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RADII OF CONVEXITY FOR CERTAIN CLASSES OF UNIVALENT ANALYTIC FUNCTIONS

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Let $P(\alpha, \beta)$ denote the class of functions $p(z)=1+b_1z+\cdots$ which are analytic and satisfy the inequality $|(p(z)-1)/\{2\beta(p(z)-\alpha)-(p(z)-1)\}|<1$ for some α, β $(0\leqq\alpha<1, 0<\beta\leqq1)$ and all $z\in E=\{z: |z|<1\}$. Also, let $P_b(\alpha, \beta)=\{p\in P(\alpha, \beta): p'(0)=2b\beta(1-\alpha), 0\leqq b\leqq 1\}$. In the present paper, we determine sharp estimates for the radii of convexity for functions in the classes $R_a(\alpha, \beta)$ and $S_a^*(\alpha, \beta)$ where $R_a(\alpha, \beta)=\{f(z)=z+a\beta(1-\alpha)z^2+\cdots: f'\in P_a(\alpha, \beta), 0\leqq a\leqq 1\}$, $S_a^*(\alpha, \beta)=\{g(z)=z+2a\beta(1-\alpha)z^2+\cdots: zg'/g\in P_a(\alpha, \beta), 0\leqq a\leqq 1\}$. The results thus obtained not only sharpen and generalize the various known results but also give rise to several new results.

1. Introduction.

Let P denote the class of functions

$$p(z)=1+b_1z+b_2z^2+\cdots \tag{1.1}$$

which are analytic and satisfy $\mathrm{Re}\,(p(z))>0$ for $z\in E\equiv\{z: |z|<1\}$. Considerable work has been done to study the various aspects of the above mentioned class (see e.g., [11], [12] and others). Some of these results have also been extended to the class $P(\alpha)$ of functions $p(z)$ which are analytic and satisfy $\mathrm{Re}\,(p(z))>\alpha$, $0\leqq\alpha<1$ for $z\in E$. If $p\in P(\alpha)$, it is easily seen that $|b_1|\leqq 2(1-\alpha)$. Further, we note that if $\tau=\exp\{-i \arg b_1\}$ then $p(\tau z)=1+|b_1|z+\cdots$ and so while studying $P(\alpha)$, there is no loss of generality if one takes the first coefficient b_1 in (1.1) to be nonnegative.

McCarty in [8] determined a lower bound on $\mathrm{Re}\, zp'(z)/p(z)$ for functions $p(z)$ in the class $P_b(\alpha)\equiv\{p\in P(\alpha): p'(0)=2b(1-\alpha), 0\leqq b\leqq 1\}$. He also applied the results obtained to determine the sharp estimates for the radii of convexity of the two classes $R_a(\alpha)$ and $S_a^*(\alpha)$ for each $a\in[0, 1]$ and $\alpha\in[0, 1)$ where

$$R_a(\alpha)=\{f(z)=z+a(1-\alpha)z^2+\cdots: f'\in P_a(\alpha)\}$$

and

$$S_a^*(\alpha)=\{g(z)=z+2a(1-\alpha)z^2+\cdots: zg'/g\in P_a(\alpha)\}\,.$$

For still another class $R'_a(\alpha)$ defined by $R'_a(\alpha)=\{f(z)=z+a(1-\alpha)z^2+\cdots: |f'(z)-1|<\alpha, 1/2<\alpha\leqq 1, z\in E\}$ Goel [4] determined the radius of convexity.

In the present paper, we propose an approach by which it is not only possible to have a unified study of the above mentioned classes but of various other classes as well. For this purpose we introduce the following classes:

$$P(\alpha, \beta) = \{p(z) = 1 + b_1 z + \cdots : |(p(z)-1)/\{2\beta(p(z)-\alpha) - (p(z)-1)\}| < 1, \quad \text{for } \alpha \in [0, 1),\ \beta \in (0, 1] \text{ and } z \in E\}$$

$$P_b(\alpha, \beta) = \{p \in P(\alpha, \beta) : p'(0) = 2b\beta(1-\alpha),\ 0 \leqq b \leqq 1\}$$

$$R_a(\alpha, \beta) = \{f(z) = z + a\beta(1-\alpha)z^2 + \cdots : f' \in P_a(\alpha, \beta),\ 0 \leqq a \leqq 1\}$$

$$S_a^*(\alpha, \beta) = \{g(z) = z + 2a\beta(1-\alpha)z^2 + \cdots : zg'/g \in P_a(\alpha, \beta),\ 0 \leqq a \leqq 1\}$$

and determine sharp estimates for the radii of convexity for functions in $R_a(\alpha, \beta)$ and $S_a^*(\alpha, \beta)$.

2. Preliminary lemmas. Let B denote the class of analytic functions $w(z)$ in E which satisfy the conditions $w(0)=0$ and $|w(z)|< 1$ for $z \in E$. We require the following lemmas:

LEMMA 1 [15]. *If $w \in B$, then for $z \in E$*

$$|zw'(z) - w(z)| \leqq \frac{|z|^2 - |w(z)|^2}{1-|z|^2}. \tag{2.1}$$

LEMMA 2. *Let $w \in B$. Then we have*

$$\operatorname{Re}\left\{\frac{zw'(z)}{(1+sw(z))(1+tw(z))}\right\} \leqq -\frac{1}{(s-t)^2}\operatorname{Re}\left\{sp(z) + \frac{t}{p(z)} - s - t\right\} + \frac{1}{(s-t)^2}\frac{r^2|sp(z)-t|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|} \tag{2.2}$$

where $p(z) = (1+tw(z))/(1+sw(z))$, $|z| = r$ and $-1 \leqq t < s \leqq 1$.

Using the estimate (2.1), the lemma follows easily. Hence we omit the proof.

LEMMA 3. *If $p(z) = (1+tw(z))/(1+sw(z))$, $w \in B$, then for each $b \in [0, 1]$ and s, t satisfying $-1 \leqq t < s \leqq 1$, $p(z)$ lies in the disc*

$$\Delta(z) \equiv \{\zeta : |\zeta - A_b| \leqq D_b\},$$

where

$$A_b = \frac{(1+br)^2 - str^2(b+r)^2}{(1+br)^2 - s^2r^2(b+r)^2};\quad D_b = \frac{(s-t)r(b+r)(1+br)}{(1+br)^2 - s^2r^2(b+r)^2}$$

and $r = |z| < 1$.

Proof. Since $p(z) = (1 + tw(z))/(1 + sw(z))$, we have

$$w(z) = \frac{1 - p(z)}{sp(z) - t} = -[bz + \cdots] = -z\phi(z) \tag{2.3}$$

where ϕ is analytic and $|\phi(z)| \leqq 1$ for $z \in E$ with $\phi'(0) = b$. Now, since $(\phi(z) - b)/(1 - b\phi(z))$ is subordinate to z, it follows that $\phi(z)$ is subordinate to $(z + b)/(1 + bz)$ and so

$$\left|\frac{1 - p(z)}{sp(z) - t}\right| \leqq |z|\frac{(|z| + b)}{(1 + b|z|)} . \tag{2.4}$$

Putting $p(z) = \xi + i\eta$, (2.4) gives

$$\left|\xi + i\eta - \frac{(1 + br)^2 - str^2(b + r)^2}{(1 + br)^2 - s^2r^2(b + r)^2}\right| \leqq \frac{(s - t)r(b + r)(1 + br)}{(1 + br)^2 - s^2r^2(b + r)^2} .$$

Hence the lemma.

LEMMA 4. *If* $p(z) = (1 + tw(z))/(1 + sw(z))$, $w \in B$, *then for* $|z| = r$, $0 \leqq r < 1$, *we have*

$$\mathrm{Re}\left\{kp(z) + \frac{t}{p(z)}\right\} - \frac{r^2|sp(z) - t|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}$$

$$\geqq \begin{cases} \dfrac{2}{1 - r^2}[\sqrt{(1 + t)(1 - tr^2)(k(1 - r^2) + 1 - s^2r^2)} - (1 - str^2)] & \text{if } R_b \leqq R^* \\ W/W^* & \text{if } R_b \geqq R^* \end{cases} \tag{2.5}$$

where

$$\begin{aligned} W = {} & t(kt + s^2)r^4 + 2bt\{(k + s) + (kt + s^2)\}r^3 \\ & + [b^2(1 + t)\{(k + t) + (kt + s^2)\} + 2t(k + s) - (s - t)^2]r \\ & + 2b\{(k + t) + t(k + s)\}r + (k + t) , \end{aligned} \tag{2.5; a}$$

$$W^* = \{1 + rb(1 + t) + tr^2\}\{1 + rb(1 + s) + sr^2\} \tag{2.5; b}$$

and $R^{*2} = (1 + t)(1 - tr^2)/(k(1 - r^2) + 1 - s^2r^2)$, $R_b = A_b - D_b$ *where* A_b, D_b *are defined as in Lemma* 3 *and* $k \geqq s$, $-1 \leqq t < s \leqq 1$.

Proof. Let $|z| = r$, *and* $p(z) = A_b + \xi + i\eta \equiv \mathrm{Re}^{i\psi}$, then $-\pi/2 < \psi < \pi/2$. Denoting the left hand side of (2.5) by

$U_b(\xi, \eta)$, we get

$$U_b(\xi, \eta) = k(A_b + \xi) + t(A_b + \xi)R^{-2} + \frac{1 - s^2r^2}{1 - r^2}[((A_b + \xi) - A_1)^2 + \eta^2 - D_1^2]R^{-1} \tag{2.6}$$

and

$$\frac{\partial U_b}{\partial \eta} = \eta R^{-4} V_b(\xi, \eta) \tag{2.7}$$

where

$$V_b(\xi, \eta) = -2t(A_b + \xi) + (D_1^2 + 2A_1(A_b + \xi) - A_1^2)\left(\frac{1 - s^2 r^2}{1 - r^2}\right)R + \left(\frac{1 - s^2 r^2}{1 - r^2}\right)R^3$$

$$= -2tR\cos\psi + (D_1^2 - A_1^2 + 2A_1 R\cos\psi)\left(\frac{1 - s^2 r^2}{1 - r^2}\right)R + \left(\frac{1 - s^2 r^2}{1 - r^2}\right)R^3 \equiv M_b(R, \psi)(\text{say})\,. \tag{2.8}$$

Since for fixed r, $0 \leqq r < 1$, $A_b - D_b$ decreases as b increases over the interval $[0, 1]$, it follows that $R \geqq R\cos\psi \geqq A_b - D_b \geqq A_1 - D_1$. Thus, for all b, $0 \leqq b \leqq 1$,

$$M_b(R, \psi) \geqq R\cos\psi\left[-2t + (D_1^2 - A_1^2 + 2A_1 R\cos\psi + R^2)\left(\frac{1 - s^2 r^2}{1 - r^2}\right)\right]$$

$$\geqq 2R\cos\psi\left[\left(\frac{1 - s^2 r^2}{1 - r^2}\right)(A_1 - D_1)^2 - t\right] > 0,$$

for all s, t satisfying $-1 \leqq t < s \leqq 1$. Thus $V_b(\xi, \eta) \equiv M_b(R, \psi)$ is positive for all points in the disc $\Delta(z)$. Now, (2.7) gives that, for every fixed ξ, $U_b(\xi, \eta)$ is increasing function of η for positive η and is a decreasing function of η for negative η. Thus, the minimum of $U_b(\xi, \eta)$ inside the disc Δ is attained on the diameter forming part of the real axis. Setting $\eta = 0$ in (2.6), we obtain

$$\min_{-1 \leqq \eta \leqq 1} U_b(\xi, \eta) \equiv N_b(R) = \left(k + \frac{1 - s^2 r^2}{1 - r^2}\right)R + \frac{(1 + t)(1 - tr^2)}{(1 - r^2)} R^{-1} - 2A_1\left(\frac{1 - r^2 s^2}{1 - r^2}\right) \tag{2.9}$$

where $R = A_b + \xi \in [A_b - D_b, A_b + D_b]$. Thus the absolute minimum of $N_b(R)$ in $(0, \infty)$ is attained at

$$R^* = \left(\frac{(1 + t)(1 - tr^2)}{k(1 - r^2) + 1 - s^2 r^2}\right)^{1/2} \tag{2.10}$$

and the value of this minimum is equal to

$$N_b(R^*) = \frac{1}{1-r^2}\left[\sqrt{(k(1-r^2)+1-s^2r^2)(1+t)(1-tr^2)} - (1-str^2)\right]. \tag{2.11}$$

Since it is easily seen that $R^* < A_1 + D_1$ and that $A_b + D_b$ is a decreasing function of b for $0 \leqq b \leqq 1$, it follows that $R^* < A_b + D_b$ for $b \in [0, 1]$; but R^* is not always greater than $A_b - D_b$. In case $R^* \notin [A_b - D_b, A_b + D_b]$, it can be easily verified that $N_b(R)$ increases with R in $[A_b - D_b, A_b + D_b]$. Thus the minimum of $N_b(R)$ on the segment $[A_b - D_b, A_b + D_b]$ is attained at $R_b = A_b - D_b$. The value of this minimum equals

$$N_b(R_b) \equiv N_b(A_b - D_b) = W/W^*,$$

where W and W^* are given by (2.5; a) and (2.5; b). Moreover $N_b(R^*) = N_b(R_b)$ for those values of k, s, and t for which $R_b = R^*$. Hence the lemma.

3. The class $R_a(\alpha, \beta)$. Let $R(\alpha, \beta)$ be the class of functions $f(z) = z + a_2z^2 + \cdots$ which are analytic and satisfy the inequality $|(f'(z)-1)/\{2\beta(f'(z)-\alpha) - (f'(z)-1)\}| < 1$ for some $\alpha, \beta (0 \leqq \alpha < 1, 0 < \beta \leqq 1)$ and $z \in E$. One of the authors [9] has shown that for $f \in R(\alpha, \beta)$, $|a_2| \leqq \beta(1-\alpha)$. Define

$$R_a(\alpha, \beta) = \{f(z) = z + a\beta(1-\alpha)z^2 + \cdots : f' \in P_a(\alpha, \beta),\ 0 \leqq a \leqq 1\}.$$

Now, we determine a sharp estimate for the radii of convexity for functions in $R_a(\alpha, \beta)$.

THEOREM 1. *Let $f \in R_a(\alpha, \beta)$, then f is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation*

$$1 + 4\alpha\beta ar + (4\alpha\beta^2a^2 - 2(1+\beta-3\alpha\beta))r^2 + 4\beta(2\alpha\beta-1)ar^3 + (2\beta-1)(2\alpha\beta-1)r^4 = 0$$

if $R_a \geqq R^*$ and

$$r_0 = [\{-\alpha\beta + \sqrt{\alpha(1-2\alpha\beta+\alpha\beta^2)}\}/(1-2\alpha\beta)]^{1/2}$$

if $R_a \leqq R^*$ where

$$R_a = \frac{1 + 2\alpha\beta ar + (2\alpha\beta-1)r^2}{1 + 2\beta ar + (2\beta-1)r^2}, \quad R^* = \left(\frac{\alpha(1-(2\alpha\beta-1)r^2)}{1-(2\beta-1)r^2}\right)^{1/2}$$

and $r = |z| < 1$. The result is sharp for each $\alpha, \beta (0 \leqq \alpha < 1, 0 < \beta \leqq 1)$ and $0 \leqq a \leqq 1$.

Proof. Since $f \in R_a(\alpha, \beta)$, an application of Schwarz's lemma gives

$$(3.1) \qquad f'(z) = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)}$$

where $w \in B$. Logarithmic differentiation of (3.1) gives

$$(3.2) \quad 1 + z\frac{f''(z)}{f'(z)} = 1 - 2\beta(1-\alpha)\left\{\frac{zw'(z)}{(1+(2\beta-1)w(z))(1+(2\alpha\beta-1)w(z))}\right\}.$$

Applying (2.2) with $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (3.2), we get

$$(3.3) \quad \operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geqq \frac{1}{2\beta(1-\alpha)}\left[\operatorname{Re}\left\{(2\beta - 1)p(z) + \frac{2\alpha\beta - 1}{p(z)}\right\} - \frac{r^2|(2\beta - 1)p(z) + 1 - 2\alpha\beta|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}\right] + \frac{1 - 2\alpha\beta}{\beta(1 - \alpha)}$$

where $p(z) = (1 + (2\alpha\beta - 1)w(z))/(1 + (2\beta - 1)w(z))$. An application of Lemma 4 with $k = s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (3.3) gives

$$(3.4) \quad \operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geqq \begin{cases} \dfrac{1}{\beta(1-\alpha)(1-r^2)}[\sqrt{4\alpha\beta^2(1-(2\beta-1)r^2)(1+(1-2\alpha\beta)r^2)} \\ \quad -(1+(1-2\alpha\beta)(2\beta-1)r^2)+(1-2\alpha\beta)(1-r^2)] \\ \quad \text{if } R_a \leqq R^*, \\ \dfrac{1+4\alpha\beta ar+(4\alpha\beta^2a^2-2(1+\beta-3\alpha\beta))r^2 + 4\beta\times(2\alpha\beta-1)ar^3+(2\beta-1)(2\alpha\beta-1)r^4}{(1+2\beta ar+(2\beta-1)r^2)(1+2\alpha\beta ar+(2\alpha\beta-1)r^2)} \\ \quad \text{if } R_a \geqq R^* \end{cases}$$

where

$$R_a = \frac{1 + 2\alpha\beta ar + (2\alpha\beta - 1)r^2}{1+2\beta ar + (2\beta - 1)r^2}, \quad R^* = \left(\frac{\alpha(1 - (2\alpha\beta - 1)r^2)}{1 - (2\beta - 1)r^2}\right)^{1/2},$$
$$0 \leqq a \leqq 1.$$

Now the theorem follows easily from (3.4).

The function given by

$$f'(z) = \frac{1 - 2\alpha\beta az + (2\alpha\beta - 1)z^2}{1 - 2\beta az + (2\beta - 1)z^2} \quad \text{if } R_a \geqq R^*,$$

and

$$f'(z) = \frac{1 - 2\alpha\beta cz + (2\alpha\beta - 1)z^2}{1 - 2\beta cz + (2\beta - 1)z^2} \quad \text{if } R_a \leqq R^*$$

where c is determined by the relation

$$\frac{1 - 2\alpha\beta cr + (2\alpha\beta - 1)r^2}{1 - 2\beta cr + (2\beta - 1)r^2} = R^* = \sqrt{\frac{\alpha(1 + (1 - 2\alpha\beta)r^2)}{(1 - (2\beta - 1)r^2)}}$$

show that the results obtained in the theorem are sharp.

Putting $\beta = 1$, in Theorem 1, we get the following result due to McCarty [8].

COROLLARY 1(a). *Each $f \in R_a(\alpha)$ maps $|z| < r_0$ onto a convex region where r_0 is the smallest positive root of the equation*

$$1 + 4\alpha ar + (6\alpha - 4 + 4\alpha a^2)r^2 + 4(2\alpha - 1)ar^3 + (2\alpha - 1)r^4 = 0$$

if $R_a \geqq R^*$ and

$$r_0 = [\{-\alpha + \sqrt{\alpha(1 - \alpha)}\}/(1 - 2\alpha)]^{1/2}$$

if $R_a \leqq R^*$, where

$$R_a = \frac{1 + 2\alpha ar + (2\alpha - 1)r^2}{1 + 2ar + r^2}, \quad R^* = \left(\frac{\alpha(1 - (2\alpha - 1)r^2)}{1 - r^2}\right)^{1/2}$$

and $r = |z| < 1$. The result is sharp for each α $(0 \leqq \alpha < 1)$ and $0 \leqq a \leqq 1$.

COROLLARY 1(b). *Let $f \in R'_a(\alpha)$, then f is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation*

$$1 + 2(1 - \alpha)ar + ((1 - \alpha)a^2 - 3\alpha)r^2 - 2\alpha ar^3 = 0$$

if $R_a \geqq R^*$ and

$$r_0 = [\{-(1 - \alpha) + \sqrt{(1 - \alpha)(1 + 3\alpha)}\}/2\alpha]^{1/2}$$

if $R_a \leqq R^*$, where

$$R_a = \frac{1 + (1 - \alpha)ar - \alpha r^2}{1 + ar}, \quad R^* = [(1 - \alpha)(1 + \alpha r^2)]^{1/2}$$

and $r = |z| < 1$. The result is sharp for each α $(0 \leqq \alpha < 1)$ and $0 \leqq a \leqq 1$.

The result is obtained by replacing α by $1 - \alpha$ and β by $1/2$ in Theorem 1. It may be noted that this result was obtained by Goel [4] under the additional restriction $1/2 \leqq \alpha \leqq 1$.

REMARK. Replacing (α, β) by $(0, 1)$, or by $(0, 1 - \delta)$, $0 \leqq \delta < 1$ or by $(0, (2\delta - 1)/2\delta)$, $1/2 < \delta \leqq 1$, or by $((1 - \gamma)/1 + \gamma, (1 + \gamma(/2)$, $0 < \gamma \leqq 1$, or by $((1 - \delta + 2\gamma\delta)/(1 + \delta), (1 + \delta)/2)$, $0 \leqq \gamma < 1, 0 < \delta \leqq 1$, we get the estimates for the radii of convexity for functions with fixed second coefficient of the classes introduced and studied by MacGregor [7], Shaffer [13], Goel [3], Caplinger and Causey [1] and the authors [6] respectively.

4. The class $S_a^*(\alpha, \beta)$. Let $S^*(\alpha, \beta)$ be the class of functions $g(z) = z + a_2z^2 + \cdots$ which are analytic and satisfy the inequality $|(zg'(z)/g(z) - 1)/\{2\beta(zg'(z)/g(z) - \alpha) - (zg'(z)/g(z)-1)\}| < 1$, for some $\alpha, \beta (0 \leqq \alpha < 1, 0 < \beta \leqq 1)$ and $z \in E$. The authors [5] have shown that for $g \in S^*(\alpha, \beta)$, $|a_2| \leqq 2\beta(1 - \alpha)$. Define

$$S_a^*(\alpha, \beta) = \{g(z) = z + 2a\beta(1-\alpha)z^2 + \cdots : zg'/g \in P_a(\alpha, \beta),\ 0 \leqq a \leqq 1\}.$$

Now, we determine a sharp estimate for the radii of convexity for functions in $S_a^*(\alpha, \beta)$.

THEOREM 2. *Let $g \in S_a^*(\alpha, \beta)$, then g is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation*

$$1 + 2\beta(3\alpha - 1)ar + (4\alpha^2\beta^2a^2 + 8\alpha\beta - 2 - 4\beta)r^2 - 2\beta(1 + \alpha - 4\beta\alpha^2)ar^3 + (1 - 2\alpha\beta)^2r^4 = 0$$

if $R_a \geqq R^*$ and

$$r_0 = [(5\alpha - 1)/\{(1 - \alpha + 4\beta\alpha^2) + 4\alpha\sqrt{(1 + \beta - 3\alpha\beta + \alpha^2\beta^2)}\}]^{1/2}$$

if $R_a \leqq R^*$, where

$$R_a = \frac{1 + 2\alpha\beta ar + (2\alpha\beta - 1)r^2}{1 + 2\beta ar + (2\beta - 1)r^2}, \quad R^* = \left(\frac{\alpha(1 + (1 - 2\alpha\beta)r^2)}{(2-\alpha)-(2\beta-\alpha)r^2}\right)^{1/2}$$

and $r = |z| < 1$. The result is sharp for each α, β $(0 \leqq \alpha < 1, 0 < \beta \leqq 1)$ and $0 \leqq a \leqq 1$.

Proof. Since $g \in S_a^*(\alpha, \beta)$, an application of Schwarz's lemma gives

$$z\frac{g'(z)}{g(z)} = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)} \tag{4.1}$$

where $w \in B$. Logarithmic differentiation of (4.1) gives

$$1 + z\frac{g''(z)}{g'(z)} = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)} - 2\beta(1 - \alpha)\left\{\frac{zw'(z)}{(1 + (2\beta - 1)w(z))(1 + (2\alpha\beta - 1)w(z))}\right\}. \tag{4.2}$$

Applying (2.2) with $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (4.2), we get

$$\text{(4.3)} \quad \operatorname{Re}\left\{1 + z\frac{g''(z)}{g'(z)}\right\} \geqq \frac{1}{2\beta(1-\alpha)}\Bigg[\operatorname{Re}\left\{(4\beta - 1 - 2\alpha\beta)p(z) + \frac{2\alpha\beta - 1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)+1-2\alpha\beta|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\Bigg] + \frac{\alpha+\alpha\beta-1}{\beta(1-\alpha)}$$

where $p(z) = (1 + (2\alpha\beta - 1)w(z))/(1 + (2\beta - 1)w(z))$. Now, an application of Lemma 4 with $k = 4\beta - 1 - 2\alpha\beta$, $s = 2\beta - 1$ and $t = 2\alpha\beta - 1$ to (4.3) gives the required results easily.

The functions given by

$$z\frac{g'(z)}{g(z)} = \frac{1 - 2\alpha\beta az + (2\alpha\beta - 1)z^2}{1 - 2\beta az + (2\beta - 1)z^2} \text{ if } R_a \geqq R^*$$

and

$$z\frac{g'(z)}{g(z)} = \frac{1 - 2\alpha\beta cz + (2\alpha\beta - 1)z^2}{1 - 2\beta cz + (2\beta - 1)z^2} \text{ if } R_a \leqq R^*$$

where c is determined by the relation

$$\frac{1 - 2\alpha\beta cr + (2\alpha\beta - 1)r^2}{1 - 2\beta cr + (2\beta - 1)r^2} = R^* = \left(\frac{\alpha(1 - (2\alpha\beta - 1))r^2}{(2-\alpha)-(2\beta-\alpha)r^2}\right)^{1/2}$$

show that the results obtained in the theorem are sharp.

Taking $\beta = 1$, in Theorem 2, we get the following result due to McCarty [8] which also includes the result obtained by Tepper [16].

COROLLARY 2(a). *Each* $g \in S_a^*(\alpha)$ *maps* $|z| < r_0$ *onto a convex region where* r_0 *is the smallest positive root of the equation*

$$1 + (6\alpha - 2)ar + (4\alpha^2a^2 + 8\alpha - 6)r^2 + (8\alpha^2 - 2\alpha - 2)ar^3 + (2\alpha - 1)^2r^4 = 0$$

if $R_a \geqq R^*$ *and*

$$r_0 = [(5\alpha - 1)/\{(4\alpha^2 - \alpha + 1) + 4\alpha\sqrt{(\alpha^2 - 3\alpha + 2)}\}]^{1/2}$$

if $R_a \leqq R^*$ *where*

$$R_a = \frac{1 + 2\alpha ar + (2\alpha - 1)r^2}{1 + 2ar + r^2}, \quad R^* = \left(\frac{\alpha(1 - (2\alpha - 1))r^2}{(2 - \alpha)(1 - r^2)}\right)^{1/2}$$

and $r = |z| < 1$. *The result is sharp for each* α $(0 \leqq \alpha < 1)$ *and* $0 \leqq a \leqq 1$.

REMARKS. (i) Replacing (α, β) by $(0, 1/2)$, or by $(0, (2\delta - 1)/2\delta)$, $1/2 < \delta \leqq 1$, or by $((1 - \gamma)/1 + \gamma, (1 + \gamma)/2)$, $0 < \gamma \leqq 1$, we may obtain the estimates for the radii of convexity for functions with fixed second coefficient of the classes introduced and studied by Eenigenburg [2], Ram Singh [14] and Padmanabhan [10] respectively.

(ii) Setting $a = 1$ in Theorem 1 and Theorem 2 we get the sharp estimates for the radii of convexity for functions in $R(\alpha, \beta)$ and $S^*(\alpha, \beta)$. These were obtained by the authors in [9] and [5] and thus also include the results obtained in [1], [2], [13] etc.

(iii) By setting $a = 0$ in Theorem 1 and Theorem 2, we may get the results for functions in $R(\alpha, \beta)$ and $S^*(\alpha, \beta)$ with missing second coefficient and in particular for odd functions in these classes.

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REFERENCES

1. T. R. Caplinger and W. M. Causey, *A class of univalent functions*, Proc. Amer. Math. Soc., **39** (1973), 357-361.

2. P. J. Eenigenburg, *A class of starlike mappings in the unit disc*, Compositio Math., **24** (1972), 235-238.

3. R. M. Goel, *A class of univalent functions whose derivatives have positive real part in the unit disc*, Nieuw Arch. Wisk., **15** (1967), 55-63.

4. ———, *A class of univalent functions with fixed second coefficients*, J. Math. Sci., **4** (1969), 85-92.

5. O. P. Juneja and M. L. Mogra, *On starlike functions of order α and type β*, Rev. Roumaine Math. Pures Appl., **23** (1978), 751-765.

6. ———, *A class of univalent functions*, Bull. Sci. Math., (To appear).

7. T. H. MacGregor, *Functions whose derivatives have positive real part*, Trans. Amer. Math. Soc., **104** (1962), 532-537.

8. C. P. McCarty, *Two radius of convexity problems*, Proc. Amer. Math. Soc., **42** (1974). 153-160.

9. M. L. Mogra, *On a class of univalent functions whose derivatives have a positive real part*, (Communicated).

10. K. S. Padmanabhan, *On certain classes of starlike functions in the unit disc*, J. Indian Math. Soc., **32** (1968), 89-103.

11. M. S. Robertson, *Variational methods for functions with positive real part*, Trans. Amer. Math. Soc., **102** (1962), 82-93.

12. ———, *Extremal problems for analytic functions with positive real part and applications*, Trans. Amer. Math. Soc., **106** (1963), 236-253.

13. D. B. Shaffer, *Distortion theorems for a special class of analytic functions*, Proc. Amer. Math. Soc., **39** (1973), 281-287.

14. R. Singh, *On a class of starlike functions II*, Ganita, **19** (1968), 103-110.

15. V. Singh and R. M. Goel, *On radii of convexity and starlikeness of some classes of functions*, J. Math. Soc. Japan, **23** (1971), 323-339.

16. D. E. Tepper, *On the radius of convexity and boundary distortion of schlicht functions*, Trans. Amer. Math. Soc., **150** (1970), 519-528.

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