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ON REPRESENTING ANALYTIC GROUPS WITH THEIR AUTOMORPHISMS

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A real or complex Lie group is said to be *faithfully representable* if it has a faithful finite-dimensional analytic representation. Let G be a real or complex analytic group, and let A denote the group of all analytic automorphisms of G, endowed with its natural structure of a real or complex Lie group. The natural semidirect product $G \rtimes A$ is a real or complex Lie group, sometimes called the *holomorph* of G. We show that if G is faithfully representable and if the maximum nilpotent normal analytic subgroup of G is simply connected then $G \rtimes A$ is faithfully representable.

This result follows quite easily from well-known representation-theoretical results and techniques. What we use is contained in [1, Ch. XVIII], and all the references given below are to this. Nominally, these references cover only the real case. However, as explained loc. cit., both the results and their proofs are almost identical in the complex case.

Thanks are due to Martin Moskowitz who drew my attention to this question and who obtained a number of special results that are consequences of the theorem below and contain suggestions for its proof.

PROPOSITION. *Let G be a faithfully representable real or complex analytic group, and let N be the maximum nilpotent normal analytic subgroup of G. If N is simply connected there is a faithful finite-dimensional analytic representation of G whose restriction to N is unipotent.*

Proof. By Theorem 4.3 (or 4.7), G is a semidirect product $B \rtimes H$, where B is solvable and simply connected, and H is reductive. The construction of B is carried out in the proof of Theorem 4.2, and this shows that, if N is simply connected, one can arrange to have $N \subset B$ (one begins with a semidirect product decomposition of the radical of G having the form $M \rtimes Q$, where Q is reductive, and M is simply connected *and contains* N).

By Theorem 3.1, there exists a faithful finite-dimensional analytic representation ρ of B whose restriction to N is unipotent. Now ρ satisfies the conditions of Theorem 2.2, so that (enlarging the representation space of ρ) one can extend ρ to a finite-dimensional analytic representation of G whose restriction to B is faithful and whose restriction to N is still unipotent. Any given faithful finite-dimensional analytic representation of G, via restriction to H and the canonical homomorphism $G \to H$, yields a finite-dimensional analytic representation of G whose kernel is precisely B. The direct sum of this and the representation obtained above satisfies the requirements of the proposition.

THEOREM. *Let G be a faithfully representable real or complex analytic group. Let N be the maximum nilpotent normal analytic subgroup of G, and let A be the group of all analytic automorphisms of G. If N is simply connected then $G \rtimes A$ is faithfully representable.*

Proof. Let B denote the group of all Lie algebra automorphisms of the Lie algebra $\mathscr{L}(G)$ of G. This is an algebraic linear group, and we denote its *algebraic* identity component by B_1. The Lie algebra of B_1 may be identified with the Lie algebra of all derivations of $\mathscr{L}(G)$. If R is the radical of G then $\mathscr{L}(R)$ is the radical of $\mathscr{L}(G)$, while $\mathscr{L}(N)$ is the maximum nilpotent ideal of $\mathscr{L}(G)$. Therefore, by a well-known result from Lie algebra theory, every derivation of $\mathscr{L}(G)$ sends $\mathscr{L}(R)$ into $\mathscr{L}(N)$. This implies that B_1 acts trivially on $\mathscr{L}(R)/\mathscr{L}(N)$.

Let A° denote the group of all elements of A whose canonical images in B belong to B_1. Then A° acts trivially on R/N. Since B_1 is normal and of finite index in B, the group A° is normal and of finite index in A.

Let I denote the group of inner automorphisms of G. Clearly, $I \subset A^\circ$. Now let S be a maximal semisimple analytic subgroup of G, and let T be the subgroup of A° consisting of the elements of A° that keep the elements of S fixed. Since every maximal semisimple analytic subgroup of G is a G-conjugate of S, every coset of I in A contains an element that stabilizes S. It follows from this that the group TI is normal in A°. Since the group of inner automorphisms of S is of finite index in the group of all analytic automorphisms of S, it follows also that TI is of finite index in A°.

Clearly, A° contains the identity component A_1 of A. If n is the index of TI in A° then a^n belongs to TI for every a in A_1. Since A_1 is an analytic group, these elements a^n generate A_1. Hence $A_1 \subset TI$. We conclude that TI is open and of finite index in A.

Now let ρ be a representation of G with the properties stated in the proposition. Let V denote the space of representative functions on G that are associated with ρ. Then V is finite-dimensional and stable under the right and left translation actions of G. If ρ'

is the semisimple representation of G that is associated with ρ then N is contained in the kernel of ρ'. Let x be an element of the radical of G, and let τ be an element of T. Then $\tau(x)x^{-1} \in N$, so that $\rho'(\tau(x)x^{-1})$ is the identity map on the representation space. Since the elements of T keep the elements of S fixed, it is now clear that Lemma 2.1 applies to our present situation and gives the result that the space $V \circ T$ spanned by the composites of the elements of V with those of T is still finite-dimensional. From the 2-sided G-stability of V, it is clear that $V \circ (TI) = V \circ T$.

Let us write J for TI, and W for $V \circ T$. We define an action of $G \rtimes J$ on W by setting $(x, \alpha)\cdot w = x\cdot(w \circ \alpha^{-1})$, where, for any function f on G, the *translate* $x \cdot f$ is defined by $(x \cdot f)(y) = f(yx)$. Thus, our definition means that the value of $(x, \alpha)\cdot w$ at y is $w(\alpha^{-1}(yx))$. In order to verify that this is indeed a representation of $G \rtimes J$ on W, it suffices to check that, compatibly with $\alpha x = \alpha(x)\alpha$ (in $G \rtimes J$), one has $\alpha\cdot(x\cdot w) = \alpha(x)\cdot(\alpha\cdot w)$. Evidently, this representation is analytic, and its restriction to G is faithful.

In order to obtain a representation of $G \rtimes A$ whose restriction to G is faithful, we use the ordinary group algebra $F[G \rtimes A]$, where F is the field of real numbers or the field of complex numbers. We form the tensor product $F[G \rtimes A] \otimes_{F[G \rtimes J]} W$ and let $G \rtimes A$ act via multiplication on the left factor. As a vector space, this module is the direct sum of $[A: J]$ copies of W, and thus is finite-dimensional. Since $A_1 \subset J$, it is clear that this representation is analytic.

Finally, we make $\mathscr{L}(G)$ into a $G \rtimes A$ – module via the canonical homomorphisms $G \rtimes A \to A \to B$, so that the kernel of this representation of $G \rtimes A$ on $\mathscr{L}(G)$ is precisely G. The direct sum of the two representations we have constructed satisfies the requirements of our theorem.

The simplest example of a faithfully representable analytic group G such that $G \rtimes A$ is not faithfully representable is the direct product $\boldsymbol{R} \times T$ of the additive group $\boldsymbol{R}$ of real numbers and the multiplicative group T of complex numbers of absolute value 1. For every real number a, this group has an analytic automorphism a^*, where $a^*(r, u) = (r, \exp(iar)u)$. If S is the space of representative functions associated with a representation of G that is not trivial on T, it is easy to see that the space spanned by the functions $f \circ a^*$ with f in S and a in $\boldsymbol{R}$ is not finite-dimensional.

We may summarize our results as follows.

Summary. *Let K be a real or complex Lie group having a semidirect product decomposition $G \rtimes H$, where G is connected and faithfully representable, and the maximum nilpotent normal analy-*

tic subgroup N of G is simply connected. There is a finite-dimensional analytic representation of K whose restriction to G is faithful and whose restriction to N is unipotent. If H is faithfully representable there is a faithful such representation.

Proof. Let A denote the group of all analytic automorphisms of G. The semidirect product decomposition of K defines an analytic homomorphism $\eta: H \to A$ and hence an analytic homomorphism η^*: $K \to G \rtimes A$. If ρ is a representation of $G \times A$ as obtained in the theorem, then $\rho \circ \eta^*$ clearly satisfies the requirements of the first statement of the summary. The second statement then follows immediately.

Reference

1. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965.

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