar{A}$ is an L-algebra on the Boolean field bundle $\beta = (A/t)_{(t \in T(A))}$, and that the map $\mathfrak{G}: A \to \bar{A}$, $x \to \bar{x}$ is an isometric isomorphism. Also $\bar{I} = id(k(\bar{I}))$. We assert that $||\bar{x} + \bar{I}|| = \sup\{||\bar{x}(s)||: s \in k(\bar{I})\}$. Let this sup be denoted ε. Now $||\bar{x} + \bar{I}|| = \inf\{||\bar{x} + \bar{y}||: \bar{y} = 0$ on $k(\bar{I})\} = \inf\{\sup_{t \in T(A)} ||x(t) + y(t)||: \bar{y} = 0$ on $k(\bar{I})\} \geqq \varepsilon$, as each of the terms of the inf $\geqq \varepsilon$. If $\varepsilon = 0$, then $\bar{x} = 0$ on $k(\bar{I})$, so $\bar{x} \in \bar{I}$, so $||\bar{x} + \bar{I}|| = 0$. Hence w.l.o.g. $\varepsilon > 0$. Let $\varepsilon_n = \varepsilon(1 + 1/n)$, for $n = 1, 2, \cdots$. Thus $\varepsilon_n > \varepsilon$, and the sets $G_n = \{t \in T(A): ||\bar{x}(t)|| < \varepsilon_n\}$ are open and contain $k(\bar{I})$, so there are clopen sets E_n such that $k(\bar{I}) \subseteqq E_n \subseteqq G_n$ (by Lemma 1.12). The elements $y_n = -\bar{x}\varphi_{E_n^c} = -\bar{x}(1 - \varphi_{E_n})$ are in $\bar{A}$, as $\bar{A}$ is an L-algebra on β. But as $y_n = 0$ on $k(\bar{I})$, so $y_n \in \bar{I}$. Now $||\bar{x} + y_n|| = \sup\{||\bar{x}(t) - \bar{x}(t)\varphi_{E_n^c}(t)||: t \in T(A)\} = \sup\{||\bar{x}(t)||: t \in E_n\} \leqq \varepsilon_n$. Also as $\varepsilon_n \to \varepsilon$, and $||\bar{x} + \bar{I}|| \leqq ||\bar{x} + y_n|| \leqq \varepsilon_n$, so $||\bar{x} + \bar{I}|| \leqq \varepsilon$, and hence $||\bar{x} + \bar{I}|| = \varepsilon$.

Thus we see from this result that if $t \in T(A)$, then as $\bar{t} = M^t$, so $||x + t|| = ||\bar{x} + M^t|| = \sup\{||\bar{x}(s)||: s \in k(M^t)\} = ||\bar{x}(t)||$. Hence we see that $||x + I|| = ||\bar{x} + \bar{I}|| = \sup\{||\bar{x}(s)||: s \in k(\bar{I})\}$, and as $k(\bar{I}) = V(I)$, we now see $||x + I|| = \sup\{||x + s||: s \in V(I)\}$.

THEOREM 2.6. *Let A be a C^*-algebra, and I a closed ideal in A. Then I and A/I are C^*-algebras also.*

Proof. Let $y + I$ be an idempotent in A/I. Then by our lemma, $||y + I|| = \sup\{||y + t||: I \subseteq t\}$. But if $t \in V(I)$, then t/I is a maximal ideal in A/I, so $y + I$ or $1 - y + I \in t/I$. Hence y or $1 - y \in t$. So $||y + t|| = 0$ or 1, and so $||y + I|| \leqq 1$.

Suppose now that $x + I \in t/I$, and $\delta > 0$. Then $x \in t$, so there is an idempotent p in t such that $||x - xp|| < \delta$. Hence $p + I$ is an idempotent in t/I, and $||(x + I)(p + I) - (x + I)|| \leqq ||px - x|| < \delta$. Thus A/I is a C^*-algebra.

Suppose first that I is a unital algebra. Then there is an idempotent $p \in A$ such that $I = pA$. Then the map $\gamma: I \to A/(1 - p)A$ $x \to x + (1 - p)A$ is an isometric isomorphism, and so I is a C^*-algebra, as $A/(1 - p)A$ is. The only part not obvious is that γ is isometric. So let $x \in I$. Then $||\gamma(x)|| = ||x + (1 - p)A|| = \sup\{||x + (1 - p)A + t/(1 - p)A||: t \in V((1 - p)A)\}$ (as $A/(1 - p)A$ is a C^*-algebra) $= \sup\{||x + t||: 1 - p \in t\} = \sup\{||x + t||: t \in T(A)\} = ||x||$ (as A is a C^*-algebra).

Suppose finally that I is nonunital. Then if p is an idempotent in I, clearly $||p|| \leqq 1$. Also as A is a C^*-algebra, $I = \mathrm{cl}(\cup\{pI: p = p^2 \in I\})$. Suppose that t is a maximal modular ideal in I, $x \in t$, and $\delta > 0$. Then there is a maximal ideal t' in A such that $t = I \cap t'$, and t' does not contain I. So as $x \in t'$, there is an idempotent $p \in t'$ such that $||x - px|| < \delta$. If $p \in I$, then $p \in t$. So suppose $p \notin I$. Now there is an idempotent q in I which is not in t'. If $r = q(1 - p)$, then r is an idempotent in I, and $r \notin t$, and $||rx|| < \delta$. Thus I is a C^*-algebra.

Suppose now that I is a nonunital Banach algebra. Define $I_e = I \oplus F$, as Banach spaces, with norm $||x + \alpha 1|| = \max(||x||, |\alpha|)$, for all $x \in I$ and $\alpha \in F$. Also define a multiplication on I_e by the rule $(x + \alpha 1)(x' + \alpha' 1) = xx' + \alpha x' + \alpha' x + \alpha' 1$. Then I_e is a unital Banach algebra containing I as a maximal ideal.

THEOREM 2.7. *Let I be a nonunital Banach algebra. Then I is a C^*-algebra if and only if I_e is a C^*-algebra.*

Proof. From Theorem 2.6 we know that if I_e is a C^*-algebra, then I is one also, as I is a closed ideal in I_e.

Suppose that I is a C^*-algebra, and t is a maximal ideal of I_e, with $x \in t$. If $t = I$, then we see from (c) of Definition 2.2 that $t = \mathrm{cl}(\cup\{pt; p = p^2 \in t\})$. It follows easily from this and the strong triangle inequality that if δ is any positive number, there is an idempotent p in t such that $\|x - xp\| < \delta$. So suppose now $t \neq I$. Then $t \cap I$ is a maximal modular ideal in I, so there is an idempotent p of I such that $\|x - xp\| < \delta$ and $p \in t$, or there is an idempotent q of I such that $\|qx\| < \delta$ and $q \notin t$. Suppose the second condition holds. Now there is an idempotent $q' \in I$, $q' \notin t$, and so $1 - q' \in t$. Let $r = 1 - qq'$. Then r is an idempotent and $\|x - rx\| < \delta$. Moreover as q, q' are not in t, $qq' \notin t$, and hence $r \in t$.

Finally suppose p is any idempotent in I_e. Then p or $1 - p \in I$, since I is a maximal ideal in I_e. So in any case $\|p\| \leqq 1$. Thus I_e is a C^*-algebra.

Examples of C^*-algebras. Before giving our list of examples, let us just make a useful definition.

DEFINITION. If A is a (not necessarily unital) Banach algebra, we call A a *V-algebra* if for all maximal modular ideals t of A, A/t is a valued field, i.e., for all $x, y \in A$, $\|x + t\|\|y + t\| = \|xy + t\|$. If A is a C^*-algebra and a V-algebra, we call A a *C^*V-algebra*. It turns out that, except for some unimportant exceptions, 'all' C^*-algebras are C^*V-algebras.

EXAMPLE 1. Let K be a complete valued field extension of F, and T any topological space. Then $C_b(T, K)$ is a C^*V-algebra over F. In particular, K and $C_b(T, F)$ are C^*V-algebras over F. Also if T is a compact space, then $C(T, F)$ is a C^*V-algebra. Recall that a *Gelfand* algebra is an algebra such that for all maximal modular ideals t of the algebra A, say, $A/t = F$. $C(T, F)$ is a Gelfand algebra for T a compact space. But if T is just any topological space, then $C_b(T, F)$ is not necessarily a Gelfand algebra, unless F is locally compact. (See e.g., [4], page 156.)

EXAMPLE 2. If T is a compact space, and A is a closed subalgebra of $C(T, F)$ with $1 \in A$, then A is a C^*V-algebra (and in fact, also a Gelfand algebra).

EXAMPLE 3. If T is a locally compact space, and $C_\infty(T, F)$ denotes the algebra of functions on T with values in F which are continuous and which vanish at ∞, normed with the sup norm, then $C_\infty(T, F)$ is a (possibly nonunital) C^*V-algebra.

EXAMPLE 4. If $(A_i)_{i \in I}$ is any family of $C^* V$-algebras, then $\bigoplus_{i \in I} A_i$ is also a $C^* V$-algebra. In particular if $(K_i)_i$ is any family of complete valued field extensions of F, then $\bigoplus_i K_i$ is a $C^* V$-algebra.

EXAMPLE 5. If A is any (not necessarily unital) $C^* V$-algebra, then the *multipliers* of A, $M(A) = : \{S: A \to A: S$ is linear and for all $x, y \in A$, $xS(y) = S(x)y\}$ is a $C^* V$-algebra also, if $T(A)$ is strongly zero-dimensional.

EXAMPLE 6. Let G be locally compact abelian group which is Hausdorff and totally disconnected. In [5] it is shown that if G is p-free and torsional, then G has an F-valued Haar integral. With this integral a non-archimedean group algebra $L(G, F)$ of G can be defined. It can be shown that $L(G, F)$ is a $C^* V$-algebra. Hence also $M(G, F) = M(L(G, F))$, the multipliers of $L(G, F)$, is a $C^* V$-algebra, and it is possible to regard this algebra as the measure algebra of G (see [2]).

EXAMPLE 7. Finally, if (T, U) is a non-archimedean uniform space, and $BUC(T, U) = \{f: T \to F: f$ is uniformly continuous and bounded$\}$, then it can be shown that $BUC(T, U)$ is a $C^* V$-algebra. The definition of a non-archimedean uniform space can be found in [4], page 27.

The proofs of many of these examples are rather long, and can be found in [2].

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