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PEIRCE IDEALS IN JORDAN ALGEBRAS

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In attempting to investigate infinite-dimensional simple Jordan algebras J having rich supplies of idempotents, it would be helpful to know that the Peirce subalgebra $J_1(e)$ relative to an idempotent e in J remains simple. This clearly holds for associative and alternative algebras because any ideal in a Peirce space is the projection of a global ideal. The corresponding result is false for Jordan algebras: there are multiplications of the ambient algebra J which send J_1 to itself (therefore leave invariant the projection of a global ideal), but are not expressible as multiplication by elements of J_1 (therefore need not leave invariant an arbitrary ideal of J_1). We show that an ideal K_1 is the projection of a global ideal iff it is invariant under the multiplications $V_{J_{1/2},J_{1/2}}$ and $U_{J_{1/2}}U_{J_{1/2}}$. This yields an explicit expression for the global ideal generated by a Peirce ideal. We then show that if J is a simple Jordan algebra with idempotent, the Peirce subalgebras J_1 and J_0 inherit simplicity.

Throughout we consider a quadratic Jordan algebra J over an arbitrary ring of scalars Φ with product

$$U_x y$$

quadratic in x and linear in y. Linearization yields a trilinear product

$$\{xyz\} = U_{x,z}y = V_{x,y}z \ .$$

(See [1] for basic results on quadratic Jordan algebras.) If e is an idempotent element of J, $e^2 = e$, then we have a *Peirce decomposition* $J = J_1 \oplus J_{1/2} \oplus J_0$ where J_1, J_0 are subalgebras. We wish to relate the ideals in these Peirce subalgebras J_i to ideals in the ambient algebra J.

Analogous results hold for Jordan triple systems. However, in this case U_e is merely an involution on J_1 rather than the identity map, and this causes such technical complications in the Peirce identities that the basic argument is lost sight of. We prefer to do the simpler Jordan algebra case first, and treat the triple system case separately [3].

We recall a few basic identities satisfied by Jordan algebras:

(0.1) $U_{U(x)y} = U_x U_y U_x$

(0.2) $U_{V(x,y)z} = U_x U_y U_z + U_z U_y U_x + V_{x,y} U_z V_{y,x} - U_{U(x)U(y)z,z}$

(0.3) $U_{V(x,y)z,z} = V_{x,y} U_z + U_z V_{y,x}$

(0.4) $U_x U_{y,z} = V_{x,y} V_{x,z} - V_{U(x)y,z}$

(0.5) $U_{y,z}U_x = V_{y,x}V_{z,x} - V_{y,U(x)z}$

(0.6) $\{xxy\} = x^2 \circ y,\ V_{x,x} = V_{x^2},\ V_{x,y} + V_{y,x} = V_{x\circ y}$.

In a Peirce decomposition we have the following identities for $i = 1, 0$ and $j = 1 - i$:

(P1) $U_{x_i \circ y_{1/2}} = U_{x_i}U_{y_{1/2}}$ on $J_j (x_i \in J_i,\ y_{1/2} \in J_{1/2})$

(P2) $U_{x_i \circ y_{1/2}} = U_{y_{1/2}}U_{x_i}$ on J_i

(P3) $U_{x_{1/2}}y_{1/2} = x_{1/2} \circ E_i(x_{1/2} \circ y_{1/2}) - y_{1/2} \circ E_j(x_{1/2}^2)$

(P4) $\{x_{1/2}a_i y_{1/2}\} = E_j(x_{1/2} \circ (a_i \circ y_{1/2}))$

(P5) $\{x_{1/2}y_{1/2}a_i\} = E_i(x_{1/2} \circ (x_{1/2} \circ a_i))$

(P6) $a_i \circ (x_{1/2} \circ b_j) = \{a_i x_{1/2} b_j\} = (a_i \circ x_{1/2}) \circ b_j$

(P7) $a_i^2 \circ x_{1/2} = a_i \circ (a_i \circ x_{1/2})\quad (V_{a_i^2} = V_{a_i}^2$ on $J_{1/2})$

(P8) $U_{a_i}b_i \circ x_{1/2} = a_i \circ (b_i \circ (a_i \circ x_{1/2}))\quad (V_{U(a_i)b_i} = V_{a_i}V_{b_i}V_{a_i}$ on $J_{1/2}))$

(P9) $\{a_i b_i x_{1/2}\} = a_i \circ (b_i \circ x_{1/2})$

where E_i denotes the Peirce projection on the Peirce space J_i.

1. Ideal-building. A subspace K of a Jordan algebra is an *ideal* if it is both an *outer ideal*

$$(1.1)\qquad U_{\hat{J}}K \subset K \qquad (U_J K \subset K,\ V_J K \subset K)$$

and an *inner ideal*

$$(1.2)\qquad U_K \hat{J} \subset K \qquad (U_K J \subset K,\ K^2 \subset K)\,.$$

Here $\hat{J} = \Phi 1 + J$ denotes the unital hull of the Jordan algebra J; if J is itself unital then $\hat{J} = J$, and the conditions $V_J K \subset K$ and $K^2 \subset K$ are superfluous $(V_x = U_{x,1},\ x^2 = U_x 1)$. A useful observation is that once K is known to be an outer ideal it is an inner ideal as soon as

$$(1.3)\qquad U_{k_i} J \subset K \text{ for some spanning set } \{k_i\} \text{ of } K\,.$$

From now on we fix an idempotent e in J and consider the corresponding Peirce decomposition

$$J = J_1 \oplus J_{1/2} \oplus J_0\,.$$

Then the unital hull $\hat{J} = \Phi 1 + J = \Phi(1 - e) + J$ can be identified with $J_1 \oplus J_{1/2} \oplus \hat{J}_0$. Note that any ideal $K \lhd J$ is invariant under the Peirce projections E_i since these are multiplication operators, therefore K is the direct sum of its Peirce components

$$K = K_1 \oplus K_{1/2} \oplus K_0 \qquad (K_i = K \cap J_i)\,.$$

Triple products of Peirce elements largely reduce to simpler bilinear products:

$$U_{x_1+x_{1/2}+x_0}(y_1 + y_{1/2} + y_0) = U_{x_1}y_1 + U_{x_{1/2}}(y_1 + y_{1/2} + y_0) + U_{x_0}y_0 + \{x_1 y_{1/2} x_0\} + \{x_1 y_1 x_{1/2}\} + \{x_0 y_0 x_{1/2}\} + \{x_1 y_{1/2} x_{1/2}\} + \{x_0 y_{1/2} x_{1/2}\}$$

$$(1.4)\qquad \begin{aligned} &= U_{x_1}y_1 + U_{x_{1/2}}(y_1 + y_0) + \{x_{1/2}\circ E_1(x_{1/2}\circ y_{1/2}) - y_{1/2}\circ E_0(x_{1/2}^2)\} \\ &+ U_{x_0}y_0 + x_1\circ(x_0\circ y_{1/2}) + x_1\circ(y_1\circ x_{1/2}) + x_0\circ(y_0\circ x_{1/2}) \\ &+ E_1((x_1\circ y_{1/2})\circ x_{1/2}) + E_0((x_0\circ y_{1/2})\circ x_{1/2}) . \end{aligned}$$

Correspondingly, the ideal conditions (1.1), (1.2) for K reduce to simpler conditions on the Peirce components K_i.

IDEAL CRITERION 1.5. A subspace $K = K_1 \oplus K_{1/2} \oplus K_0$ is an ideal of a Jordan algebra $J = J_1 \oplus J_{1/2} \oplus J_0$ iff for $i = 1, 0, j = 1 - i$

(C1) K_i is an ideal in J_i

(C2) $E_i(J_{1/2}\circ K_{1/2}) \subset K_i$

(C3) $J_i\circ K_{1/2} \subset K_{1/2}$

(C4) $K_i\circ J_{1/2} \subset K_{1/2}$

(C5) $U_{J_{1/2}}K_i \subset K_j$

(C6) $U_{k_{1/2}}\hat{J}_i \subset K_j$ for some spanning set $\{k_{1/2}\}$ of $K_{1/2}$.

If $1/2 \in \Phi$ the conditions (C5), (C6) are superfluous.

Proof. Clearly these inclusions are all necessary by the Peirce relations and the fact that any product involving a factor from an ideal falls back in that ideal.

A routine calculation shows (C1)-(C5) suffice to establish outerness: $U_{\hat{J}}K \subset K$ follows from (1.4) since $U_{\hat{J}_i}K_i \subset K_i$ by (C1); $U_{J_{1/2}}K_i \subset K_j$ by (C5); $J_{1/2}\circ E_i(J_{1/2}\circ K_{1/2}) \subset K_{1/2}$ by (C2), (C4); $K_{1/2}\circ E_0(J_{1/2}^2) \subset K_{1/2}$ by (C3); $J_1\circ(\hat{J}_0\circ K_{1/2}) \subset K_{1/2}$ by (C 3) (noting $\hat{J}_0\circ K_{1/2} = \Phi e_0\circ K_{1/2} + J_0\circ K_{1/2} = \Phi K_{1/2} + J_0\circ K_{1/2}$ since $e_0\circ x_{1/2} = x_{1/2}$); $\hat{J}_i\circ(K_i\circ J_{1/2}) \subset K_{1/2}$ by (C4), (C3); $E_i(J_{1/2}\circ(\hat{J}_i\circ K_{1/2})) \subset K_i$ by (C3), (C2).

Once we have outerness, innerness (1.3) follows for the spanning set of elements $k_i \in K_i (i = 1, 0)$ and the given $k_{1/2} \in K_{1/2}$ since $U_{K_i}\hat{J} = U_{k_i}\hat{J}_i \subset K_i$ by (C1), $U_{k_{1/2}}\hat{J}_i \subset K_j$ by (C6), and $U_{k_{1/2}}J_{1/2} = k_{1/2}\circ E_1(k_{1/2}\circ J_{1/2}) - J_{1/2}\circ E_0(k_{1/2}^2) \subset K_{1/2}$ by (C3), (C4), and $E_0(k_{1/2}^2) = U_{k_{1/2}}e_1 \in K_0$ by (C6).

Since $2U_x = U_{x,x}$ and always $U_{J_{1/2},J_{1/2}}K_i = E_j(J_{1/2}\circ(K_i\circ J_{1/2})) \subset K_j$ by (C4), (C2), $U_{J_{1/2},K_{1/2}}\hat{J}_i = E_j(J_{1/2}\circ(J_i\circ K_{1/2})) \subset K_j$ by (C3), (C2), we see that (C5), (C6) are consequences of (C2)-(C4) when $1/2 \in \Phi$.

REMARK 1.6. In characteristic 2 situations we cannot dispense with (C5) and (C6)—they really are necessary in addition to the other conditions. For example, if J is the special Jordan algebra $\Phi e_{11} + \Phi(e_{12} + e_{21}) + \Phi e_{22}$ of symmetric 2×2 matrices over Φ of characteristic 2, then relative to $e = e_{11}$ we have $J_1 = \Phi e_{11}$, $J_{1/2} = \Phi(e_{12} + e_{21})$, $J_0 = \Phi e_{22}$ so $J_{1/2}\circ J_{1/2} = 2\Phi(e_{12} + e_{21})^2 = 0$, and thus (C2) is automatic for any K. If we take $K_1 = K_0 = 0$, $K_{1/2} = J_{1/2}$ then (C1)-(C5) hold trivially, but not (C6) since $U_{J_{1/2}}J_i = \Phi U_{e_{12}+e_{21}}e_{ii} = \Phi e_{jj} = J_j \neq 0$. Thus (C6) is not a consequence of the other conditions. If we take $K = \lambda\Phi e_{11}$, $K_{1/2} = \lambda\Phi(e_{12} + e_{21})$, $K_0 = \lambda^2\Phi_{22}$ for noninvertible λ in a domain Φ of characteristic 2, then (C1), (C2)-(C4) hold trivially, as does (C6) by

$$U_{\lambda(e_{12}+e_{21})}(\Phi e_{ii}) = \lambda^2 \Phi e_{jj} ,$$

but (C5) is not a consequence since $U_{e_{12}+e_{21}}(\lambda\Phi e_{11}) = \lambda\Phi e_{22} \not\subset \lambda^2\Phi e_{22} = K_0$.

Next we introduce the key notions of invariance. An ideal K_i in a Peirce space J_i ($i = 1, 0$) is *invariant* if it is both *U-invariant*

$$(1.7) \qquad U_{J_{1/2}} U_{J_{1/2}} K_i \subset K_i$$

and *V-invariant*

$$(1.8) \qquad V_{J_{1/2}, J_{1/2}} K_i = E_i(J_{1/2} \circ (J_{1/2} \circ K_{1/2})) \subset K_i .$$

By the Peirce relations and (P5) the maps $U_{x_{1/2}} U_{y_{1/2}}$ and $V_{x_{1/2}, y_{1/2}}$ map J_i into itself, though in general they cannot be compressed into a multiplication from J_i.

V-invariance is the more fundamental notion, and goes a long way towards ensuring U-invariance. For example, the special case $z = y$ in (0.4) shows

$$(1.9) \qquad 2U_x U_y = V_{x,y} V_{x,y} - V_{U(x)y,y} ,$$

so whenever we can divide by 2 V-invariance implies U-invariance.

We can flip an invariant ideal from one diagonal Peirce space to the other.

FLIPPING LEMMA 1.10. *If K_i is an ideal in a Peirce space J_i (i=1, 0) then $K_j = U_{J_{1/2}} K_i$ is an ideal in J_j. If K_i is V-invariant or U-invariant, so is the flipped ideal K_j.*

Proof. K_j is outer since $U_{\hat{J}_j} K_j = U_{\hat{J}_j} U_{J_{1/2}} K_i = U_{\hat{J}_j \circ J_{1/2}} K_i$ (by (P1)) $\subset U_{J_{1/2}} K_i = K_j$ as in (1.1), and for the spanning set of elements $k_j = U_{y_{1/2}} k_i$ we have by (0.1) $U_{k_j} J_j = U_{y_{1/2}} U_{k_i} U_{y_{1/2}} J_j$ (by (0.1)) $\subset U_{J_{1/2}} U_{K_i} J_i \subset U_{J_{1/2}} K_i = K_j$, so by (1.3) K_j is an ideal. If K_i is V-invariant so is K_j, since by (0.3) $V_{J_{1/2}, J_{1/2}} K_j = V_{J_{1/2}, J_{1/2}} U_{J_{1/2}} K_i \subset \{U_{V(J_{1/2}, J_{1/2}) J_{1/2}, J_{1/2}} - U_{J_{1/2}} V_{J_{1/2}, J_{1/2}}\} K_i \subset U_{J_{1/2}} K_i + U_{J_{1/2}}(V_{J_{1/2}, J_{1/2}} K_i) \subset U_{J_{1/2}} K_i$ (by V-invariance of K_i) $= K_j$, and K_j trivially inherits U-invariance

$$U_{J_{1/2}} U_{J_{1/2}} K_j = U_{J_{1/2}} U_{J_{1/2}} U_{J_{1/2}} K_i \subset U_{J_{1/2}} K_i$$

(by U-invariance) $= K_j$.

Now we are ready to establish the main result of this section, describing the global ideal generated by an invariant Peirce ideal.

PROJECTION THEOREM 1.11. *An ideal K_i in a Peirce space*

J_i $(i = 1, 0)$ *is the Peirce projection of a global ideal* K *in* J *iff* K_i *is invariant. In this case the ideal generated by* K_i *takes the form*

$$K = K_i \oplus K_i \circ J_{1/2} \oplus U_{J_{1/2}} K_i .$$

If $1/2 \in \Phi$ *we have* $U_{J_{1/2}} K_i = E_j(J_{1/2} \circ (K_i \circ J_{1/2}))$.

Proof. We have already noted that if K_i is the projection of an ideal K then by the Peirce relations and invariance of K under all multiplications from J, K_i must be invariant. We must establish the converse. Since the ideal generated by K_i must certainly certain the above products, if we can show the above K actually is an ideal then we will have exhibited K_i as the projection of an ideal K which is thus precisely the ideal generated by K_i.

We verify the conditions of the Ideal Criterion (1.5). K_i is an invariant ideal in J_i by hypothesis, and $K_j = U_{J_{1/2}} K_i$ is an invariant ideal in J_j by the Flipping Lemma 1.10. Thus (C1) holds. For (C2), note $E_i(J_{1/2} \circ K_{1/2}) = E_i(J_{1/2} \circ (J_{1/2} \circ K_i)) = \{J_{1/2} J_{1/2} K_i\} = V_{J_{1/2}, J_{1/2}} K_i \subset K_i$ by (P5) and *V-invariance,* also $E_j(J_{1/2} \circ K_{1/2}) = \{J_{1/2} K_i J_{1/2}\} \subset U_{J_{1/2}} K_i = K_j$ by (P4). For (C3), $J_j \circ K_{1/2} = J_j \circ (K_i \circ J_{1/2}) = K_i \circ (J_j \circ J_{1/2}) \subset K_i \circ J_{1/2} = K_{1/2}$ by (P6), while $J_i \circ K_{1/2} = J_i \circ (K_i \circ J_{1/2}) = (J_i \circ K_i) \circ J_{1/2} - K_i \circ (J_i \circ J_{1/2}) \subset K_i \circ J_{1/2} = K_{1/2}$ by (P7) and the fact that $K_i \lhd J_i$. For (C4) we have $K_i \circ J_{1/2} = K_{1/2}$ by definition, and $K_j \circ J_{1/2} = U_{J_{1/2}} K_i \circ J_{1/2} \subset - U_{J_{1/2}} J_{1/2} \circ K_i + J_{1/2} \circ \{K_i J_{1/2} J_{1/2}\}$ (linearized (0.6)) $\subset J_{1/2} \circ K_i + J_{1/2} \circ V_{J_{1/2}, J_{1/2}} K_i = J_{1/2} \circ K_i = K_{1/2}$ by *V-invariance* of K_i. For (C5), $U_{J_{1/2}} K_i = K_j$ by definition, while $U_{J_{1/2}} K_i = U_{J_{1/2}} U_{J_{1/2}} K_i \subset K_i$ by *U-invariance* of K_i. For (C6), the spanning elements $k_{1/2} = k_i \circ y_{1/2}$ satisfy $U_{k_i \circ y_{1/2}} \hat{J}_i = U_{y_{1/2}} U_{k_i} \hat{J}_i \subset U_{J_{1/2}} K_j = K_j$ by (P2) and $K_i \lhd J_i$, similarly $U_{k_i \circ y_{1/2}} \hat{J}_j = U_{k_i} U_{y_{1/2}} \hat{J}_j \subset U_{k_i} J_i \subset K_i$ by (P1) and $K_i \lhd J_i$. Thus (C1–C6) hold, and K is an ideal.

EXAMPLE 1.12. The connector ideal generated by an off-diagonal Peirce space $J_{1/2}$ is

$$I(J_{1/2}) = U_{J_{1/2}} J_0 \oplus J_{1/2} \oplus U_{J_{1/2}} J_1 .$$

Proof. It suffices to verify conditions (C1–C6) of (1.5): (C3–C6) are automatic since $K_{1/2} = J_{1/2}$, $K_j = U_{J_{1/2}} \hat{J}_i$; (C1) follows from the Flipping Lemma 1.10 applied to $\hat{J}_i$ in J; (C2) follows from $E_i(J_{1/2} \circ J_{1/2}) = \{J_{1/2} \hat{e}_j J_{1/2}\} \subset U_{J_{1/2}} \hat{e}_j \subset K_i$ by (P4).

EXAMPLE 1.13. If Z_i denotes the kernel of the Peirce specialization of J_i on $J_{1/2}$,

$$Z_i = \{z_i \in J_i \mid z_i \circ J_{1/2} = 0\}$$

then $Z = Z_1 \oplus Z_0$ is an ideal in J which annihilates the connector ideal, $U_Z I(J_{1/2}) = 0$.

Proof. Any time K has $K_{1/2} = 0$ the conditions (C2), (C3), (C6) become vacuous and (C4) becomes the condition $K_i \subset Z_i$. If we take $K_i = Z_i$ (C4) is thus satisfied, as is (C1) since the Peirce specialization is a homomorphism of J_i into End $(J_{1/2})$ by (P7), (P8) and therefore its kernel is an ideal. Moreover, these are interchanged by $U_{J_{1\ 2}}$ as in (C5) since $U_{x_{1/2}} z_i \circ y_{1/2} = V_{y_{1/2}} U_{x_{1/2}} z_i = \{U_{x_{1/2}, y_{1/2} \circ x_{1/2}} - U_{x_{1/2}} V_{y_{1/2}}\} z_i$ (by (0.3) with $x = 1$) $= \{x_{1/2} z_i E_i(y_{1/2} \circ x_{1/2})\} - U_{x_{1\ 2}} V_{y_{1/2}} z_i = (x_{1/2} \circ z_i) \circ E_i(y_{1/2} \circ x_{1/2}) - U_{x_{1/2}}(y_{1/2} \circ z_i) = 0$ by (P9) if $z_i \circ x_{1/2} = z_i \circ y_{1/2} = 0$.

Thus Z is an ideal in $J \cdot U_Z I(J_{1/2}) = 0$ since by (1.4) we have $U_{z_1 + z_0}(k_1 + k_{1/2} + k_0) = U_{z_1} k_1 + U_{z_0} k_0 + z_1 \circ (y_{1/2} \circ z_0) = 0$ where $U_{z_i} K_i = U_{z_i} U_{J_{1/2}} J_j = U_{z_i \circ J_{1/2}} J_j$ by (P1) and $Z_i \circ J_{1/2} = 0$.

PROPOSITION 1.14. *If J is a prime Jordan algebra and $e \neq 1, 0$ a proper idempotent, then $J_{1/2} \neq 0$ and the Peirce specializations of J_1 and J_0 on $J_{1/2}$ are faithful* (*hence J_1, J_0 are special Jordan algebras*).

Proof. If $J_{1/2} = 0$ then $J = J_1 \boxplus J_0$ would be a direct sum of ideals, whereupon primeness would force $J = J_1$(hence $e = 1$) or $J = J_0$ (hence $e = 0$). Thus $J_{1/2}$ cannot vanish if e is proper. Then $U_Z I(J_{1/2}) = 0$ for $I(J_{1/2}) \neq 0$ forces $Z = 0$ by primeness.

Thus in any prime exceptional Jordan algebra J, as soon as we examine a proper piece $J_1(e)$ or $J_0(e)$ it is special (in some sense J has no smaller exceptional pieces), and exceptionality results only from the way J_1 and J_0 are tied together via $J_{1/2}$

In §4 we will see that when J is simple the same is true of J_1 and J_0, so J is built up of pieces which are simple and special.

Note that if J is simple and e proper we have $J_{1/2} \neq 0$ by 1.14, so by simplicity $I(J_{1/2}) = J$ and by (1.12) we have

$$(1.15) \qquad U_{J_{1/2}} \hat{J}_0 = J_1\,, \quad U_{J_{1/2}} J_1 = J_0\,.$$

We can improve on this by removing the hat from J_0. To do this we need to look at the ideal generated by J_0. Trivially J_i is an invariant ideal in J_i, and $J_1 \circ J_{1/2} = e \circ J_{1/2} = J_{1/2}$, so by 1.11 we have

EXAMPLE 1.16. The ideal in J generated by a diagonal Peirce space $J_i(e)$ is

$$(i = 1) \quad I(J_1) = J_1 \oplus J_{1/2} \oplus U_{J_{1/2}} J_1$$
$$(i = 0) \quad I(J_0) = J_0 \oplus J_0 \circ J_{1/2} \oplus U_{J_{1/2}} J_0\,.$$

If J is simple then $e \neq 0$ implies $J_1 \neq 0$ and hence $I(J_1) = J_1$, once more leading to $U_{J_{1/2}}J_1 = J_0$. If we knew $e \neq 1$ implied $J_0 \neq 0$ we could similarly deduce $I(J_0) = J$ by simplicity and hence $U_{J_{1/2}}J_0 = J_1$ (without the hat).

Surprisingly, it takes a bit of arguing to establish $J_0 \neq 0$. Suppose in fact $J_0 = 0$. Then for $z_{1/2} \in J_{1/2}$ we would have $z_{1/2}^2 \in J_1 + J_0 = J_1$, and $z_1 = z_{1/2}^2$ would be trivial since $U_{z_1}J = U_{z_1}J_1 = U_{z_{1/2}}U_{z_{1/2}}J_1 \subset U_{z_{1/2}}J_0 = 0$. But a simple J with idempotent is not nil and therefore has no trivial elements, so $z_{1/2}^2 = 0$ and $z_{1/2}\circ w_{1/2} = 0$ for all $z_{1/2}$, $w_{1/2} \in J_{1/2}$. But then by (1.4) $U_{z_{1/2}}w_{1/2} = z_{1/2}\circ E_1(z_{1/2}\circ w_{1/2}) - w_{1/2}\circ E_0(z_{1/2}^2) = 0$, so $U_{z_{1/2}}J_{1/2} = 0$, and since already $U_{z_{1/2}}J_1 \subset J_0 = 0$ we have $U_{z_{1/2}}J = 0$ and $z_{1/2}$ would be trivial. Again J has no trivial elements, so $z_{1/2} = 0$, $J_{1/2} = 0$, contradicting 1.14.

PROPOSITION 1.17. *If J is a simple Jordan algebra and $e \neq 1, 0$ a proper idempotent, then*

$$U_{J_{1/2}}J_0 = J_1\,, \quad U_{J_{1/2}}J_1 = J_0\,.$$

2. **Invariance.** To construct global ideals we must begin with invariant Peirce ideals. We now turn to the question of conditions under which an ideal is automatically invariant. Throughout this section we will be concerned with ideals K_i in a diagonal Peirce space $J_i (i = 1, 0)$.

While $V_{J_{1/2},J_{1/2}}K_i$ and $U_{J_{1/2}}U_{J_{1/2}}K_i$ are not in general contained in K_i, they are in some sense contained in the "square root" and "fourth root" of K_i: $V_{J_{1/2},J_{1/2}}$ maps K_i^2 into K_i, and $U_{J_{1/2}}U_{J_{1/2}}$ maps K_i^4 into K_i. More precisely, we have the following useful technical result.

LEMMA 2.1. *For any ideal $K_i \lhd J_i$ we have*

$$V_{J_{1/2},J_{1/2}}(U_{K_i}\hat{J}_i) \subset K_i \tag{2.2}$$

and

$$U_{J_{1/2}}U_{J_{1/2}}(U_{U(K_i)\hat{J}_i}\hat{J}_i) \subset K_i\,. \tag{2.3}$$

In general, for $x, y \in J_{1/2}$, $k \in K_i$, $a \in \hat{J}_i$ we have

$$V_{x,y}U_k a = U_{V(x,y)k,k}a - U_k V_{y,x}a \in K_i \tag{2.4}$$

$$\begin{aligned} U_x U_y U_k a &= U_{\{xyk\}}a - U_k U_y U_x a - V_{x,y}U_k V_{y,x}a \\ &\quad + U_{k,U(x)U(y)k}a \subset U_{\{xyk\}}a + K_i \end{aligned} \tag{2.5}$$

so that whenever $k \in K_i$ is V-invariant, $\{xyk\} = V_{x,y}k \in K_i$, then $U_k\hat{J}_i$ is U-invariant, $U_xU_y(U_ka) \in K_i$.

Proof. For (2.4) we have by (0.3) $V_{x,y}U_k a U_{\{xyk\},k}a - U_k V_{y,x}a \in U_{J_i,K_i}a - U_{K_i}V_{y,x}a \subset K_i$ whenever $K_i \lhd J_i$. For (2.5) we use (0.2): $U_{\{xyk\}}a = [U_x U_y U_k + U_k U_y U_x + V_{x,y} U_k V_{y,x} - U_{k,U(x)U(y)k}]a \equiv U_x U_y U_k a$ modulo K_i since $U_k U_y U_x a \in U_{K_i} J_i \subset K_i$, $V_{x,y} U_k V_{y,x} a \in V_{x,y} U_{K_i} J_i \subset K_i$ by (2.4), and $U_{k,U(x)U(y)k}a \in U_{K_i,J_i}a \subset K_i$. Applying (2.4) to $k \in K_i$, $a \in \hat{J}_i$ yields (2.2), and applying (2.5) to $k \in U_{K_i}\hat{J}_i$ (so $\{xyk\} \equiv 0$ by (2.2)) yields (2.3).

EXAMPLE 2.6. If B_i, C_i are invariant ideals in J_i so is their product $U_{B_i}C_i$.

Proof. For V-invariance apply (2.4), for U-invariance apply (2.5).

EXAMPLE 2.7. If K_i is an idempotent ideal in J_i, $U_{K_i}\hat{J}_i = K_i$, then K_i is invariant.

EXAMPLE 2.8. If B_α are invariant ideals in J_i so is their sum $\sum B_\alpha$ and their intersection $\cap B_\alpha$.

EXAMPLE 2.9. For any ideal $K_i \lhd J_i$ the infinite Penico derived ideal $P^\infty(K_i) = \bigcap P^n(K_i)$ is an invariant ideal ($P^{n+1}(K_i) = P(P^n(K_i))$ where $P(L_i) = U_{L_i}\hat{J}_i$). Similarly for the infinite derived ideal $D^\infty(K_i)$ (where $D(L_i) = U_{L_i}L_i$). Thus either K_i contains a nonzero invariant ideal, or else it is ∞-nilpotent: $P^\infty(K_i) = 0$.

Proof. V-invariance of $P^\infty(K_i)$ follows from (2.2),

$$V_{J_{1/2},J_{1/2}}(P^{n+1}(K_i)) \subset P^n(K_i)\ ,$$

and U-invariance from (2.3), $U_{J_{1/2}}U_{J_{1/2}}(P^{n+2}(K_i)) \subset P^n(K_i)$. For $D^\infty(K_i)$ we use (2.4) to get V-invariance, $V_{J_{1/2},J_{1/2}}D^{n+1}(K_i) \subset D^n(K_i)$ and (2.5) to get U-invariance, $U_{J_{1/2}}U_{J_{1/2}}D^{n+2}(K_i) \subset D^n(K_i)$ (note $V_{x,y}U_{d_{n+1}}V_{y,x}d'_{n+1} \in V_{x,y}U_{d_{n+1}}D^n \subset V_{x,y}D^{n+1} \subset D^n$ by the relation for the V's).

We have seen in the Flipping Lemma 1.10 that one way of obtaining an invariant Peirce ideal to is flip an invariant ideal by $U_{J_{1/2}}$. Another way of obtaining an invariant Peirce ideal is to take the kernel of $U_{J_{1/2}}$ instead of the image.

KERNEL LEMMA 2.10. *Ker* $U_{J_{1/2}} = \{z \in J_i \mid U_{J_{1/2}}z = \mid U_{J_{1/2}}U_z\hat{J}_i = 0\}$ *is an invariant ideal in* J_i.

Proof. $K_i = \operatorname{Ker} U_{J_{1/2}}$ is trivially U-invariant ($U_{J_{1/2}}U_{J_{1/2}}K_i = 0$), and is V-invariant because by 0.3 $U_{J_{1/2}}(V_{x_{1/2},y_{1/2}}z) \subset \{U_{\{y_{1/2}x_{1/2}J_{1/2}\},J_{1/2}} - V_{y_{1/2},x_{1/2}}U_{J_{1/2}}\}z = 0$, and by (0.2) and (0.3)

$$\begin{aligned} &U_{J_{1/2}}U_{V(x_{1/2},y_{1/2})z}\hat{J}_i \\ &\quad = U_{J_{1/2}}\{U_{x_{1/2}}U_{y_{1/2}}U_z + U_zU_{y_{1/2}}U_{x_{1/2}} + V_{x_{1/2},y_{1/2}}U_zV_{y_{1/2},x_{1/2}} \\ &\quad\quad - U_{U(x_{1/2})U(y_{1/2})z,z}\}\hat{J}_i \subset U^2_{J_{1/2}}(U_{J_{1/2}}U_z\hat{J}_i) + U_{J_{1/2}}(U_zJ_i) \\ &\quad\quad + \{U_{\{y_{1/2}x_{1/2}J_{1/2}\},J_{1/2}} - V_{y_{1/2},x_{1/2}}U_{J_{1/2}}\}U_zJ_i - 0 = 0\,. \end{aligned}$$

K_i is a linear subspace since for $z, w \in K_i$ we have $U_{J_{1\,2}}U_{z+w}\hat{J}_i = U_{J_{1/2}}(U_z + U_w + U_{z,w})\hat{J}_i$ where by (0.3) $U_{J_{1/2}}U_{z,w}\hat{J}_i = U_{J_{1/2}}V_{w,\hat{J}_i}z = \{U_{\{\hat{J}_i,wJ_{1/2}\}J_{1/2}} - V_{\hat{J}_i,w}U_{J_{1/2}}\}z = 0$. It is an outer ideal since $U_{J_{1/2}}(U_{\hat{J}_i}z) = U_{J_{1/2}\circ\hat{J}_i}z \subset U_{J_{1/2}}z = 0$ by (P2), $U_{J_{1/2}}U_{U(\hat{J}_i)z}\hat{J}_i = U_{J_{1/2}}U_{\hat{J}_i}U_zU_{\hat{J}_i}\hat{J}_i \subset U_{J_{1/2}}U_z\hat{J}_i = 0$ by (0.1), (P2), and is an inner ideal since $U_{J_{1/2}}(U_z\hat{J}_i) = 0$, $U_{J_{1/2}}(U_{U(z)\hat{J}_i})\hat{J}_i = U_{J_{1/2}}U_zU_{\hat{J}_i}U_z\hat{J}_i \subset U_{J_{1/2}}U_z\hat{J}_i = 0$ by (0.1).

We can easily show that a strongly semiprime ideal is invariant. Recall that K_i is *strongly semiprime* in J_i if $\bar{J}_i = J_i/K_i$ is strongly semiprime in the sense of having no trivial elements $U_{\bar{z}_i}\bar{J}_i = \bar{0}$; this is equivalent to $U_{z_i}J_i \subset K_i \Leftrightarrow z_i \in K_i$.

THEOREM 2.11. *Any strongly semiprime ideal $K_i \lhd J_i$ is invariant.*

Proof. For $x, y \in J_{1/2}$, $k \in K_i$ we have $\{xyk\} \in K_i \Leftrightarrow U_{\{xyk\}}J_i \subset K_i$ (strong semiprimeness) $\Leftrightarrow U_xU_yU_kJ_i \subset K_i$ (using (2.5)) $\Rightarrow U_{U(x)U(y)k}J_i = U_xU_yU_k(U_yU_xJ_i) \subset U_xU_yU_kJ_i \subset K_i$ (by (0.1)) $\Leftrightarrow U_xU_yk \in K_i$. This shows V-invariance implies U-invariance. Further, since $\{xy(U_ka)\} \in K_i$ by (2.2) it shows $U_xU_y(U_ka) \in K_i$, i.e., $U_kU_yU_kJ_i \subset K_i$, hence by the above $\{xyk\} \in K_i$, establishing V-invariance.

Since any maximal ideal in a unital algebra is strongly semiprime (the quotient is simple with unit, therefore contains no nil ideals, therefore contains no trivial elements), we have the important

COROLLARY 2.12. *Any maximal ideal M_1 in J_1 is invariant.*

This immediately shows that J_1 is simple if J is. We return to this in §4, where we use a flipping argument to deduce that J_0 is simple as well. In the remainder of this section we undertake a more delicate analysis to show K_i is invariant if it is merely *semiprime* in J_i (in the sense that $\bar{J}_i$ is semiprime), or even if it has no trivial ideals $U_{\bar{B}_i}\bar{J}_i = \bar{0}$ (this is equivalent to $U_{B_i}\hat{J}_i \subset K_i \Rightarrow B_i \subset K_i$ for $B_i \lhd J_i$).

LEMMA 2.13. *If K_i is an ideal in J_i then $H(K_i) = K_i + V_{J_{1/2},J_{1/2}}K_i + U_{J_{1/2}}U_{J_{1/2}}K_i$ is again an ideal in J_i. In fact, for any particular $x, y \in J_{1/2}$ the subspaces*

$$\begin{aligned} K_i^{(1)} &= K_i + U_x U_y U_{K_i} \hat{J}_i \\ K_i^{(2)} &= K_i + V_{x,y} K_i + U_x U_y U_{K_i} \hat{J}_i \\ K_i^{(3)} &= K_i + V_{x,y} K_i + U_x U_y K_i \end{aligned}$$

are ideals in J_i *with*

$$K_i \subset K_i^{(1)} \subset K_i^{(2)} \subset K_i^{(3)}$$

and with each trivial modulo the preceding:

$$U_{K_i^{(3)}} \hat{J}_i \subset K_i^{(2)} , \quad U_{K_i^{(2)}} \hat{J}_i \subset K_i^{(1)} , \quad U_{K_i^{(1)}} \hat{J}_i \subset K_i .$$

Proof. Since $H(K_i)$ is just the sum of all $K_i^{(3)}$ for all possible $x,y \in J_{1/2}$, it suffices to prove the $K_i^{(j)}$ are ideals.

We first show each $K_i^{(j)}$ is an outer ideal: $U_{\hat{J}_i} K_i^{(j)} \subset K_i^{(j)}$. For $a \in \hat{J}_i$ and $k \in L_i \lhd J_i$ we have

$$\begin{aligned} U_a V_{x,y} k &= \{U_{\{yxa\},n} - V_{y,x} U_a\} k && \text{(by (0.3))} \\ &= U_{\{yxa\},a} k - V_{y\circ x} U_a k - V_{x,y} U_a k && \text{(by (0.6))} \\ &\in U_{\hat{J}_i} L_i - V_{J_i} U_{\hat{J}_i} L_i - V_{x,y} L_i \subset L_i + V_{x,y} L_i \\ U_a U_x U_y k &= \{U_{\{axy\}} - U_y U_x U_a - V_{y,x} U_a V_{x,y} + U_{a,U(y)U(x)a}\} k && \text{(by (0.2))} \\ &= \{U_{\{axy\}} + (U_x U_y - U_{x\circ y} + V_{x,y} V_{y,x} - V_{U(x)y^2}) U_a \\ &\quad -(V_{x\circ y} - V_{x,y}) U_a V_{x,y} + U_{a,U(y)U(x)a}\} k && \text{(by (0.2), (0.6))} \\ &= \{U_{\{axy\}} + U_x U_y U_a -(U_{x\circ y} + V_{U(y)x^2}) U_a + V_{x,y} U_{a,\{yxa\}} \\ &\quad - V_{x\circ y} U_a V_{x,y} + U_{a,U(y)U(x)a}\} k && \text{(by 0.3))} \\ &\in U_{J_i} L_i + U_x U_y L_i - (U_{J_i} + V_{J_i}) U_{\hat{J}_i} L_i + V_{x,y} U_{\hat{J}_i} L_i \\ &\quad - V_{J_i} U_{\hat{J}_i} V_{x,y} L_i + U_{\hat{J}_i, J_i} L_i \\ &\subset L_i + U_x U_y L_i - L_i + V_{x,y} L_i - V_{J_i} U_{\hat{J}_i} V_{x,y} L_i + L_i \\ &\subset L_i + V_{x,y} L_i + U_x U_y L_i \end{aligned}$$

(using our previous calculation to move V_{J_i}, $U_{\hat{J}_i}$ past $V_{x,y}$). Taking $L_i = K_i$ shows $K_i^{(3)}$ is outer, while $L_i = U_{K_i} \hat{J}_i \subset K_i$ shows $K_i^{(2)}$, $K_i^{(1)}$ are outer (using (2.2) for $K_i^{(1)}$).

Now we show the $K_i^{(j)}$ are inner, in fact the stronger assertion that each is trivial modulo its predecessor: $U_{K_i^{(j)}} \hat{J}_i \subset K_i^{(j-1)} \subset K_i^{(j)}$. For $j = 1$ we have $K_i^{(1)} \equiv U_x U_y U_{K_i} J_i$ modulo the ideal $K_i^{(0)} = K_i$, so

$$\begin{aligned} U_{K_i}(1) \hat{J}_i &\equiv U_{U(x)U(y)U(K_i)\hat{J}_i} \hat{J}_i = U_y U_y U_{U(K_i)\hat{J}_i} U_x U_x \hat{J}_i \\ &\subset U_x U_y U_{U(K_i)\hat{J}_i} J_i \subset K_i \equiv 0 && \text{(by (2.3))} \end{aligned}$$

so $U_{K_i^{(1)}} \hat{J}_i \subset K_i$. In particular, $K_i^{(1)}$ is inner and thus an ideal. Once $K_i^{(1)}$ is an ideal we have for $j = 2$ that $K_i^{(2)} \equiv V_{x,y} K_i$ modulo $K_i^{(1)}$, so

$$U_{K_i^{(2)}} \hat{J}_i \equiv U_{\{xyK_i\}} \hat{J}_i \equiv U_x U_y U_{K_i} \hat{J}_i \subset K_i^{(1)} \equiv 0 \qquad \text{(by (2.5))}$$

so $U_{K_i^{(2)}}\hat{J}_i \subset K_i^{(1)}$ and $K_i^{(2)}$ too is an ideal. Then we have $K_i^{(3)} \equiv U_x U_y K_i$ modulo the ideal $K_i^{(2)}$, so
$U_{K_i^{(3)}}\hat{J}_i \equiv U_{U(x)U(y)K_i}\hat{J}_i = U_x U_y U_{K_i} U_y U_x \hat{J}_i \subset U_y U_y U_{k_i} J_j \subset K_i^{(2)} \equiv 0$ so $U_{K_i^{(3)}}\hat{J}_i \subset K_i^{(2)}$ and $K_i^{(3)}$ is also an ideal trivial modulo its predecessor.

Our calculations show each $U_x U_y K_i$ is an ideal and each $K_i + V_{x,y}K_i$ is an outer ideal; if $1/2 \in \Phi$ outer ideals are ideals, so $U_{J_{1/2}}U_{J_{1/2}}K_i$ and $K_i + V_{J_{1/2},J_{1/2}}K_i$ are both ideals in this case.

REMARK 2.14. If invertible elements are dense one can show

$$B(J_{1/2}, J_{1/2})K_1 \lhd J_1 \qquad (B(x, y) = I + V_{x,y} + U_x U_y) .$$

Indeed, $U_a B(x, U_a y)z = B(U_a x, y)U_a z$ shows for invertible $x_1 \in J_1$ that

$$\begin{aligned} U_{x_1}B(J_{1/2}, J_{1/2})K_1 &= U_{e_0+x_1}B(J_{1/2}, x_1 \circ J_{1/2})K_1 \\ &= U_a B(J_{1/2}, U_a J_{1/2})K_1 (a = e_0 + x_1) = B(U_a J_{1/2}, J_{1/2})U_a K_1 \\ &= B(J_{1/2}, J_{1/2})U_{x_1}K_1 \subset B(J_{1/2}, J_{1/2})K_1 , \end{aligned}$$

hence if such x_1 are dense BK_1 is outer, and it is inner since for the spanning set of $B(x_{1/2}, y_{1/2})k_1$ we have $U_{B(x,y)k}J_1 = B(x, y)U_k B(y, x)J_1 \subset B(x, y)U_{k_1}J_1 \subset B(x, y)K_1$. It is not known if this holds in general. If Φ is a field with more than two elements then $B(J_{1/2}, J_{1/2})K_i$ is just $K_i + V(J_{1/2}, J_{1/2})K_i + U(J_{1/2})U(J_{1/2})K_i$ and thus is certainly an ideal.

Now we can establish invariance of semiprime ideals.

THEOREM 2.15. *Any semiprime ideal $K_i \lhd J_i$ is invariant.*

Proof. Semiprimeness means J_i/K_i contains no trivial ideals. But then $K_i = K_i^{(0)} \subset K_i^{(1)} \subset K_i^{(2)} \subset K_i^{(3)}$ with $K_i^{(j+1)}/K_i^{(j)}$ trivial forces in turn $K_i = K_i^{(1)} = K_i^{(2)} = K_i^{(3)}$. This shows $V_{x,y}K_i \subset K_i$ and $U_x U_y K_i \subset K_i$ for any particular $x, y \in J_{1/2}$, and thus K_i is V-and U-invariant.

REMARK 2.16. We have established invariance of K_i as long as $\bar{J}_i = J_i/K_i$ contains no ideals $\bar{L}_i$ consisting entirely of trivial elements (i.e., $U_{L_i}\hat{J}_i \subset K_i \Rightarrow L_i \subset K_i$). It is not known whether an algebra without such ideals is necessarily semiprime; this holds whenever $1/2 \in \Phi$ since $\bar{L}_i^2 = \bar{0}$ implies $2U_{\bar{L}_i}\hat{\bar{J}}_i = \bar{L}_i \circ (\bar{L}_i \circ \hat{\bar{J}}_i) - \bar{L}_i^2 \circ \hat{\bar{J}}_i = \bar{0}$.

3. **The invariant hull.** If we have no specific information about a given ideal $K_i \lhd J_i$ which allows us to conclude it is invariant, we must enlarge it by applying all possible V's and U's until the result is invariant. The *invariant hull* $\mathrm{Inv}(K_i)$ of the ideal K_i is the smallest invariant ideal containing K_i.

In (1.9) we saw that V-invariance implies U-invariance when $1/2 \in \Phi$. More generally,

PROPOSITION 3.1. *The subalgbra $E(\mathscr{U}, \mathscr{V})$ of* End(J_i) *generated by the restrictions to J_i of $V_{J_{1/2}, J_{1/2}}$ and $U_{J_{1/2}} U_{J_{1/2}}$ reduce to $\mathscr{U} + \mathscr{V}$ where $\mathscr{U}$ is the linear span of all operators*

$$U_{x_1} U_{y_1} \cdots U_{x_n} U_{y_n}$$

and $\mathscr{V}$ the linear span of all

$$V_{x_1, y_1} \cdots V_{x_n, y_n}$$

where x_i, y_i belong to some spanning set for $J_{1/2}$. Further, $2\mathscr{U} \subset \mathscr{V}$.

Proof. The Jordan identities (0.4), (0.5) show that the partially linearized U-operators $U_x U_{y,z}$ and $U_{y,z} U_x$ can be replaced by products of V-operators: $U_x U_{y,z} \in \mathscr{V}$, $U_{y,z} U_x \in \mathscr{V}$. In particular, for $y = z$ we see as in (1.9)

$$2U_x U_y \in \mathscr{V}.$$

These together with the further Jordan identities

$$(0.8) \qquad U_x U_y V_{z,w} = U_{\{xyz\},x} U_{w,y} - V_{z,y} U_x U_{w,y} - U_x U_{U(y)z,w} \in \mathscr{V}$$

$$(0.9) \qquad V_{w,z} U_y U_x = U_{w,y} U_{\{xyz\},x} - U_{w,y} U_x V_{y,z} - U_{U(y)z,w} U_x \in \mathscr{V}$$

show that any mixed term involving a product of U's with at least one V factors, or 2 times any product of U's can be expressed solely in terms of V's,

$$\mathscr{U}\mathscr{V} + \mathscr{V}\mathscr{U} \subset \mathscr{V}, \quad 2\mathscr{U} \subset \mathscr{V}.$$

Thus the subalgabra generated by $\mathscr{U}$ and $\mathscr{V}$ reduces to $\mathscr{U} + \mathscr{V}$ with $2U \subset \mathscr{V}$.

Since $V_{x,y}$ is bilinear in x, y, if $\{u_i\}$ spans $J_{1/2}$ then the V_{u_i, u_j} span $V_{J_{1/2}, J_{1/2}}$, and $U_{J_{1/2}} U_{J_{1/2}}$ is spanned by the $U_{u_i} U_{u_j}$ modulo terms $U_{u_i} U_{u_i, u_k}$, $U_{u_j, u_k} U_{u_i}$, $U_{u_i, u_j} U_{u_k, u} \in \mathscr{V}$.

REMARK 3.2. For $x, y \in J_{1/2}$ we have an operator identity on J_i

$$U_x U_x = U_{x^2} = U_{E_i(x^2)}, \; U_x U_y + U_y U_x + U_{x,y}^2 = U_{E_i(x \circ y)} + U_{E_i(x^2), E_i(y^2)}$$

showing $U_{x_1} U_{x_2} \cdots U_{x_{2n}}$ is an alternating function of the variables $x_i \in J_{1/2}$ modulo products with fewer U's and either more V's or more multiplications from J_i (which automatically leave any ideal $K_i \lhd J_i$ invariant). Thus $\mathscr{U}$ is spanned modulo $\mathscr{V}$ and $\mathscr{M}(J_i)$ by all $U_{u_1}U_{u_2}\cdots U_{u_{2n}}$ for $u_1 < \cdots < u_{2n}$ in some ordered spanning set for $J_{1/2}$.

THEOREM 3.3. *The invariant hull of a given ideal $K_i \lhd J_i$ is*

$$\mathrm{Inv}\,(K_i) = \mathscr{U}K_i + \mathscr{V}K_i = \sum_{k=0}^{\infty} V^k_{J_{1/2},J_{1/2}}K_i + \sum_{m=0}^{\infty} U^{2m}_{J_{1/2}}K_i\ .$$

If $1/2 \in \Phi$ this reduces to $\sum V^k_{J_{1/2},J_{1/2}}K_i$.

Proof. A subspace is U- and V-invariant iff it is invariant under the subalgebra generated by all U's and V's, which by 3.1 is just $\mathscr{U} + \mathscr{V}$, so $\mathscr{U}K_i + \mathscr{V}K_i$ is the invariant closure of K_i. To see this remains an ideal in J_i if K_i is to begin with, note that this invariant closure can also be represented as $\mathrm{Inv}\,(K_i) = \sum_{n=0}^{\infty} H^n(K_i)$ where $H(L_i) = L_i + V_{J_{1/2},J_{1/2}}L_i + U_{J_{1/2}}U_{J_{1/2}}L_i$, where by Lemma 2.13 each $H^n(K_i)$ is an ideal and therefore their sum is too.

If $1/2 \in \Phi$ we can dispense with the U's by 3.1.

REMARK 3.4. By our comments 3.2, if $J_{1/2}$ is finitely spanned we need only take a finite number of powers $U^{2m}_{J_{1/2}}$.

REMARK 3.5. $\mathrm{Inv}\,(K_i)$ is Baer-radical modulo K_i since it is a union of $H^n(K_i)$, where $H^n(K_i)$ is Baer-radical modulo $H^{n-1}(K_i)$ (being the sum over all $x, y \in J_{1/2}$ of ideals $K_i^{(3)} = K_i + V_{x,y}K_i + U_xU_yK_i$ nilpotent modulo K_i by (2.13)). Once more this shows that if K_i is semiprime in J_i then $\mathrm{Inv}\,(K_i) = K_i$ and K_i is invariant.

We can, if compelled, write down explicitly the ideal generated by a diagonal Peirce ideal.

THEOREM 3.6. *If K_i is an ideal in a Peirce space J_i $(i = 1, 0)$ of a Jordan algebra J, then the ideal it generates in J is*

$$I(K_i) = I_i \oplus I_{1/2} \oplus I_j$$

$$I_i = \mathrm{Inv}\,(I_i) = (\mathscr{V} + \mathscr{U})K_i = \sum_{j,k=0}^{\infty} (V^j_{J_{1/2},J_{1/2}} + U^{2k}_{J_{1/2}}\}K_i$$

$$I_{1/2} = V_{J_{1/2}}\,\mathrm{Inv}\,(K_i) = V_{J_{1/2}}\mathscr{V}K_i = V_{J_{1/2}}\left\{\sum_{j=0}^{\infty} V^j_{J_{1/2},J_{1/2}}\right\}K_i$$

$$I_j = U_{J_{1/2}}\,\mathrm{Inv}\,(K_i) = (\mathscr{V} + \mathscr{U})U_{J_{1/2}}K_i = \mathrm{Inv}\,(U_{J_{1/2}}K_i)$$

$$= \sum_{j,k=0}^{\infty} \{V^j_{J_{1/2},J_{1/2}} + U^{2k}_{J_{1/2}}\}U_{J_{1/2}}K_i\ .$$

Proof. The ideal generated by K_i coincides with the ideal generated by its invariant hull $\mathrm{Inv}\,(K_i) = (\mathscr{V} + \mathscr{U})K_i$ by (3.3), so by

(1.11) $I_i = \operatorname{Inv}(K_i)$, $I_{1/2} = V_{J_{1/2}} \operatorname{Inv}(K_i)$, $I_j = U_{J_{1/2}} \operatorname{Inv}(K_i)$. Note that $U_{J_{1/2}} \operatorname{Inv}(K_i) = U_{J_{1/2}}(\mathscr{V} + \mathscr{U})K_i = (\mathscr{V} + \mathscr{U})U_{J_{1/2}}K_i$ since $U_{J_{1/2}}U^{2k}_{J_{1/2}} = U^{2k}_{J_{1/2}}U_{J_{1/2}}$ shows $U_{J_{1/2}}\mathscr{U} = \mathscr{U}U_{J_{1/2}}$, and $U_{J_{1/2}}V_{J_{1/2},J_{1/2}} + V_{J_{1/2},J_{1/2}}U_{J_{1/2}} \subset U_{\{J_{1/2}J_{1/2}J_{1/2}\},J_{1/2}} \subset U_{J_{1/2}}$ by (0.3) shows $U_{J_{1/2}}\mathscr{V} = \mathscr{V}U_{J_{1/2}}$. Note further that

$$V_{J_{1/2}}\mathscr{U} \subset V_{J_{1/2}}\mathscr{V},\ V_{J_{1/2}} \operatorname{Inv}(K_i) = V_{J_{1/2}}\mathscr{V}K_i$$

because

$$V_{J_{1/2}}U^2_{J_{1/2}} \subset V_{J_{1/2}} \sum_{j=0}^{2} V^j_{J_{1/2},J_{1/2}}$$

follows from the following obscure Jordan identity:

$$\begin{aligned}(0.10) \qquad V_xU_yU_z = V_{U(z)U(y)x} - V_zV_{U(y)x,z} + V_{U(z)y}V_{y,x} - V_zV_{y,z}V_{y,x} \\ - V_{U(\{xyz\},z)y} + V_zV_{y,\{xyz\}} + V_{\{xyz\}}V_{y,z}\end{aligned}$$

(or else substitute 1 in (0.5), $V_yV_{x,y} = V_xU_y + V_{U(y)}$ to see $V_{J_{1/2}}U_{J_{1/2}} \subset V_{J_{1/2}}V_{J_{1/2},J_{1/2}} + V_{J_{1/2}} \subset V_{J_{1/2}}\mathscr{V}$, so

$$\begin{aligned}V_{J_{1/2}}U_{J_{1/2}}U_{J_{1/2}} &\subset V_{J_{1/2}}\mathscr{V}U_{J_{1/2}} \subset V_{J_{1/2}}(U_{J_{1/2}}\mathscr{V} + U_{J_{1/2}}) \\ &\subset (V_{J_{1/2}}\mathscr{V})\mathscr{V} + V_{J_{1/2}}\mathscr{V} = V_{J_{1/2}}\mathscr{V}) .\end{aligned}$$

EXAMPLE 3.7. The largest invariant ideal contained in $K_i \lhd J_i$ is the invariant kernel

$$\begin{aligned}\operatorname{Inv}\ker(K_i) &= \{z \in K_i \mid E(\mathscr{U}, \mathscr{V})z \subset K_i\} \\ &= \{z \in K_i \mid V^n_{J_{1/2},J_{1/2}}z,\ U^{2m}_{J_{1/2}}z \in K_i \text{ for all } n, m\} .\end{aligned}$$

Proof. Certainly if z belongs to an invariant ideal $I_i \lhd K_i$ so do all V^nz and $U^{2m}z$, so z belongs to $\operatorname{Inv}\ker(K_i) = Z_i$. Conversely, Z_i is clearly a linear subspace which is invariant, $E(\mathscr{U}, \mathscr{V})Z_i \subset Z_i$. It remains to show Z_i is an ideal.

Z_i is outer: the identities (0.3), (0.2) show

$$\begin{aligned}VU_{\hat{J}_i} \subset U_{\hat{J}_i} + U_{\hat{J}_i}V \subset U_{\hat{J}_i}E(\mathscr{U}, \mathscr{V}),\ U^2U_{\hat{J}_i} \subset U_{\hat{J}_i}U^2 + U_{\hat{J}_i} \\ + VU_{\hat{J}_i}V \subset U_{\hat{J}_i}U^2 + U_{J_i} + U_{\hat{J}_i}E(\mathscr{U}, \mathscr{V})V \subset U_{\hat{J}_i}E(\mathscr{U}, \mathscr{V}) ,\end{aligned}$$

and hence by induction $E(\mathscr{U}, \mathscr{V})(U_{\hat{J}_i}Z_i) \subset U_{\hat{J}_i}(E(\mathscr{U}, \mathscr{V})Z_i) \subset U_{\hat{J}_i}K_i \subset K_i$. Therefore $U_{\hat{J}_i}Z_i \subset Z_i$.

Z_i is inner: the identities (0.3), (0.2) show $VU_{Z_i}\hat{J}_i \subset U_{Z_i}V\hat{J}_i + U_{V(Z_i),Z_i}\hat{J}_i \subset U_{Z_i}\hat{J}_i$ (since Z_i is V-invariant), $U^2(U_{Z_i}\hat{J}_I) \subset \{U_{Z_i}U^2 + U_{Z_i} + VU_{Z_i}V\}\hat{J}_i$ (since Z_i is U, V-invariant) $\subset U_{Z_i}\hat{J}_i$, hence by induction $E(\mathscr{U}, \mathscr{V})(U_{Z_i}\hat{J}_i) \subset U_{Z_i}\hat{J}_i \subset U_{K_i}\hat{J}_i \subset K_i$ and $U_{Z_i}\hat{J}_i \subset Z_i$.

EXAMPLE 3.8. We give a straightforward example of Jordan algebra having noninvariant Peirce ideals. Let D be an associative algebra with involution $*$, and let D' be an ample subspace ($D' \subset H(D, *)$ is symmetric, contains 1, and has $xD'x^* \subset D'$ for all $x \in D$: if $1/2 \in \Phi$ then $D' = H(D, *)$). Then the algebra $J = H(D_n, D')$ of hermitian nxn matrices over D with diagonal entries in D' forms a Jordan algebra with idempotent $e = e_{11}$. Here a subspace $K_1 = K'[11]$ of the Peirce space $J_1 = D'[11]$ is an ideal iff K' is a Jordan ideal in D',

(i) (outer ideal) $x'k'x' \in K'$ for all $x' \in D'$, $k' \in K'$

(ii) (inner ideal) $k'x'k' \in K'$.

On the other hand, such a K_1 is V-invariant iff K is closed under traces,

(iii) (V-invariant) $t(DK') \subset K'$: $xk' + k'x^* \in K'$ for $x \in D$, $k' \in K'$

and U-invariant iff it is closed under norms,

(iv) (U-invariant) $xk'x^* \in K'$ for all $x \in D$, $k' \in K'$.

These follow from the general rules $V(a[1j], b^*[1j])c[11] = t(abc)[11]$ and

$$U(a[1j])U(b^*[1j])c[11] = abcb^*a^*[11]$$

and $U(1[aj], d[1k])U(b^*[1j], f^*[1k])c[11] = (abcf^*d^* + dfcb^*a^*)[11]$. In this case U-invariance implies V-invariance (and conversely if $1/2 \in \Phi$), and the invariant hull of K_1 is

$$\operatorname{Inv}(K_1) = K_1 + U_{J_{1/2}}U_{J_{1/2}}K_1 = \{\textstyle\sum xK'x^*\}[11] .$$

For example, if we take $D = M_2(\Phi)$ a split quaternion algebra over a ring Φ and $D' = \Phi 1$, then K' is an ideal of D' iff (i)$\Phi^2K' \subset K'$, (ii) $\Phi K'^2 \subset K'$, and K' is V-invariant iff (iii) $t(D)K' = \Phi K' \subset K'$, and K' is U-invariant iff (iv) $n(D)K' = \Phi K' \subset K'$. If $1/2 \in \Phi$ or Φ is a field all ideals K' of D' are invariant, but if $\Phi = \boldsymbol{Z}[x]$, $K' = \boldsymbol{Z}x^2 + x^4\boldsymbol{Z}[x] + 2\boldsymbol{Z}[x]$ then one easily verifies that K' is a Jordan ideal in $\boldsymbol{Z}[x]$ which is not an associative ideal (and hence not invariant).

In this example we obtained the invariant hull from a single application of $U_{J_{1/2}}U_{J_{1/2}}$ because the coordinates of $J_{1/2} = \sum D[1j]$ are closed under multiplication. To construct examples where the invariant hull requires all $V^n_{J_{1/2}, J_{1/2}}$ and $U^{2m}_{J_{1/2}}$ we take subalgebras where the coordinates of $J_{1/2}$ are not closed. From now on our examples will sit inside $H(D_2, D')$.

EXAMPLE 3.9. (All V's are necessary.) Let $D = \Lambda(V) \otimes \Phi[\varepsilon]$ be the ring of dual numbers ($\varepsilon^2 = 0$) over the exterior algebra $\Lambda(V)$ on an infinite-dimensional vector space V over a field Φ of characteristic $\neq 2$, with canonical reversal involution fixing V. (Thus the symmetric elements are spanned by the elements of $\Lambda^n(V)$ for $n \equiv 0$ or $1 \bmod 4$.)

Then the set $H(D_2)$ of all 2×2 matrices with entries in the associative coordinate ring D forms a Jordan algebra. We take $\tilde{J}$ to be the subalgebra

$$\begin{aligned}\tilde{J} &= \varepsilon H(D_2) + (V \wedge V)[12] \\ &= \varepsilon H(D)[11] + \{V \wedge V + \varepsilon D\}[12] + \varepsilon H(D)[22]\end{aligned}$$

and $J = \tilde{J} + \Phi 1[11]$ the subalgebra obtained by tacking on $e = 1[11]$. Thus $H(D_2) \supset J \supset \tilde{J} \supset \varepsilon H(D_2)$.

Since $\tilde{J}_1 = \varepsilon J_1$ is trivial ($U_{\tilde{J}_1}\tilde{J}_1 = \tilde{J}_1^2 = 0$ since $\varepsilon^2 = 0$), *any* subspace $K_1 \subset \tilde{J}_1$ is an ideal in J_1. However, only certain subspaces will be invariant:

$$\begin{aligned}V_{u_1\wedge u_2[12], u_3\wedge u_4[12]}k[11] &= 2u_1 \wedge u_2 \wedge u_3 \wedge u_4 \wedge k[11] \\ U_{u_1\wedge u_2[12], u_3\wedge u_4[12]}k[11] &= -2u_1 \wedge u_2 \wedge u_3 \wedge u_4 \wedge k[11] \\ U_{u_1\wedge v_1[12]}k[11] &= 0\ .\end{aligned}$$

Thus a subspace $K_1 = \varepsilon K[11]$ will be invariant only if the subspace K of $H(D)$ is closed under multiplication by the degree 4 part of the exterior algebra (generated by all $u_1 \wedge u_2 \wedge u_3 \wedge u_4$ for $u_i \in V$). If $K = \Phi v_0$ then $V_{u_1\wedge v_1[12], w_1\wedge t_1[12]} \cdots V_{u_n\wedge v_n[12], w_n\wedge t_n[12]}K_1 = \varepsilon\Phi u_1 \wedge v_1 \wedge w_1 \wedge t_1 \wedge \cdots \wedge u_n \wedge v_n \wedge w_n \wedge t_n \wedge v_0[11] \subset \varepsilon\Lambda^{4n+1}(V)[11]$, from which it is clear that arbitrarily high powers of $V_{J_{1/2}, J_{1/2}}$ are needed to generate the arbitrarily long elements $\varepsilon u_1 \wedge u_2 \wedge \cdots \wedge u_{4n} \wedge v_0[11]$ in $\mathrm{Inv}(K_1)$.

EXAMPLE 3.10. (All U's are necessary.) Again we take $H(D_2)$ for D an associative algebra with involution, but this time D is a "square root" of an exterior algebra $\Lambda(V)$ on an infinite-dimensional vector space V over a field Φ of characteristic 2. If V has basis $\{v_1, v_2, \cdots\}$ we let $D = \Phi[x_1, x_2, \cdots]$ be a commutative polynomial ring (with identity involution) where $x_i^2 = v_i$, $v_i^2 = 0$. Note

$$D^2 \subset \Phi[x_1^2, x_2^2, \cdots] = \Phi[v_1, v_2, \cdots] \cong \Lambda(V),\ (D^2)^2 = 0\ .$$

Let

$$\tilde{J} = H(D_2^2) + \{\textstyle\sum \Phi x_i\}[12] = D^2[11] + \{\sum \Phi x_i + D^2 x_i\}[12] + D^2[22]$$

and $J = \tilde{J} + \Phi e[11]$. Again $\tilde{J}_1 = D^2[11]$ is trivial since the characteristic is 2 and $(D^2)^2 = 0$, so any subspace $K_1 \subset \tilde{J}_1$ is an ideal in J_1. Here V-invariance is automatic,

$$V_{a[12], b[12]}c[11] = 2abc[11] = 0\ .$$

U-invariance of $K_1 = K[11]$ means closure of K under even products of v_i's, since

$$U_{a[12]}U_{b[12]}c[11] = a^2b^2c[11]\ ,$$

and if $a = \sum \alpha_i x_i + \sum d_i^2 x_i$ then $a^2 = \sum \alpha_i^2 v_i$. From this it is clear that arbitrarily large powers $U_{x_1[12]} U_{x_2[12]} \cdots U_{x_{2n}[12]} v_0[11] = v_1 v_2 \cdots v_n v_0[11]$ are needed to obtain the invariant hull of $K_1 = \Phi v_0[11]$.

4. **Simplicity of J_1 and J_0.** We use our constructions to show that Peirce subalgebras J_1 and J_0 inherit simplicity from J. The basic idea of the proof is easily stated. Since a simple algebra J contains no proper ideals K, there are no proper projections in the Peirce subalgebras J_1 and J_0, consequently by 1.11 there are no proper *invariant* ideals in J_1 or J_0. Since J_1 has unit element e there exist (by the usual Zornification) maximal ideals K_1, necessarily strongly semiprime in J_1 by (2.12), so any maximal K_1 is invariant and therefore zero; but $K_1 = 0$ maximal means J_1 is simple.

For the nonunital algebra J_0 we cannot use this argument, but we can make use of the simplicity of J_1: any ideal K_0 in J_0 is flipped into an ideal $K_1 = U_{J_{1/2}} K_0$ in J_1. If this image is zero the same holds for the invariant hull of K_0, forcing this hull to be zero and $K_0 = 0$. If on the other hand the image is all of J_1 then the same holds for K_0^3; but the double flip of K_0^3 is contained in K_0, which forces $K_0 = J_0$. This means J_0 is simple.

Now to fill in the details.

MAIN THEOREM 4.1. *If e is an idempotent in a simple Jordan algebra J then the Peirce subalgebras $J_1(e)$ and $J_0(e)$ are also simple.*

Proof. The result is vacuous if $e = 0$ ($J_1 = 0$, $J_0 = J$), so we may assume $e \neq 0$. Then J is not nil, $\mathrm{Nil}(J) \neq J$, so by simplicity $\mathrm{Nil}(J) = 0$ and in particular J contains no trivial elements. Each J_i inherits this strong semiprimeness since an element trivial in J_i is trivial in all of $J (U_{z_i} J = U_{z_i} J_i)$, therefore J_i is not trivial and will be simple if it has no proper ideals. We know J_i contains no proper *invariant* ideals, and we must deduce it has no proper ideals whatsoever.

We have already seen this is true for J_1 thanks to its unit e, so consider J_0. Suppose we have an ideal $K_0 \lhd J_0$. By the Flipping Lemma 1.10 the image $K_1 = U_{J_{1/2}} K_0$ is an ideal in J_1, so by what we have just shown it must either be J_1 or 0.

First consider the case $K_1 = U_{J_{1/2}} K_0 = 0$. Then $K_0 \subset \mathrm{Ker}\, U_{J_{1/2}}$, which by the Kernel Lemma 2.10 is an invariant ideal of J_0. Such an invariant ideal can only be J_0 or 0, and it is not all of J_0 since $U_{J_{1/2}} J_0 \neq 0$ by (1.17), so $\mathrm{Ker}\, U_{J_{1/2}}$ must be 0 and K_0 was 0 to begin with. So far we have shown that $K_1 = 0$ implies $K_0 = 0$.

Now consider the case $K_1 = U_{J_{1/2}} K_0 = J_1$. Since J_0 is strongly semiprime it has no nilpotent ideals, so $K_0 \neq 0 \Rightarrow K_0' = U_{K_0} \hat{J}_0 \neq 0 \Rightarrow K_0'' = U_{K_0'} \hat{J}_0 \neq 0$. But by the previous case $K_0'' \neq 0$ implies $K_1'' =$

$U_{J_{1/2}}K_0''$ is nonzero and therefore all of J_1. Thus by (1.17) $J_0 = U_{J_{1/2}}J_1 = U_{J_{1/2}}(U_{J_{1/2}}K_0'')$. On the other hand, $U_{J_{1/2}}U_{J_{1/2}}K_0'' = U_{J_{1/2}}U_{J_{1/2}}(U_{U(K_0)\hat{J}_0}\hat{J}_0) \subset K_0$ by (2.3), so we have $K_0 = J_0$. This shows $K_1 = J_1$ implies $K_0 = J_0$. Thus $K_0 \lhd J_0$ implies $K_0 = 0$ or $K_0 = J_0$, and J_0 too is simple.

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