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NORMAL EXPECTATIONS AND INTEGRAL DECOMPOSITION OF TYPE III VON NEUMANN ALGEBRAS

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Let M be a σ-finite type III von Neumann algebra with separating and cyclic vector ζ (on a not necessarily separable Hilbert space), let C be the center of M, let e be the projection corresponding to the subspace generated by $C\zeta$, and let $\tau(x)$ be the unique element in C with $\tau(x)e=exe$ for x in M. For χ in the spectrum X of C, let ρ_χ be the canonical representation of the state $\tau_\chi(x)=\tau(x)^\wedge(\chi)$. The integral $\int\tau_\chi(x)d\nu(\chi)$ induces the central decomposition of M. A separable C^*-algebra B of M is found so that $\rho_\chi(M)''$ has a σ-weakly continuous projection of norm one on $\rho_\chi(B)''$, and $\rho_\chi(B)''$ is a type III factor on an open dense set of X. It is shown that $\rho_\chi(M)''$ is type III and that τ_χ has a decomposition (in the sense of Choquet-Bishop-de Leeuw) as an integral of type III functionals quasi-supported by primary type III functionals for χ in the open dense set.

1. Introduction. One may write every *normal* (i.e., σ-weakly continuous positive linear) functional ϕ of a von Neumann algebra M as an integral of a field of linear functionals over a base space. Several different choices are possible. The field of states (i.e., of positive functionals ψ with $\psi(1) = 1$) can be taken, and the measure can be taken to be a Borel measure *quasi-supported* in the sense of Choquet-Bishop-de Leeuw by the *primary* functionals (i.e., functionals whose canonical representations produce factor von Neumann algebras) in that every Baire set disjoint from the set of primary functionals has measure 0 [26], [37], [44], [45]. It appears there is not much information on whether the measure is supported in some way by functionals whose type (i.e., functionals whose canonical representations have type) corresponds to the type of the algebra M.

One may decompose ϕ in another way. The algebra M may be considered as a Banach module over its center C. The functional ϕ can be written as $(\phi|C)\circ\Phi$ where Φ is in the positive cone $M_\sim^+$ (i.e., $\Phi(M^+)\subset C^+$) of the space $M_\sim$ of σ-weakly continuous C-module homomorphisms of M into C. Defining the field of functionals $\{\phi_\chi|\chi\in X\}$ over the spectrum X of C by $\phi_\chi(x) = \Phi(x)^\wedge(\chi)$, one gets the representation

$$\phi(x) = \int\phi_\chi(x)^\wedge d\mu(\chi)$$

where μ is the usual spectral measure on X given by $\phi(x) = \int x^{\wedge}(\chi)d\mu(\chi)$ for x in C. Here $x^{\wedge}$ is the Gelfand transform of x [10]. One can also write the canonical representation ρ_ϕ of ϕ as the direct integral of the canonical representations $\{\rho_\chi\}$ of the field of functionals $\{\phi_\chi\}$ so that the Hilbert space $H(\phi)$ of ρ_ϕ corresponds to the direct integral of the field of Hilbert spaces of the canonical representations ρ_χ. In the usual framework, the algebra M is generated by C and a countable*-subalgebra $\{x_i\}$ of M over the rational complex number C_r and the components of the direct integral decomposition of the Hilbert space consist of the fields $\{\rho_\chi(x_i)\zeta_\chi\}_i$, where ζ_χ is the cyclic vector of $\rho_\chi(M)$ with

$$\omega_{\zeta_\chi}(\rho_\chi(x)) = (\rho_\chi(x)\zeta_\chi, \zeta_\chi) = \phi_\chi(x)$$

for x in M, and the components of the direct integral decomposition of the algebra is the von Neumann algebra on clos $\{\rho_\chi(x_i)\zeta_\chi\}$ generated by the restrictions of the operators $\{\rho_\chi(x_i)\}_i$ (cf. [5], [14], [15], [19], [20], [33]). Here, however, we seek information on the von Neumann algebra $\rho_\chi(M)''$ generated by $\rho_\chi(M)$ on $H_\chi = H(\phi_\chi)$ even in the general case where the algebra is not countably generated over its center. We have already proved that there is a map (χ) of X into disjoint quasi-equivalence classes of representations of M such that, for every Ψ in $M_\sim^+$ with $\Psi(1) = 1$, the canonical representation induced by $\Psi(\cdot)^{\wedge}(\chi)$ is in the class (χ) except possibly for a nowhere dense set [11]. Also if M is of type I (resp. type II), then the classes (χ) are classes of type I (resp. type II) factor representations [11], [31].

In this article, we study the classes (χ) of representations for type III algebras. We show that every σ-finite von Neumann algebra M has a σ-weakly continuous projection of norm 1 onto a countably generated algebra N of the same type. If M is of type III, the algebra N can be chosen to have a weakly dense separable *-subalgebra B so that there is a faithful σ-weakly continuous projection of norm one of $\rho_\chi(M)''$ reduced modulo the projection q_χ of H_χ onto clos $\rho_\chi(M)\zeta_\chi$ onto $\rho_\chi(B)''_{q_\chi}$ and $\rho_\chi(B)''_{q_\chi}$ is a type III factor for all x except possibly a nowhere dense set. For any type III algebra M, we show that $\rho_\chi(M)''$ has a faithful σ-weakly continuous projection of norm one onto a particularly simple kind of direct sum of type III factors except possibly for a nowhere dense set. Finally, we decompose the states ϕ_χ into an integral over a convex w^*-compact set of type III states with regard to a Borel measure quasi-supported by the type III primary states except possibly for a nowhere dense set of X.

The main technical tool is the decomposition of generalized and modular Hilbert algebras. The framework differs to some extent from the recent studies of Sutherland [33], Lance [19], [20] and

Jurzak [14], [15], in that the interest here is in nowhere dense sets rather than sets of measure 0. However, the framework is analogous in the sense that we have need only for the decomposition of countably generated generalized Hilbert algebras to obtain the decomposition of B given by $\{\rho_\chi(B)\}$. The main contribution is the construction of the correct countably generated algebra needed to obtain information about $\rho_\chi(M)''$, via the relationship between $\rho_\chi(B)''$ and $\rho_\chi(M)''$. In any case we show that the decomposition of generalized Hilbert algebras associated with faithful normal linear functionals can be obtained without any countable field assumptions of Sutherland, Lance or Jurzak. The assumptions are needed only to obtain the information that component generalized Hilbert algebras produce factors.

2. **Preliminaries.** An algebra $\mathscr{A}$ over the complex numbers C with involution $\zeta \to \zeta^\#$ is called a *modular Hilbert algebra* if $\mathscr{A}$ has an inner product (ζ, η) and a complex one parameter automorphism group $\Delta(\lambda)$ satisfying the following axioms:

(I) $(\xi\zeta, \eta) = (\eta, \xi^\#\zeta)$;

(II) for every $\zeta \in \mathscr{A}$, the map $\eta \to \zeta\eta$ is continuous on $\mathscr{A}$;

(III) the subalgebra $\mathscr{A}^2$ generated by $\zeta\eta$ for ζ, η in $\mathscr{A}$ is dense in $\mathscr{A}$;

(IV) $(\Delta(\lambda)\zeta)^\# = \Delta(-\bar{\lambda})\zeta^\#$ for all $\lambda \in C$, ζ in $\mathscr{A}$;

(V) $(\Delta(\lambda)\zeta, \eta) = (\zeta, \Delta(\bar{\lambda})\eta)$;

(VI) $(\Delta(1)\zeta^\#, \eta^\#) = (\eta, \zeta)$;

(VII) $(\Delta(\lambda)\zeta, \eta)$ is an analytic function of λ on C; and

(VIII) for every real t, the set $(1 + \Delta(t))\mathscr{A}$ is dense in $\mathscr{A}$.

If $\mathscr{A}$ is an algebra with involution $\#$ over C and admits an inner product which satisfies (I)-(III) and the following condition, then $\mathscr{A}$ is called a *generalized Hilbert* algebra:

(IX) the involution is a preclosed conjugate linear operator of $\mathscr{A}$.

Let $\mathscr{A}$ be a generalized Hilbert algebra. Let $J\Delta^{1/2}$ be the polar decomposition of the closure of $\#$. For each ζ in $\mathscr{A}$, there is a unique bounded linear operator $\pi(\zeta)$ in the completion H of $\mathscr{A}$ in the inner product such that $\pi(\zeta)\eta = \zeta\eta$ for all η in $\mathscr{A}$. Let $\mathscr{A}'$ be the space of all ζ in the domain $\mathscr{D}(\Delta^{-1/2})$ of $\Delta^{-1/2}$ such that there is a bounded linear operator $\pi'(\zeta)$ of H satisfying the relation $\pi'(\zeta)\eta = \pi(\eta)\zeta$ for all η in $\mathscr{A}$. Let $\mathscr{A}''$ be the space of all ζ in $\mathscr{D}(\Delta^{1/2})$ such that there is a bounded linear operator $\pi(\zeta)$ such that $\pi(\zeta)\eta = \pi'(\eta)\zeta$ for all η in $\mathscr{A}'$. The sets $\mathscr{A}'$, $\mathscr{A}''$ are generalized Hilbert algebras with involution $\zeta^\flat = J\Delta^{-1/2}$ and $\zeta^\# = J\Delta^{1/2}\zeta$ respectively. If $\mathscr{A}$ is equal to $\mathscr{A}''$, then $\mathscr{A}$ is said to be *full*. If $\mathscr{S}$ is a subset of $\mathscr{A}''$ (resp. $\mathscr{A}'$), let $\pi(\mathscr{S})$ (resp. $\pi'(\mathscr{S})$) be the set of all $\pi(\zeta)$

(resp. $\pi'(\zeta)$) for ζ in $\mathscr{S}$. Let $\mathscr{L}(\mathscr{A})$ (resp. $\mathscr{L}(\mathscr{A}'')$) be the double commutant $\pi(\mathscr{A})''$ (resp. $\pi(\mathscr{A}'')''$) of $\pi(\mathscr{A})$ (resp. $\pi(\mathscr{A}'')$) on H and let $\mathscr{R}(\mathscr{A}')$ be the double commutant of $\pi'(\mathscr{A}')$ on H; then $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{A}'')$ and the commutant $\mathscr{L}(\mathscr{A})'$ of $\mathscr{L}(\mathscr{A})$ is $\mathscr{R}(\mathscr{A}')$. The algebra $\mathscr{L}(\mathscr{A})$ is called the *left von Neumann algebra of* $\mathscr{A}$.

Now let $\mathscr{A}$ be full. The operator Δ^{it} is a unitary operator on H, for every t in the set $\boldsymbol{R}$ of real numbers, that maps $\mathscr{A}$ onto $\mathscr{A}$ and

$$t \longrightarrow \sigma_t(x) = \Delta^{it} x \Delta^{-it}$$

is a strongly continuous one parameter automorphism group for $\mathscr{L}(\mathscr{A})$ (resp. $\mathscr{R}(\mathscr{A}')$) called the *modular automorphism* group. If ζ is in $\mathscr{A}$, then $\sigma_t(\pi(\zeta)) = \pi(\Delta^{it}\zeta)$, for every $t \in \boldsymbol{R}$. The unitary involution J maps $\mathscr{A}$ onto $\mathscr{A}'$, and satisfies $J(\zeta\eta) = J(\eta)J(\zeta)$ for ζ, η in $\mathscr{A}$. Furthermore, the map $x \rightarrow JxJ$ is an anti-isomorphism of $\mathscr{L}(\mathscr{A})$ onto $\mathscr{R}(\mathscr{A}')$ and satisfies

$$J\pi(\zeta)J = \pi'(J\zeta)$$

for ζ in $\mathscr{A}$ [34].

If $\mathscr{A}$ is a full generalized Hilbert algebra, let $\mathscr{A}_0$ be the set of all ζ in $\mathscr{A}$ such that

(a) $\zeta \in \{\mathscr{D}(\Delta^\lambda) \mid \lambda \in \boldsymbol{C}\}$;

(b) $\Delta^\lambda\zeta \in \mathscr{A}$ for $\lambda \in \boldsymbol{C}$; and

(c) the function $\lambda \rightarrow (\Delta^\lambda\zeta, \eta)$ is analytic on $\boldsymbol{C}$ for every η in $\mathscr{A}$.

One may replace (c) by either of the equivalent conditions [21]:

(c′) the function $\lambda \rightarrow (\Delta^\lambda\zeta, \eta)$ is analytic for every η in the completion H of $\mathscr{A}$, or

(c″) the function $\lambda \rightarrow \pi(\Delta^\lambda\zeta)$ is a holomorphic function of $\boldsymbol{C}$ into $\mathscr{L}(\mathscr{A})$.

The set $\mathscr{A}_0$ is a modular Hilbert algebra called the *maximal modular* Hilbert whose involution is the involution of $\mathscr{A}$ restricted to $\mathscr{A}_0$ and whose one parameter automorphism group is the restriction of $\Delta(\lambda) = \Delta^\lambda$. The algebra $\mathscr{A}_0$ is *equivalent* to $\mathscr{A}$ in the sense that $\mathscr{A}_0$ is dense in $\mathscr{A}$ and $\mathscr{A}_0'' = \mathscr{A}$ [34].

It is important to notice that the full generalized Hilbert algebra, and its maximal modular Hilbert algebra is a module over the center of its left von Neumann algebra and satisfies

$$\pi(x\zeta) = x\pi(\zeta) \tag{1}$$

for every x in the center [3, Lemma 4.10].

Let M be a von Neumann algebra on the Hilbert space H. A *weight* ϕ of H is a map of H^+ into $[0, \infty]$ such that $\phi(x + y) = \phi(x) + \phi(y)$ for x and y in M^+ and $\phi(\lambda x) = \lambda\phi(x)$ for $\lambda > 0$ and x in M^+.

A weight ϕ is said to be *normal* if there is a family $\{\omega_\alpha\}$ of normal functionals on M such that lub $\omega_\alpha(x) = \phi(x)$ for every x in M^+. The weight is said to be *faithful* if $\phi(x) = 0$ implies $x = 0$ and *semi-finite* if the set $N_\phi^* N_\phi$ is weakly dense in M where

$$N_\phi = \{x \in M \mid \phi(x^*x) < \infty\} \qquad [3] .$$

Let ϕ be a faithful semi-finite normal weight on M. Let $\Lambda = \Lambda_\phi$ denote the linear injection of $\mathscr{A} = N_\phi^* \cap N_\phi$ onto a dense subset of a Hilbert space $H(\phi)$ so that

$$\phi(x^*y) = (\Lambda(y), \Lambda(x))$$

for every x, y in $\mathscr{A}$. The set $\Lambda(\mathscr{A})$, called the *generalized left Hilbert algebra associated with* ϕ, is a full generalized Hilbert algebra with involution $\#$ given by $\Lambda(x)^\# = \Lambda(x^*)$. There is a faithful normal representation $\rho = \rho_\phi$ of M on $H(\phi)$ given by $\rho(x)\Lambda(y) = \Lambda(xy)$ for x, y in M. The algebra $\rho(M)$ is equal to $\mathscr{L}(\mathscr{A})$. If ϕ is a functional, then ρ is the canonical representation induced by ϕ [6, § 2]. The weight ϕ on $\mathscr{A}$ satisfies the KMS *boundary conditions* with respect to the automorphism group $\sigma_t(x) = \Delta^{it} x \Delta^{-it}$ in the sense that

(1) $\phi \cdot \sigma_t = \phi$;

(2) if x and y are in $\mathscr{A}$, then there is a continuous bounded function h on the strip $\{\lambda \in \boldsymbol{C} \mid 0 \leqq \operatorname{Im} \lambda \leqq 1\}$ which is holomorphic in the interior and satisfies the boundary conditions

$$h(t) = \phi(\sigma_t(x)y)$$

and

$$h(t + i) = \phi(y\sigma_t(x))$$

for all real t ([3], [22]).

The following observation is important in our later calculations.

LEMMA 1. *The maximal modular Hilbert algebra $\mathscr{A}_0$ of the full generalized Hilbert algebra $\mathscr{A}$ is invariant under* $(1 + \Delta)^{-1}$.

Proof. If ζ is in $\mathscr{A}_0$, then the element

$$(1 + \Delta)^{-1} \Delta^\lambda \zeta = \Delta^\lambda (1 + \Delta)^{-1} \zeta$$

is contained in $\mathscr{A}$ for all α in $\boldsymbol{C}$ (because $\mathscr{A}_0$ is a subset of $\mathscr{A}'$ [34, Lemma 8.1]), and the function

$$\lambda \longrightarrow (\Delta^\lambda (1 + \Delta)^{-1} \zeta, \eta) = (\Delta^\lambda \zeta, (1 + \Delta)^{-1} \eta)$$

is an entire function.

Let $\mathscr{K}(\boldsymbol{R})$ be the algebra of all continuous complex-valued functions h on $\boldsymbol{R}$ whose support supp h is compact. Let $\mathscr{E}$ be the subset of $\mathscr{K}(\boldsymbol{R})$ given by

$$\mathscr{E} = \text{linear span}\left\{h * k(s) = \int h(t)k(s-t)dt \,|\, h, k \in \mathscr{K}(\boldsymbol{R})\right\}.$$

Let l_n $(n = 1, 2, \cdots)$ be the function equal to 1 on $[-n, n]$, 0 on $(-\infty, -n-1) \cup (n+1, \infty)$ and linear in between. Let k_n $(n = 1, 2, \cdots)$ be a function $\mathscr{K}(\boldsymbol{R})^+$ with supp k_n contained in $[-n^{-1}, n^{-1}]$ with $\|k_n\|_1 = 1$. Let $\mathscr{E}_0$ be the smallest subalgebra of $\mathscr{E}$ over the field of rational complex numbers $\boldsymbol{C}_r$ containing functions of the form $l_m h_1 * l_n h_2$, where h_1, h_2 are polynomials with coefficients in $\boldsymbol{C}_r$, that is invariant under the involution $h \to \tilde{h}$ given by $h(t) = h(-t)^-$ and multiplication by the functions k_n. For any h in $\mathscr{E}$ there is a sequence $\{h_n\}$ in $\mathscr{E}_0$ and a compact subset in $\boldsymbol{R}$ such that all the sets supp h_n are contained in this compact set and such that $\lim \|h_n - h\|_\infty = 0$.

DEFINITION 2. A subset of a modular Hilbert algebra is said to be invariant (resp. quasi-invariant) if it is invariant under J, $\Delta^\lambda(\lambda \in \boldsymbol{C})$, $h(\log \Delta)$ $(h \in \mathscr{E})$ and $(1 + \Delta)^{-1}$ (resp. J, $\Delta^\lambda(\lambda \in \boldsymbol{C}_r)$, $h(\log \Delta)$ $(h \in \mathscr{E}_0)$, and $(1 + \Delta)^{-1}$).

We note that the maximal modular Hilbert algebra of a generalized Hilbert algebra is invariant.

Our constructions in § 4 are based on the following observation.

LEMMA 3. *An invariant subalgebra of a modular Hilbert algebra $\mathscr{B}$ is a modular Hilbert algebra whose modular operator and unitary involution are the restrictions of the corresponding operators of $\mathscr{B}$.*

Proof. The subalgebra satisfies all the properties for modular Hilbert algebras with the possible exception of properties III and VIII. However, property III holds due to Lemma 5.1 of [34] while property VIII holds since the subalgebra is invariant under $h(\log \Delta)$ $(h \in \mathscr{E})$ (cf. [34, proof p. 55]).

Let $\mathscr{S}$ be a subset of the generalized Hilbert algebra $\mathscr{A}$ (resp. $\mathscr{A}'$), the C^*-algebra generated by $\pi(\mathscr{S})$ (resp. $\pi'(\mathscr{S})$) will be denoted by $\mathscr{L}^*(\mathscr{S})$ (resp. $\mathscr{R}^*(\mathscr{S})$). If $\mathscr{S}$ is an invariant subalgebra of $\mathscr{A}_0$, then $\pi(\mathscr{S})$ (resp. $\pi'(\mathscr{S})$) is a^*-subalgebra of $\mathscr{L}(\mathscr{A}_0)$ (resp. $\mathscr{R}(\mathscr{A}_0)$) and so $\mathscr{L}^*(\mathscr{S})$ (resp. $\mathscr{R}^*(\mathscr{S})$) is the norm closure of $\pi(\mathscr{S})$ (resp. $\pi'(\mathscr{S})$).

3. Decomposition of modular Hilbert algebras. In this section we consider the decomposition of a modular Hilbert algebra formed from a faithful normal functional on a von Neumann algebra. Let M be a von Neumann algebra with center C on the Hilbert space H. If e is a maximal abelian projection in the commutant C' of C on H, then there is, for every x in C', a unique element $\tau_e(x) = \tau(x)$ in C such that $\tau(x)e = exe$. The map τ is in the set $C'^{+}_{\sim}$ of σ-weakly continuous C-module homomorphisms ϕ of C' into C such that $\phi(x^*x) \geqq 0$ for all $x \in C'$. Let χ be a point in the spectrum χ of C, let $[\chi]$ be the closed two-sided ideal in C' generated by χ, and let $x(\chi)$ denote the image of x in C' in the algebra $C'(\chi) = C'/[\chi]$ under the canonical homomorphism. Then the set $C'e(\chi)$ is a Hilbert space under the inner product

$$(xe(\chi), ye(x)) = \tau(y^*x)^{\wedge}(\chi)$$

and the map $x \to x(\chi)$ defines a representation of C' on the Hilbert space $C'e(\chi)$. The representation is the canonical representation induced by the state $\tau_\chi(x) = \tau(x)^{\wedge}(\chi)$ on C'. The restriction ρ_χ of this representation to M on its invariant subspace H_χ spanned by the vectors $Me(\chi) = \{xe(\chi) \mid x \in M\}$ is the canonical representation induced by the state τ_χ restricted to M. If A is a $*$-subalgebra of M, the *von Neumann algebra* $\rho_\chi(A)''$ *generated* by $\rho_\chi(A)$ will be denoted by A_χ [10]. A more abstract equivalent interpretation of M_χ can be found in [31] and [32].

Let ϕ be a faithful normal functional on M. The canonical representation ρ of M on $H(\rho)$ is a σ-weakly continuous isomorphism and the image $\rho(M) = \rho(M)''$ of M has a cyclic and separating vector ζ_0 such that $\omega \cdot \rho = \phi$ where $\omega = \omega_{\zeta_0}$. We now identify M with $\rho(M)$, H with $H(\rho)$ and ϕ with ω, thereby assuming that M has a cyclic and separating vector ζ_0 on H. Let e be the maximal abelian projection in C' whose range is clos $\{x\zeta_0 \mid x \in C\}$ and let $\tau = \tau_e$. We notice that $\omega \cdot \tau = \omega$.

The next proposition may be viewed as a generalization of the results of Sutherland [35], Lance [19] and Jurzak [14] in that no assumption about the existence of a countable field is made.

PROPOSITION 4. *Let M be a von Neumann algebra with center C on the Hilbert space H, let ζ_0 be a separating and cyclic vector for M, and let $\mathscr{B}$ be an invariant subalgebra of the maximal modular Hilbert algebra $\mathscr{A}_0$ of the generalized Hilbert algebra $M\zeta_0$. Let e be the maximal abelian projection in C' corresponding to the subspace generated by $C\zeta_0$, let χ be in the spectrum of C, and let $p = p_\chi$ be the projection of $C'e(\chi)$ onto its subspace $K = K_\chi$ generated by $\mathscr{L}^*(\mathscr{B})e(\chi)$. Then the projection p is in $\mathscr{L}^*(\mathscr{B})(\chi)' \cap \mathscr{R}^*(\mathscr{B})(\chi)' \cap \{\Delta^{it}(\chi)\}'$, the vector $e(\chi)$ is a cyclic and separating vector for the algebras $A = \mathscr{L}^*(\mathscr{B})(\chi)''_p$ and $\mathscr{R}^*(\mathscr{B})(\chi)''_p$, the maps $\sigma^\chi_t(x) = \Delta^{it}(\chi)x\Delta^{-it}(\chi)$*

($t \in \boldsymbol{R}$) form the modular automorphism group of the generalized Hilbert algebra $Ae(\chi)$, and the commutant of A is $\mathscr{R}^*(\mathscr{B})(\chi)''_p$.

Proof. If ζ is in $\mathscr{B}$, then the relation

$$\pi(\zeta)x\zeta_0 = \pi'(\zeta)x\zeta_0$$

holds for every x in C; consequently, the element $\pi(\zeta)e(\chi)$ is equal to $\pi'(\zeta)e(\chi)$ and the subspace K is identical with the subspace generated by $\mathscr{R}^*(\mathscr{B})e(\chi)$.

We now show $\Delta^{it}(\chi)$ commutes with p. First we notice that Δ^{it} is in C' and thus $\Delta^{it}(\chi)$ is an operator in the algebra $\mathscr{B}(C'e(\chi))$ of bounded linear operators on $C'e(\chi)$. Since

$$\Delta^{it}(\chi)\pi(\zeta)e(\chi) = \Delta^{it}\pi(\zeta)\Delta^{-it}e(\chi) = \pi(\Delta^{it}\zeta)e(\chi)$$

for ζ in $\mathscr{B}$, we have that $\Delta^{it}(\chi)p$ is equal to $p\Delta^{it}(\chi)p$ and so p commutes with $\Delta^{it}(\chi)$.

The functions

$$\sigma_t(x) = \Delta^{it}x\Delta^{-it} \qquad (t \in \boldsymbol{R})$$

are automorphisms of $\mathscr{L}^*(\mathscr{B})$ due to the invariance of $\mathscr{B}$ (cf. [34, Corollary 9.1]), and the automorphisms $\{\sigma_t\}$ form a one parameter automorphism group strongly continuous in the sense that, for every x in $\mathscr{L}^*(\mathscr{B})$, $\sigma_t(x)$ converges to x in the norm topology whenever t tends to 0. We show that the positive linear functional τ_χ of $\mathscr{L}^*(\mathscr{B})$ satisfies the KMS boundary conditions with respect to the automorphism group $\{\sigma_t\}$. First the functional τ_χ is invariant under σ_t since

$$\tau_\chi(\sigma_t(x)) = (x\Delta^{-it}e(\chi), \Delta^{-it}e(\chi)) = (xe(\chi), e(\chi)) = \tau_\chi(x)$$

for all x in $\mathscr{L}^*(\mathscr{B})$. Now given ζ, η in $\mathscr{B}$, we have that

$$h(\lambda) = \tau(\pi(\Delta^{i\lambda}\zeta)\pi(\eta))$$

is a holomorphic function of $\boldsymbol{C}$ into C such that

$$h(t) = \tau(\pi(\Delta^{it}\zeta)\pi(\eta)) = \tau(\sigma_t(\pi(\zeta))\pi(\eta))$$

and

$$h(t+i) = \tau(\pi(\Delta^{it-1}\zeta)\pi(\eta)) = \tau(\pi(\eta)\sigma_t(\pi(\zeta)))$$

for all real t due to the fact that

$$\begin{aligned}(\tau(\pi(\Delta^{it-1}\zeta)\pi(\eta)x\zeta_0, y\zeta_0) &= (\pi(\Delta^{it-1}\zeta)\pi(\eta)x\zeta_0, y\zeta_0)\\ &= (\pi(x\eta)\zeta_0, \pi(J\Delta^{it-1/2}(y\zeta))\zeta_0)\\ &= (x\eta, J\Delta^{it-1/2}y\zeta)\end{aligned}$$

$$= (\Delta^{it} y\zeta, J\Delta^{1/2} x\eta)$$
$$= (\tau(\pi(\eta)\sigma_t(\pi(\zeta)))x\zeta_0, y\zeta_0)$$

for all x, y self-adjoint in C (cf. relation (1)). The function $h(\lambda)$ is bounded on $0 \leqq \operatorname{Im} \lambda \leqq 1$ because

$$\|h(\lambda)\| \leqq \|\pi(\eta)\| \operatorname{lub}\{\|\pi(\Delta^{\beta}\zeta)\| \mid -1 \leqq \beta \leqq 0\} < +\infty$$

and so

$$\|h(\lambda)\| \leqq \|\pi(\eta)\| \|\pi(\zeta)\|$$

by the Phragmen-Lindelöf theorem. For every x, y in $\mathscr{L}^*(\mathscr{B})$, there are sequences $\{\zeta_n\}$, $\{\eta_n\}$ in $\mathscr{B}$ with $\lim \pi(\zeta_n) = x$, $\lim \pi(\eta_n) = y$ and $\|\pi(\zeta_n)\| \leqq \|x\|$ $\|\pi(\eta_n)\| \leqq \|y\|$. Consequently, there is a bounded continuous function $k(\lambda)$ of $0 \leqq \operatorname{Im} \lambda \leqq 1$ into C, which is holomorphic on $0 < \operatorname{Im} \lambda < 1$ and which satisfies the boundary conditions

$$k(t) = \tau(\sigma_t(x)y)$$

and

$$k(t + i) = \tau(y\sigma_t(x))$$

for real t due to the Phragmen-Lindelof theorem. The function $k(\lambda)\hat{}(\chi)$ is continuous and bounded on $0 \leqq \operatorname{Im} \lambda \leqq 1$ and it is analytic on $0 < \operatorname{Im} \lambda < 1$ since the existence of limit of the difference quotient $(k(\lambda + h) - k(\lambda))/h$ in the norm of C implies the existence of

$$\lim (k(\lambda + h)\hat{}(\chi) - k(\lambda)\hat{}(\chi))/h .$$

Hence, the functional τ_χ on $\mathscr{L}^*(\mathscr{B})$ satisfies the KMS boundary conditions with respect to $\{\sigma_t\}$. Since the representation $x \to x(\chi)p$ of $\mathscr{L}^*(\mathscr{B})$ on K is identified with the canonical representation of $\mathscr{L}^*(\mathscr{B})$ induced by τ_χ, the vector $e(\chi)$ is separating and cyclic for $A = \mathscr{L}^*(\mathscr{B})(\chi)''_p$ and the modular automorphisms $\{\sigma_t^\chi\}$ for A with regard to $\omega_\chi = \omega_{e(\chi)}$ are given by

$$(2) \qquad \sigma_t^\chi(px(\chi)p) = p\sigma_t(x)(\chi)p = \Delta^{it}(\chi)px(\chi)p\Delta^{-it}(\chi)$$

for x in $\mathscr{L}^*(\mathscr{B})$ [34, Theorem 13.3], and thus,

$$\sigma_t^\chi(x) = \Delta^{it}(\chi)x\Delta^{-it}(\chi)$$

for x in A. Setting Δ_χ equal to the modular operator of $Ae(\chi)$, we have that $\Delta^{it}(\chi)p = \Delta_\chi^{it}$ for every t in $\boldsymbol{R}$ because each $\Delta^{it}(\chi)p$ leaves $e(\chi)$ invariant.

In order to show that the commutant A' of A is equal to $\mathscr{R}^*(\mathscr{B})(\chi)''p$, we show that the unitary involution J_χ of the generalized Hilbert algebra $Ae(\chi)$ is given by $J_\chi\pi(\zeta)e(\chi) = \pi(J\zeta)e(\chi)$ for ζ in $\mathscr{B}$. For this we also need to consider the modular operators.

There is a positive self-adjoint operator a_j ($j = 1, 2$) on K satisfying

$$a_j\pi(\zeta)e(\chi) = \pi(\Delta^{1/j}\zeta)e(\chi) \tag{3}$$

for ζ in $\mathscr{B}$. For x, y in C, we have that

$$\begin{aligned}(\tau(\pi(\eta)^*\pi(\Delta^\lambda\zeta))x\zeta_0, y\zeta_0) &= (\pi(\Delta^\lambda x\zeta)\zeta_0, \pi(y\eta)\zeta_0)\\ &= (\Delta^\lambda(x\zeta), y\eta)\\ &= (x\zeta, \Delta(\bar{\lambda})y\eta)\\ &= (\tau(\pi(\Delta(\bar{\lambda})\eta)^*\pi(\zeta))x\zeta_0, y\zeta_0)\end{aligned}$$

for every ζ, η in $\mathscr{B}$ and λ in $\boldsymbol{C}$ since $\mathscr{A}_0$ is a C-module and Δ^λ commutes with every element of C. This means that

$$\tau(\pi(\eta)^*\pi(\Delta^{1/j}\zeta)) = \tau((\pi(\Delta^{1/j}\eta)^*\pi(\zeta)))$$

and that

$$\tau(\pi(\zeta)^*\pi(\Delta^{1/j}\zeta)) \geqq 0$$

for $j = 1, 2$. Hence, reducing this modulo χ, we see that relation (3) defines a positive symmetric operator on the dense linear manifold $\pi(\mathscr{B})e(\chi)$ of K, and a_j can be set equal to its Friedrichs extension (cf. [25, § 124]). There is also a positive bounded operator b on K such that

$$b\pi(\zeta)e(\chi) = \pi((1 + \Delta)^{-1}\zeta)e(\chi) \tag{4}$$

for all ζ in $\mathscr{B}$. Indeed, because $(1 + \Delta)^{-1}\mathscr{B}$ is equal to $\mathscr{B}$, arguments similar to the preceding ones give a positive symmetric operator on K with domain $\pi(\mathscr{B})e(\chi)$ satisfying (4). The operator is bounded because

$$\begin{aligned}&(\tau(\pi((1 + \Delta)^{-1}\zeta)^*\pi((1 + \Delta)^{-1}\zeta))x\zeta_0, x\zeta_0)\\ &\quad = \|(1 + \Delta)^{-1}\pi(\zeta)x\zeta_0\|^2 \leqq \|\pi(\zeta)x\zeta_0\|\end{aligned}$$

for all ζ in $\mathscr{B}$ and x in C implies

$$\|\pi((1 + \Delta)^{-1}\zeta)e(\chi)\| \leqq \|\pi(\zeta)e(\chi)\|$$

for all ζ, and so the operator has a unique extension b satisfying (4).

Considering the fact that

$$(1 + a_1)b\pi(\zeta)e(\chi) = \pi(\zeta)e(\chi)$$

for every ζ in $\mathscr{B}$, and the fact that self-adjoint operators are maximal, we see that $1 + a_1$ is equal to b^{-1}. Thus, the operator a_1 is equal to the closure of the restriction of a_1 to $\pi(\mathscr{B})e(\chi)$. Again, by considering the restrictions to $\pi(\mathscr{B})e(\chi)$ and the maximality of self-adjoint operators, we get that $a_2^2 = a_1$. This means that $a_2 = a_1^{1/2}$ due to the uniqueness of the positive square root, and consequently, that the closure of a_2 restricted to $\pi(\mathscr{B})e(\chi)$ is a_2.

The existence of a conjugate linear isometry J_1 of K onto K such that $J_1^2 = 1$ and

$$J_1\pi(\zeta)e(\chi) = \pi(J\zeta)e(\chi)$$

for $\zeta \in \mathscr{B}$ is verified in the same manner as the previous paragraph. We can now verify that J_1 and a_1 are the respective unitary involution and modular operator J_χ and Δ_χ of $Ae(\chi)$. We have that

$$J_1 a_2 \pi(\zeta)e(\chi) = \pi(J\Delta^{1/2}\zeta)e(\chi) = \pi(\zeta)^* e(\chi)$$

for every ζ in $\mathscr{B}$. Given an element x in A, there is a sequence $\{\zeta_n\}$ of elements in $\mathscr{B}$ such that $\{\pi(\zeta_n)e(\chi)\}$ converges to $xe(\chi)$ and $\{\pi(\zeta_n)^*e(\chi)\}$ converges to $x^*e(\chi)$. Because J_1^2 is the identity and because a_2 is a closed operator, the element $xe(\chi)$ is in the domain of a_2 and

$$J_1 a_2 xe(\chi) = x^* e(\chi) = J_\chi \Delta_\chi^{1/2} xe(\chi)$$

for all x in A. We showed earlier in the proof that the closure of the graph of $a_2 | \pi(\mathscr{B})e(\chi)$ is the graph of a_2; it is also known that the closure of the graph of $\Delta_\chi^{1/2} | Ae(\chi)$ is the graph of $\Delta_\chi^{1/2}$ [34, Lemma 9.1]. Therefore, we get that $\mathscr{D}(\Delta_\chi^{1/2}) = \mathscr{D}(a_2)$, and that

$$J_1 a_2 = J_\chi \Delta_\chi^{1/2}$$

on the common domain. The uniqueness of the polar decomposition proves that $J_1 = J_\chi$.

We now can show that A' is equal to $\mathscr{R}^*(\mathscr{B})(\chi)_p''$. It is known that $J_\chi A J_\chi = A'$ by the theory of modular Hilbert algebras [34, Theorem 10.1]. However, noticing that

$$(5) \qquad J_\chi p\pi(\zeta)(\chi)pJ_\chi \pi(\eta)e(\chi) = p\pi'(J\zeta)(\chi)p\pi(\eta)e(\chi)$$

for ζ, η in $\mathscr{B}$, we see that $\mathscr{R}^*(\mathscr{B})(\chi)_p''$ contains a weakly dense subset $J_\chi p\pi(\mathscr{B})(\chi)pJ_\chi$ of A' and consequently, A' itself. Since the reverse inclusion relation is apparent, we conclude that A' and $\mathscr{R}^*(\mathscr{B})(\chi)_p''$ coincide.

A linear map ε of a C^*-algebra B onto its nonzero *-subalgebra D is said to be a *projection of norm one* if ε is bounded of norm one and if $\varepsilon(x) = x$ for every x in D (cf. [38]). The projection ε of norm one is said to be *faithful* if $\varepsilon(x^*x) = 0$ implies $x = 0$ for x in B.

A von Neumann algebra B is said to be *compatible* with its von Neumann subalgebra D if there is a faithful σ-weakly continuous projection of norm one of B onto D [1, 6.1.4]. It is important for the sequel that only type III algebras are compatible with their type III subalgebras ([40, Theorem 3], [29, 2.6.5]). An explicit form for a σ-weakly continuous projection ε of norm of B onto D can be given: B can be represented as a von Neumann algebra on a Hilbert space so that there is an isometry v in the commutant of D (on the Hilbert space) such that $\varepsilon(x) = v^*xv$ for x in B ([2], [18], [30]).

We preserve the notation of Proposition 4 in the next corollary.

COROLLARY 5. *The projection $q = q_\chi$ of H_χ onto the subspace generated by $M_\chi' e(\chi)$ is in the algebra $(\mathscr{L}^*(\mathscr{B})_\chi)' \cap M_\chi$ and there is a faithful σ-weakly continuous projection $\varepsilon = \varepsilon_\chi$ of norm one of the von Neumann algebra $(M_\chi)_q$ onto its von Neumann subalgebra $(\mathscr{L}^*(\mathscr{B})_\chi)_q$ such that $\omega_{e(\chi)} \cdot \varepsilon = \omega_{e(\chi)}$. In particular the set $p'M_\chi p'$ is equal to $\mathscr{L}^*(\mathscr{B})_\chi p'$, where $p' = p_\chi'$ is the projection of H_χ onto K_χ.*

Proof. We first show that q is in the commutant of $\mathscr{L}^*(\mathscr{B})_\chi$. Denoting the projection of $C'e(\chi)$ onto $H(\chi)$ by s, we have

$$M(\chi)_s' = (M(\chi)_s'')' = M_\chi'$$

[5; I, §2, Proposition 1]. For x in $M(\chi)'$ and ζ in $\mathscr{B}$, the vector $\rho_\chi(\pi(\zeta))sxse(\chi)$ is in $M_\chi' e(\chi)$ since

$$\begin{aligned}\rho_\chi(\pi(\zeta))sxse(\chi) &= s\pi(\zeta)(\chi)sxse(\chi)\\ &= sx\pi(\zeta)e(\chi)\\ &= sx\pi'(\zeta)e(\chi)\\ &= sx\pi'(\zeta)(\chi)se(\chi)\ ,\end{aligned}$$

and so every y in $\mathscr{L}^*(\mathscr{B})_\chi$ maps qH_χ into itself. This means that q is in $(\mathscr{L}^*(\mathscr{B})_\chi)'$, which we denote simply as $\mathscr{L}^*(\mathscr{B})_\chi'$.

The central support of the projection p' of H_χ onto K in the algebra $\mathscr{L}^*(\mathscr{B})_\chi'$ is the same as that of q. In fact, the central support of p' corresponds to the projection of H_χ onto the closure of

$$\mathscr{L}^*(\mathscr{B})_\chi' \mathscr{L}^*(\mathscr{B})e(\chi) = \mathscr{L}^*(\mathscr{B})_\chi' s\mathscr{R}^*(\mathscr{B})(\chi)se(\chi)$$

which is simply the closure of $\mathscr{L}^*(\mathscr{B})_\chi' e(\chi)$, while the central support of q corresponds to the projection of H_χ onto the closure of $\mathscr{L}^*(\mathscr{B})_\chi' M_\chi' e(\chi)$, which again is simply the closure of $\mathscr{L}^*(\mathscr{B})_\chi' e(\chi)$ (cf. [5, I, §1, Proposition 7, Corollary 2]). Since it is known [5, I, §2, Proposition 2] that there is an isomorphism Φ of $p'\mathscr{L}^*(\mathscr{B})_\chi p'$ onto $q\mathscr{L}^*(\mathscr{B})_\chi q$ given by

$$\Phi(p'xp') = qxq$$

for $x \in \mathscr{L}^*(\mathscr{B})_\chi$, it is only necessary to show that the set $p'M_\chi p'$ is equal to $p'\mathscr{L}^*(\mathscr{B})_\chi p'$ in order to complete the proof. In fact, the map ε is then as the composition of two σ-weakly continuous maps Φ and $x \to p'xp'$ given by

$$\varepsilon(x) = \Phi(p'xp')$$

so that ε satisfies the relation

$$\begin{aligned}\omega_{e(\chi)}(\varepsilon(x)) &= (\Phi(p'xp')p'e(\chi),\ p'e(\chi)) \\ &= (p'xp'e(\chi),\ e(\chi)) = \omega_{e(\chi)}(x)\ ,\end{aligned}$$

which also implies that ε is faithful. Now it is sufficient to show that $p'\rho_\chi(x)p'$ is in the weakly closed set $p'\mathscr{L}^*(\mathscr{B})_\chi p'$ for x in M because $p'\rho_\chi(M)p'$ is weakly dense in $p'M_\chi p'$. We have that

$$x(\chi)\pi'(\zeta)(\chi) = \pi'(\zeta)(\chi)x(\chi)$$

for every ζ in $\mathscr{B}$, and thus, by reducing to the subspace K, we have

$$\begin{aligned}p'\rho_\chi(x)p'p\pi'(\zeta)(\chi)p\xi &= px(\chi)p\pi'(\zeta)(\chi)p\xi \\ &= p\pi'(\zeta)(\chi)px(\chi)p\xi \\ &= p\pi'(\zeta)(\chi)pp'\rho_\chi(x)p'\xi\end{aligned}$$

for every ξ in K since the projection p of $C'e(\chi)$ onto K is in $\mathscr{R}^*(\mathscr{B})(\chi)'$ (Proposition 4). However, the algebra $\mathscr{R}^*(\mathscr{B})(\chi)''_p$ is equal to $\mathscr{L}^*(\mathscr{B})'_\chi$. Therefore, the element $p'\rho_\chi(x)p'$ is in $\mathscr{L}^*(\mathscr{B})_\chi$.

The next corollary will be used later.

COROLLARY 6. *For every x in $\mathscr{L}^*(\mathscr{A}_0)$ and y in M there is a bounded continuous function h of the strip $I = \{\lambda \in \boldsymbol{C} \mid 0 \leqq \operatorname{Im}\lambda \leqq 1\}$ into C, which is holomorphic on the interior $\{\lambda \in \boldsymbol{C} \mid 0 \leqq \operatorname{Im}\lambda < 1\}$ and which satisfies the boundary conditions*

$$h(t) = \tau(\sigma_t(x)y) \quad \textit{and} \quad h(t+i) = \tau(y\sigma_t(x))$$

for all $t \in \boldsymbol{R}$.

Proof. Let $\{\zeta_n\}$ be a sequence in $\mathscr{A}_0$ such that $\{\pi(\zeta_n)\}$ converges to x. The function h_n ($n = 1, 2, \cdots$) defined by

$$h_n(\lambda) = \tau(\pi(\Delta^{i\lambda}\zeta_n)y)$$

on $\boldsymbol{C}$ is holomorphic due to the existence of a Cauchy integral representation and bounded on the strip I due to the fact

$$\|h_n(\alpha + i\beta)\| \leqq \|y\|\operatorname{lub}\{\|\pi(\Delta^{-\beta}\zeta_n)\| \mid 0 \leqq \beta \leqq 1\} < +\infty\ .$$

The boundary conditions

$$h_n(t) = \tau(\sigma_t(\pi(\zeta_n))y) \quad \text{and} \quad h_n(t+i) = \tau(y\sigma_t\pi(\zeta_n))$$

($t \in \boldsymbol{R}$) are satisfied because for every z, w in C there is an entire function

$$k(\lambda) = (\tau(\pi(\Delta^{i\lambda}\zeta_n)y)z\zeta_0, w\zeta_0) = (\pi(\Delta^{i\lambda}(w^*\zeta_n))zy\zeta_0, \zeta_0)$$

that satisfies the boundary conditions

$$k(t) = (\sigma_t(\pi(w^*\zeta_n))zy\zeta_0, \zeta_0) = (h_n(t)z\zeta_0, w\zeta_0)$$

and

$$k(t+i) = ((zy)\sigma_t(\pi(w^*\zeta_n))\zeta_0, \zeta_0) = (\tau(y\sigma_t(\pi(\zeta_n)))z\zeta_0, w\zeta_0)$$

for $t \in \boldsymbol{R}$ so that

$$((h_n(t+i) - \tau(y\sigma_t(\pi(x))))z\zeta_0, w\zeta_0) = 0$$

for $t \in \boldsymbol{R}$.

Now the Phragmen-Lindelöf theorem implies that

$$||h_n(\lambda) - h_m(\lambda)|| \leqq ||\pi(\zeta_n) - \pi(\zeta_m)|| \, ||y||$$

for all λ in I. Therefore, the sequence $\{h_n\}$ converges uniformly on I to a bounded continuous function h of I into C, holomorphic on the interior of I, and satisfying the required boundary conditions.

4. **Projections of norm one.** In this section until further notice, let M be a von Neumann algebra with center C on the Hilbert space H, let X be the spectrum of C, let ζ_0 be a cyclic and separating vector for M, and let e be the maximal abelian projection in C' with range clos $C\zeta_0$. Let $\tau = \tau_e$ and let ρ_χ be the canonical representation on the Hilbert space H_χ induced by τ_χ on M for every χ in X. We show that M has a faithful σ-weakly continuous projection of norm one onto its countably generated von Neumann subalgebra of the same type. If M is of type III, we show the subalgebra can be chosen as the weak closure of a separable C^*-algebra of the form $\mathscr{L}^*(\mathscr{B})$, with $\mathscr{B}$ an invariant subalgebra of the maximal modular Hilbert algebra of the generalized Hilbert algebra $M\zeta_0$, so that $\rho_\chi(\mathscr{L}^*(\mathscr{B}))''$ is a type III factor except for perhaps a nowhere dense set of χ in X. Then the algebra $\rho_\chi(M)'' = M_\chi$ reduced to the closure of $\rho_\chi(M)'e(\chi)$ has a faithful σ-weakly continuous projection of norm 1 on its type III factor subalgebra due to the results of § 3, and consequently the algebra M_χ is type III. We show that every normal functional on a type III von Neumann can be written as an integral of a continuous field $\{\phi_\chi\}$ of functionals, whose canonical representations are in certain equivalence classes of type III representations. Furthermore, we show the functionals ϕ_χ can be represented in the sense of Choquet-Bishop-de Leeuw as integrals over w^*-compact sets

of type III functionals of M whose extreme points are primary functionals.

We now begin the construction of the countably generated algebras.

LEMMA 7. *Let $\mathscr{A}$ be a generalized Hilbert algebra and let $\mathscr{A}_0$ be the maximal modular Hilbert algebra of $\mathscr{A}$. If $\mathscr{S}$ is a countable subset of $\mathscr{A}_0$, then there is a countable subalgebra $\mathscr{T}$ of $\mathscr{A}_0$ over $\boldsymbol{C}_r$ and a subalgebra $\mathscr{U}$ of $\mathscr{A}_0$ such that*

(1) $\mathscr{S} \subset \mathscr{T} \subset \mathscr{U}$;

(2) $\mathscr{T}$ *is quasi-invariant;*

(3) $\mathscr{U}$ *is invariant; and*

(4) $\pi(\mathscr{T})$ *is dense in* $\mathscr{L}^*(\mathscr{U})$.

Proof. For any subsets $\mathscr{T}$ of $\mathscr{A}_0$ and $\mathscr{F}$ of $\mathscr{E}$ and any subfield $\boldsymbol{C}_0$ of $\boldsymbol{C}$, let $a(\mathscr{T}, \mathscr{F}, \boldsymbol{C}_0)$ be the subalgebra of $\mathscr{A}_0$ over $\boldsymbol{C}_0$ algebraically generated by the set

$$\mathscr{S}(\mathscr{T}, \mathscr{F}, \boldsymbol{C}_0) = \{J^k(1 + \Delta)^{-1} h(\log \Delta)^l \Delta^\lambda \zeta \mid \zeta \in \mathscr{T}; h \in \mathscr{F}; \lambda \in \boldsymbol{C}_0; k, l = 0, 1\} .$$

The set $\mathscr{S}(\mathscr{T}, \mathscr{F}, \boldsymbol{C}_0)$ is invariant under J and Δ^λ $(\lambda \in \boldsymbol{C}_0)$ for $\boldsymbol{C}_0$ equal to $\boldsymbol{C}_r$ or $\boldsymbol{C}$ and $\mathscr{F}$ equal to $\mathscr{E}_0$ or $\mathscr{E}$ because $\Delta(\lambda)J = J\Delta(-\bar{\lambda})$, and so the algebra $a(\mathscr{T}, \mathscr{F}, \boldsymbol{C}_0)$ is invariant under J and Δ^λ $(\lambda \in \boldsymbol{C}_0)$ because $J(\zeta\eta) = J\eta J\zeta$ and $\Delta^\lambda(\zeta\eta) = \Delta^\lambda\zeta\Delta^\lambda\eta$ for ζ, η in $\mathscr{A}_0$. Letting $\mathscr{T}_0 = \mathscr{U}_0$ be the countable sets $a(\mathscr{S}, \mathscr{E}_0, \boldsymbol{C}_r)$, we define inductively two increasing sequences

$$\mathscr{T}_i = a(\mathscr{T}_{i-1}, \mathscr{E}_0, \boldsymbol{C}_r) , \qquad \mathscr{U}_i = a(\mathscr{U}_{i-1}, \mathscr{E}, \boldsymbol{C})$$

$(1 \leqq i < \infty)$ of subalgebras $\mathscr{A}_0$ over $\boldsymbol{C}_r$ and $\boldsymbol{C}$ respectively such that $\mathscr{T}_i \subset \mathscr{U}_i$ for all i. We verify that $\pi(\mathscr{T}_n)$ is dense in $\mathscr{L}^*(\mathscr{U}_n)$ and that $\pi'(\mathscr{T}_n)$ is dense in $\mathscr{R}^*(\mathscr{U}_n)$. Recalling that π is a homomorphism of $\mathscr{A}_0$ into the algebra of bounded operators, we see that it is sufficient for us to show that the closure $\pi(\mathscr{T}_n, \mathscr{E}_0, \boldsymbol{C}_r)$ of $\pi(\mathscr{S}(\mathscr{T}_n, \mathscr{E}_0, \boldsymbol{C}_r))$ contains $\pi(\mathscr{S}(\mathscr{U}_n, \mathscr{E}, \boldsymbol{C}))$ using the fact that $\pi(\mathscr{T}_n)$ and $\pi'(\mathscr{T}_n)$ are dense in $\pi(\mathscr{U}_n)$ and $\pi'(\mathscr{U}_n)$ respectively. The density of $\pi'(\mathscr{T}_{n+1})$ in $\mathscr{R}^*(\mathscr{U}_{n+1})$ follows from this because the map $\pi'(\zeta) \to \pi(J\zeta)$ for ζ in $\mathscr{A}_0$ can be extended to a conjugate linear isometric isomorphism of $\mathscr{R}(\mathscr{A}_0)$ onto $\mathscr{L}(\mathscr{A}_0)$ carrying $\pi'(\mathscr{T}_{n+1})$ and $\pi'(\mathscr{U}_{n+1})$ onto $\pi(\mathscr{T}_{n+1})$ and $\pi(\mathscr{U}_{n+1})$ respectively.

Let $h \in \mathscr{E}_0$ and $\lambda \in \boldsymbol{C}_r$; we show first that $\pi(J^k(1 + \Delta)^{-1}h(\log \Delta)\Delta^\lambda\zeta)$ is in $\pi(\mathscr{T}_n, \mathscr{E}_0, \boldsymbol{C}_r)$ for every $\zeta \in \mathscr{U}_n$. Because the elements $\eta = \Delta^\lambda\zeta$, $J\Delta^\lambda\zeta$ and $\Delta^{\lambda-1/2}\zeta$ are in $\mathscr{U}_n$, there are sequences $\{\zeta_m\}$ and $\{\xi_m\}$ in $\mathscr{T}_n$ such that

$$\lim \pi(\zeta_m) = \pi(J\eta)$$

and

$$\lim \pi'(\xi_m) = \pi'(\Delta^{1/2}\eta) .$$

We have that

$$\begin{aligned}||\pi((1 + \Delta)^{-1}J\zeta_m) - \pi((1 + \Delta)^{-1}\eta)|| \\ &\leqq 2^{-1}||\pi'(J\zeta_m - \eta)|| \\ &\leqq 2^{-1}||\pi(\zeta_m - J\eta)||\end{aligned}$$

and

$$\begin{aligned}||\pi(J(1 + \Delta)^{-1}\Delta^{1/2}\xi_m) - \pi(J(1 + \Delta)^{-1}\eta)|| \\ &= ||\pi(J(1 + \Delta)^{-1}(\Delta^{1/2}\xi_m - \eta))^*|| \\ &= ||\pi(\Delta^{-1/2}J^2(1 + \Delta)^{-1}(\Delta^{1/2}\xi_m - \eta))|| \\ &\leqq 2^{-1}||\pi'(\xi_m - \Delta^{1/2}\eta)||\end{aligned}$$

[34, Lemma 8.1], and consequently, that

$$\lim \pi((1 + \Delta)^{-1}J\xi_m) = \pi((1 + \Delta)^{-1}\eta)$$

and

$$\lim \pi(J(1 + \Delta)^{-1}\Delta^{1/2}\xi_m) = \pi(J(1 + \Delta)^{-1}\eta) .$$

Estimating the norms, we see that

$$\begin{aligned}||\pi(h(\log \Delta)(1 + \Delta)^{-1}(J\zeta_m - \eta))|| \\ &\leqq \int|\hat{h}(t)|\,dt\,||\pi((1 + \Delta)^{-1}(J\zeta_m - \eta))||\end{aligned}$$

for every m, and thus, that

$$\lim \pi((1 + \Delta)^{-1}h(\log \Delta)J\zeta_m) = \pi((1 + \Delta)^{-1}h(\log \Delta)\Delta^{\lambda}\zeta) ,$$

and similarly, that

$$\begin{aligned}\lim \pi(J(1 + \Delta)^{-1}h(\log \Delta)\Delta^{1/2}\xi_m) \\ &= \lim \pi(\tilde{h}(\log \Delta)J(1 + \Delta)^{-1}\Delta^{1/2}\xi_m) \\ &= \pi(\tilde{h}(\log \Delta)J(1 + \Delta)^{-1}\eta) \\ &= \pi(J(1 + \Delta)^{-1}h(\log \Delta)\Delta^{\lambda}\zeta) .\end{aligned}$$

It is now clear that $\pi(\mathscr{T}_n, \mathscr{E}_0, C_r)$ contains all elements of the form

$$(6) \qquad \pi(J^k(1 + \Delta)^{-1}h(\log \Delta)^l\Delta^{\lambda}\zeta)$$

for $\zeta \in \mathscr{U}_n$, $h \in \mathscr{E}_0$, $\lambda \in C_r$, $k, l = 0, 1$.

For $\zeta \in \mathscr{U}_n$, the map

$$\lambda \longrightarrow \pi(J^k(1 + \Delta)^{-1}h(\log \Delta)^l\Delta^{\lambda}\zeta) = \pi(\Delta^{\beta}J^k(1 + \Delta)^{-1}h(\log \Delta)^l\zeta)$$

is continuous since $J^k(1 + \Delta)^{-1}h(\log \Delta)^l\zeta$ is in $\mathscr{A}_0$. Here β is λ if k is 0 and β is $-\bar{\lambda}$ if k is 1. Hence, every element of the form (6) for $\zeta \in \mathscr{U}_n$, $\lambda \in C$ and h in $\mathscr{E}_0$ is in $\pi(\mathscr{T}_n, \mathscr{E}_0, C_r)$.

We now can show every element of the form (6) for arbitrary $\zeta \in \mathscr{U}_n$, $\lambda \in \boldsymbol{C}$, $h \in \mathscr{E}$, $k, l = 0, 1$ is in $\pi(\mathscr{T}_n, \mathscr{E}_0, \boldsymbol{C}_r)$. It is sufficient to show that there is, for given ζ in $\mathscr{A}_0$ and h in $\mathscr{E}$, a sequence $\{h_n\}$ in $\mathscr{E}_0$ with

$$\lim \pi(h_m(\log \varDelta)\zeta) = \pi(h(\log \varDelta)\zeta)$$

due to the fact that $Jh(\log \varDelta)$ is equal to $\tilde{h}(\log \varDelta)J$ and that $\mathscr{E}_0$ is invariant under the involution $\sim$. There is a sequence $\{h_m\}$ in $\mathscr{E}_0$ and a closed bounded interval I of $\boldsymbol{R}$ with $\operatorname{supp} h_m \subset \operatorname{supp} h + I$ and $\|h_m - h\|_\infty < m^{-1}$ for $m = 1, 2, \cdots$. The convolution $k_m * h$ of the approximate identity k_m with h is in $\mathscr{E}$ and

$$\|\pi(h(\log \varDelta)\zeta) - \pi((k_m * h)(\log \varDelta)\zeta)\| \leqq \|\pi(\zeta)\| \|(h - k_m * h)\hat{}\|_1 .$$

The sequence $\{(f - k_m * h)\hat{}\}$ tends to 0 in $L^1(\boldsymbol{R})$ by the Dominated Convergence Theorem. Indeed, we have that

$$\limsup \|(h - k_m * h)\hat{}\|_\infty \leqq \limsup \|h - k_m * h\|_1 = 0 ,$$

and that

$$\begin{aligned} |(h - k_m * h)\hat{}| &\leqq (|\hat{k}_m| + 1)|\hat{h}| \\ &\leqq (\|k_m\|_1 + 1)|\hat{h}| \leqq 2|\hat{h}| \end{aligned}$$

for every m. Then we get that

$$\begin{aligned} &\|\pi(h(\log \varDelta)\zeta) - \pi((k_m * h_p)(\log \varDelta)\zeta)\| \\ &\qquad \leqq \|\pi((h - k_m * h)(\log \varDelta)\zeta)\| + \|\pi((k_m * h - k_m * h_p)(\log \varDelta)\zeta)\| \\ &\qquad \leqq \|\pi(\zeta)\|(\|(h - k_m * h)\hat{}\|_1 + \|\hat{k}_m(\hat{h} - \hat{h}_p)\|_1) \end{aligned}$$

and

$$\begin{aligned} \|\hat{k}_m(\hat{h} - \hat{h}_p)\|_1 &\leqq \|\hat{k}_m\|_1 \|(h - h_p)\hat{}\|_\infty \\ &\leqq \|\hat{k}_m\|_1 \|h - h_p\|_1 \\ &\leqq p^{-1}\|\hat{k}_m\|_1 \text{ meas } (\operatorname{supp} h + I) . \end{aligned}$$

Thus, by choosing m and then p, we can get $\pi((k_m * h_n)(\log \varDelta)\zeta)$ arbitrarily close to $\pi(h(\log \varDelta)\zeta)$. This proves $\pi(\mathscr{S}(\mathscr{U}_n, \mathscr{E}, \boldsymbol{C}))$ is contained in $\pi(\mathscr{T}_n, \mathscr{E}_0, \boldsymbol{C})$ and completes the induction step.

We now let $\mathscr{T} = \cup \mathscr{T}_n$ and $\mathscr{U} = \cup \mathscr{U}_n$. It is clear $\mathscr{T}$ is a quasi-invariant countable subalgebra of $\mathscr{A}_0$ over $\boldsymbol{C}_r$, that $\mathscr{U}$ is an invariant subalgebra of $\mathscr{A}_0$, that $\mathscr{S} \subset \mathscr{T} \subset \mathscr{U}$, and that $\pi(\mathscr{T})$ is dense in $\mathscr{L}^*(\mathscr{U})$.

The next lemma is needed in order to construct factor representations.

LEMMA 8. *Let N be a weakly dense *-subalgebra of M and let $\mathscr{S}$ be a countable subset of M. Then there is an open dense subset*

X_0 of X and a countable subset $\mathscr{T}$ of N with the following properties: given x, y in $\mathscr{S}$, χ in X_0 with $xe(\chi) \neq 0$, and $\varepsilon > 0$, then there is a finite subset $u_1, u_2, \cdots, u_n$ of $\mathscr{T}$ and a finite subet $v_1, v_2, \cdots, v_n$ of M' such that

(i) $||\sum u_i v_i xe(\chi) - ye(\chi)|| < \varepsilon$

and

(ii) $||\sum u_i v_i(\chi)|| \leqq ||ye(\chi)||/||xe(\chi)||$.

Proof. Let x and y in $\mathscr{S}$. The subset

$$X(x, m) = \text{clos}\{\chi \in X \mid ||xe(\chi)|| > m^{-1}\}$$

$(m = 1, 2, \cdots)$ is an open and closed subset of the hyperstonean space X due to the continuity of the map $\chi \to ||xe(\chi)||$ [8, Lemma 9]. For natural numbers m, n, we show that there is an open dense subset $X(x, y, m, n) = X'$ of $X(x, m)$ and a countable subset $\mathscr{T}(x, y, m, n) = \mathscr{T}'$ of N, such that for every χ in X', there are finite subsets $\{u_i\}$ of $\mathscr{T}'$ and $\{v_i\}$ of M' such that

$$||\sum u_i v_i xe(\chi) - ye(\chi)|| < n^{-1}$$

and

$$||\sum u_i v_i(\chi)|| \leqq ||ye(\chi)||/||xe(\chi)|| .$$

Let p be the projection in C whose Gelfand transform is the characteristic function of $X(x, m)$. There are partial isometries u and v in C'_p with $u^*u = ep$ and $v^*v \leqq ep$, and there are elements c and d in C_p with $xep = cuep$ and $yep = dvep$ due to the polar decomposition. Because $|c\hat{\ }(\chi)| = ||xe(\chi)||$ for χ in X', there is a c' in C_p with $c'c = p$. There is a sequence $\{w_i\}$ in the unit sphere of the strongly dense *-algebra of C' generated by N and M' such that

$$\lim w_i uu^*p = vu^*p \qquad \text{(strongly)}$$

on account of the Kaplansky Density Theorem and the fact that the abelian projection uu^*p corresponds to the closure of the linear manifold $Cup\zeta_0$. There is a sequence $\{p_i\}$ of orthogonal projections of C of sum p and subsequences $\{w_{ij}\}$ of $\{w_i\}$ such that

$$\lim w_{ij} uu^*p_i = vu^*p_i = vu^*(uu^*p_i)$$

in the norm for each i [11, Lemma 1]. Because w_i can be written as

$$w_i = \sum \{u_j v_j \mid j \in I_i\}$$

where the u_i are in N, the v_i are in M', and I_i is a finite index set, the totality $\mathscr{T}'$ of all such u_i is a countable set. Furthermore, the set X' of all χ in X such that $p_i^\wedge(\chi) = 1$ for some i is an open dense subset of $X(x, m)$.

We now show that $\mathscr{T}'$ and X' have the correct properties. Let χ be in X'. Thers is no loss of generality in the assumption that $d^\wedge(\chi) \neq 0$. There is a p_i with $p_i^\wedge(\chi) = 1$ and a w_{ij} with

$$||w_{ij}ue(\chi) - ve(\chi)|| \leqq ||(w_{ij}uu^* - vu^*)p_i|| \, ||ue(\chi)|| < (n|d^\wedge(\chi)|)^{-1} .$$

Setting w equal to $d^\wedge(\chi)c'^\wedge(\chi)w_{ij}$, we have that

$$||wxe(\chi) - ye(\chi)|| < n^{-1}$$

and

$$||w(\chi)|| \leqq |d^\wedge(\chi)|/|c^\wedge(\chi)| \leqq ||ye(\chi)||/||xe(\chi)|| .$$

Thus $\mathscr{T}'$ and X' have the correct properties.

The countable subset

$$\mathscr{T} = \{\mathscr{T}(x, y, m, n)) \,|\, x, y \in \mathscr{S}, m, n = 1, 2, \cdots\}$$

and any dense open subset X_0 of the set

$$\cap \{\cup \{X(x, y, m, n) \,|\, m = 1, 2, \cdots\} \cup X(x) \,|\, x, y \in \mathscr{S}, n = 1, 2, \cdots\} ,$$

whose complement is nowhere dense, satisfy the conditions of the lemma. Here $X(x)$ is the complement in X of the closure of the set $\{\chi \in X \,|\, ||xe(\chi)|| > 0\}$.

We are now able to prove the main construction lemma.

LEMMA 9. *Let $\mathscr{S}$ be a countable subset of the maximal modular Hilbert algebra $\mathscr{A}_0$ of the generalized Hilbert algebra $M\zeta_0$, and let $\{\theta_t\}$ be a strongly continuous one parameter automorphism group of M. Then there is a countable subalgebra $\mathscr{T}$ of $\mathscr{A}_0$ over C_r, a subalgebra $\mathscr{U}$ of $\mathscr{A}_0$, and an open dense subset X_0 of X with the following properties:*

(i) $\mathscr{S} \subset \mathscr{T} \subset \mathscr{U}$;

(ii) *$\mathscr{T}$ is quasi-invariant;*

(iii) *$\mathscr{U}$ is invariant;*

(iv) *$\pi(\mathscr{T})$ is dense in $\mathscr{L}^*(\mathscr{U})$;*

(v) *$\mathscr{L}^*(\mathscr{U})''$ is invariant under $\{\theta_t\}$; and*

(vi) *τ_χ restricted to $\mathscr{L}^*(\mathscr{U})$ is a primary state fox every χ in X_0.*

Proof. We may assume that ζ_0 is in $\mathscr{S}$. By induction we construct sequences $\{\mathscr{T}_n\}$ and $\{\mathscr{V}_n\}$ of countable subalgebras of $\mathscr{A}_0$ over C_r, a sequence $\{\mathscr{U}_n\}$ of subalgebras of $\mathscr{A}_0$, a sequence $\{N_n\}$ of von Neumann algebras, and a sequence $\{X_n\}$ of subsets of X such that for all $n = 1, 2, \cdots$

(1) $\mathscr{S} \subset \mathscr{T}_n \subset \mathscr{V}_n \subset \mathscr{T}_{n+1}$;
(2) $\mathscr{T}_n \subset \mathscr{U}_n \subset \mathscr{U}_{n+1}$;
(3) $\pi(\mathscr{T}_n)$ is dense in $\mathscr{L}^*(\mathscr{U}_n)$;
(4) $\mathscr{L}^*(\mathscr{U}_n)'' \subset N_n \subset \mathscr{L}^*(\mathscr{V}_n)''$;
(5) $\mathscr{T}_n$ is quasi-invariant;
(6) $\mathscr{U}_n$ is invariant;
(7) N_n is invariant under θ_t $(t \in \boldsymbol{R})$;
(8) $X_n \supset X_{n+1}$ and X_n is open and dense; and
(9) given x, y in $\pi(\mathscr{V}_n)$, χ in X_n with $xe(\chi) \neq 0$, and $\varepsilon > 0$, then there are $x_1, \cdots, x_m$ in $\pi(\mathscr{T}_{n+1})$ and $y_1, \cdots, y_m$ in M' with

$$||\sum x_i y_i xe(\chi) - ye(\chi)|| < \varepsilon$$

and

$$||\sum x_i y_i(\chi)|| \leqq ||ye(\chi)||/||xe(\chi)|| .$$

Let $\mathscr{T}_1$ be a quasi-invariant countable subalgebra of $\mathscr{A}_0$ over $\boldsymbol{C}_r$, and let $\mathscr{U}_1$ be an invariant subalgebra of $\mathscr{A}_0$ such that $\mathscr{S} \subset \mathscr{T}_1 \subset \mathscr{U}_1$ and such that $\pi(\mathscr{T}_1)$ is dense in $\mathscr{L}^*(\mathscr{U}_1)$ (Lemma 7). The von Neumann algebra N_1 generated by

$$\{\theta_t(x) \mid x \in \pi(\mathscr{T}_1), t \in \boldsymbol{R}_r\} ,$$

where $\boldsymbol{R}_r$ is the field of rational real numbers, is invariant under the automorphism group $\{\theta_t\}$ and is equal to the von Neumann algebra generated by

$$\{\theta_t(x) \mid x \in \mathscr{L}^*(\mathscr{U}_1), t \in \boldsymbol{R}_r\} .$$

There is a countable subset of $\mathscr{A}_0$ and thus a countable #-subalgebra $\mathscr{V}_1$ over $\boldsymbol{C}_r$ of $\mathscr{A}_0$ containing $\mathscr{T}_1$ such that $\pi(\mathscr{V}_1)''$ contains N_1 (Lemma 7). There is a countable subset $\mathscr{W}_1$ of $\mathscr{A}_0$ and an open dense subset X_1 of X such that, for any x, y $\pi(\mathscr{V}_1)$, $\chi \in X_1$ with $xe(\chi) \neq 0$, and any $\varepsilon > 0$, there are $x_1, \cdots, x_n$ in $\pi(\mathscr{W}_1)$ and $y_1, y_2, \cdots, y_n$ in M' with

$$||\sum x_i y_i xe(\chi) - ye(\chi)|| < \varepsilon$$

and

$$||\sum x_i y_i(\chi)|| \leqq ||ye(\chi)||/||xe(\chi)|| .$$

Let $\mathscr{T}_2$ be a quasi-invariant countable subalgebra of $\mathscr{A}_0$ over $\boldsymbol{C}_r$ containing $\mathscr{V}_1 \cup \mathscr{W}_1$. We may now repeat the construction starting with $\mathscr{T}_2$. So we may assume that sequences of algebras satisfying (1)-(9) have been constructed.

We show that $\mathscr{T} = \cup \mathscr{T}_n$ and $\mathscr{U} = \cup \mathscr{U}_n$ satisfy conditions (i)-(vi). From (1) and (2) we conclude that $\mathscr{T}$ and $\mathscr{U}$ are subalgebras of $\mathscr{A}_0$ over $\boldsymbol{C}_r$ and $\boldsymbol{C}$ respectively and from (1), (2), (5), (6) we conclude that $\mathscr{T}$ is quasi-invariant and that $\mathscr{U}$ is invariant. Since each $\pi(\mathscr{T}_n)$

is dense in $\mathscr{L}^*(\mathscr{U}_n)$, the set $\pi(\mathscr{T})$ is dense in $\mathscr{L}^*(\mathscr{U})$. For each $t \in \boldsymbol{R}$ and x in $\mathscr{L}^*(\mathscr{U}_n)$, wə have that $\theta_t(x)$ is contained in $\mathscr{L}^*(\mathscr{U})''$ on account of (1), (2), (4), (7). Because $\mathscr{L}^*(\mathscr{U})$ is weakly dense in the weakly closed algebra $\mathscr{L}^*(\mathscr{U})''$ and because θ_t is weakly continuous, the algebra $\mathscr{L}^*(\mathscr{U})''$ is invariant under θ_t.

Finally, we verify statement (vi). Because X is hyperstonean, the intersection of the sets X_n contains an open dense subset X_0. We show that τ_χ is a primary state of $\mathscr{L}^*(\mathscr{U})$ for every χ in X_0. Let x, y be in $\pi(\mathscr{T}_i)$ with $xe(\chi)$ nonzero. There are $x_1, \cdots, x_n$ in $\pi(\mathscr{T}_{i+1})$ and $y_1, y_2, \cdots, y_n$ in M' such that

$$||\sum x_j y_j xe(\chi) - ye(\chi)|| < \varepsilon$$

and

$$||\sum x_j y_j(\chi)|| < ||ye(\chi)||/||xe(\chi)||$$

from (9). Each operator $y_j(\chi)$ on $C'e(\chi)$ is in the commutant $\mathscr{L}^*(\mathscr{U})(\chi)'$ of the von Neumann algebra $\mathscr{L}^*(\mathscr{U})(\chi)''$. Setting p equal to the projection of $C'e(\chi)$ on the subspace K_χ generated by $\mathscr{L}^*(\mathscr{U})(\chi)e(\chi)$, we get that the operators $z_j = py_j(\chi)p$ are in $(\mathscr{L}^*(\mathscr{U})(\chi)_p)'$, and satisfy the relations

$$||\sum px_j(\chi)pz_j xe(\chi) - ye(\chi)|| < \varepsilon$$

and

$$||\sum px_j(\chi)pz_j|| \leqq ||ye(\chi)||/||xe(\chi)|| \, .$$

Thus, the von Neumann algebra on K_χ generated by $\mathscr{L}^*(\mathscr{U})(\chi)_p$ and its commutant is transitive on K_χ. Indeed, the arguments of [16] (cf. [24], Chapter 4, § 9) can be applied to the dense subset $\cup\, \pi(\mathscr{T}_j)e(\chi)$ of vectors in K_χ. We have already identified the action of $\mathscr{L}^*(\mathscr{U})(\chi)$ on K_χ with the canonical representation of τ_χ. Thus, the functional τ_χ on $\mathscr{L}^*(\mathscr{U})$ is a primary state.

We now analyze the center in the situation described by the previous lemma.

PROPOSITION 10. *Let $\mathscr{B}$ be an invariant subalgebra containing ζ_0 of the maximal modular algebra of the generalized Hilbert algebra $M\zeta_0$. If the set of χ in X such that the restriction of τ_χ to $\mathscr{L}^*(\mathscr{B})$ is not a primary state is nowhere dense in X, then the center of the von Neumann algebra $\mathscr{L}^*(\mathscr{B})''$ generated by $\mathscr{L}^*(\mathscr{B})$ on H is contained in C.*

Proof. The C-module $\mathscr{B}'$ generated by $\mathscr{B}$ is an invariant subalgebra of $\mathscr{A}_0$ and the algebra $\mathscr{L}^*(\mathscr{B}')$ is equal to the C^*-algebra generated by $\mathscr{L}^*(\mathscr{B})$ and C (cf. relation (1)). The state τ_χ is a primary state on $\mathscr{L}^*(\mathscr{B})$ if and only if it is a primary state on $\mathscr{L}^*(\mathscr{B}')$ since the Hilbert spaces of the canonical representations of both functionals can be identified with the closure of $\mathscr{L}^*(\mathscr{B})(\chi)e(\chi)$ in $C'e(\chi)$ and the images of both canonical representations is $\mathscr{L}^*(\mathscr{B})(\chi)$ restricted to this invariant subspace. We notice that the center of $\mathscr{L}^*(\mathscr{B})''$ is contained in that of $\mathscr{L}^*(\mathscr{B}')''$. So there is no loss of generality in the assumption that $\mathscr{B}$ is a C-module. Accordingly, we must prove that the center of $\mathscr{L}^*(\mathscr{B})''$ is C.

Let c be an element in the center of $\mathscr{L}^*(\mathscr{B})''$ with $0 \leqq c \leqq 1$. We prove that c is in C. For χ in X, let π_χ denote the canonical rapresentation of $\mathscr{L}^*(\mathscr{B})$ induced by τ_χ and let ψ_χ be a positive linear functional on $\pi_\chi(\mathscr{L}^*(\mathscr{B}))$ such that

$$\psi_\chi(\pi_\chi(x)) = \tau_\chi(cx)$$

for every x in $\mathscr{L}^*(\mathscr{B})$. Since we have

$$|\psi_\chi(\pi_\chi(x))| \leqq \tau_\chi(x^*x)^{1/2}\tau_\chi(c^2)^{1/2} \leqq ||\pi_\chi(x)e(\chi)|| ,$$

we can apply the Hahn Banach theorem to find an extension of ψ_χ to a linear functional on $\pi_\chi(\mathscr{L}^*(\mathscr{B}))''$, which we again denote by ψ_χ, such that

$$||\psi_\chi(x)|| \leqq ||xe(\chi)||$$

for x in $\pi_\chi(\mathscr{L}^*(\mathscr{B}))''$. The preceding relation implies that ψ_χ is a strongly continuous, positive functional majorized by the vector state $\omega_\chi(x) = (xe(\chi), e(\chi))$ on $\pi_\chi(\mathscr{L}^*(\mathscr{B}))''$. We show that ψ_χ satisfies the KMS boundary conditions with regard to the modular automorphism group $\sigma_t^\chi(x) = \Delta_\chi^{it}x\Delta_\chi^{-it}$ of $\pi_\chi(\mathscr{L}^*(\mathscr{B}))''$ (cf. Proposition 4).

First we show that ψ_χ is invariant under σ_t^χ. Let K be the subspace of H generated by $\mathscr{L}^*(\mathscr{B})\zeta_0$ and let p be the projection of H on K. The algebra $\mathscr{B}$ is a modular Hilbert algebra with completion K. The modular automorphism group of $\mathscr{L}(\mathscr{B})$ is $x \to \Delta^{it}x\Delta^{-it}p$ for x in $\mathscr{L}^*(\mathscr{B})''_p$, which is identified with $\mathscr{L}(\mathscr{B})$ due to Lemma 3. For u and v in C and x in $\mathscr{L}^*(\mathscr{B})$, we have

$$\begin{aligned}(\tau(c\sigma_t(x))u\zeta_0, v\zeta_0) &= (c\sigma_t(x)u\zeta_0, v\zeta_0)\\ &= (pcp(\Delta^{it}x\Delta^{-it})ppup\zeta_0, pvp\zeta_0)\\ &= (\Delta^{it}(pcpxp)\Delta^{-it}ppup\zeta_0, pvp\zeta_0)\\ &= (\sigma_t(cx)u\zeta_0, v\zeta_0)\\ &= (\tau(cx)u\zeta_0, v\zeta_0) .\end{aligned}$$

This means that

$$\tau_\chi(c\sigma_t(x)) = \tau_\chi(cx)$$

for x in $\mathscr{L}^*(\mathscr{B})$, and thus that

$$\begin{aligned}\psi_\chi(\sigma_t^\chi(\pi_\chi(x))) &= \tau_\chi(c\sigma_t(x)) \\ &= \tau_\chi(cx) = \psi_\chi(\pi_\chi(x))\end{aligned}$$

for x in $\mathscr{L}^*(\mathscr{B})$ (cf. relation (2)). By the strong continuity of σ_t^χ and ψ_χ, we conclude that ψ_χ is invariant under σ_t^χ.

Now, for x and y in $\mathscr{L}^*(\mathscr{B})$ there is a bounded continuous function h of the strip $I = \{\lambda \in \boldsymbol{C} \mid 0 \leqq \operatorname{Im} \lambda \leqq 1\}$ into C, which is holomorphic on the interior $0 < \operatorname{Im} \lambda < 1$ of I and satisfies the boundary conditions

$$h(t) = \tau(\sigma_t(x)cy) = \tau(c\sigma_t(x)y)$$

and

$$h(t + i) = \tau(cy\sigma_t(x))$$

for $t \in \boldsymbol{R}$ (Corollary 6). The function

$$h_\chi(\lambda) = h(\lambda)\hat{\ }(\chi)$$

is a bounded continuous function on I, holomorphic on the interior, and satisfies the boundary conditions

$$h_\chi(t) = \psi_\chi(\sigma_t^\chi(\pi_\chi(x))\pi_\chi(y))$$

and

$$h_\chi(t + i) = \psi_\chi(\pi_\chi(y)\sigma_t^\chi(\pi_\chi(x)))$$

for $t \in \boldsymbol{R}$. This is enough to insure that ψ_χ satisfies the KMS boundary conditions on $\pi_\chi(\mathscr{L}^*(\mathscr{B}))''$ with respect to $\{\sigma_t^\chi\}$ (cf. proof [34, Theorem 13.3]).

For every χ such that τ_χ is a primary state of $\mathscr{L}^*(\mathscr{B})$, there is a real number α_χ such that $\alpha_\chi \omega_\chi = \psi_\chi$ [34, Theorem 15.4]. Since α_χ can be written as

$$\alpha_\chi = \alpha_\chi \tau(1)\hat{\ }(\chi) = \psi_\chi(\pi_\chi(1)) = \tau(c)\hat{\ }(\chi)$$

for all χ in an open dense subset of X, there is an element d in C such that

$$\tau_\chi(cx) = d\hat{\ }(\chi)\tau_\chi(x) = \tau_\chi(dx)$$

for all x in $\mathscr{L}^*(\mathscr{B})$ and χ in X. There is a sequence $\{c_n\}$ in $\mathscr{L}^*(\mathscr{B})$ that converges strongly to c. We now can conclude that c is equal to d and thus is in C because

$$\|(d - c)^*\zeta_0\| = \lim\,(\tau((d - c)(d - c_n)^*)\zeta_0, \zeta_0) = 0\ .$$

Thus, the center of $\mathscr{L}^*(\mathscr{B})''$ is contained in C.

We can now show that every type II_1 (resp. type II_∞) von Neumann algebra has a σ-weakly continuous projection of norm one onto its countably generated type II_1 (resp. type II_∞) subalgebra. A more detailed version is needed for the subsequent proof of the analogous theorem for type III algebras.

PROPOSITION 11. *Let M be a type II_1 (resp. type II_∞) von Neumann algebra, let ψ be a faithful normal finite (resp. semi-finite) trace on M and let $\{\theta_t\}$ be a strongly continuous one parameter automorphism group of M. Then there is a subalgebra $\mathscr{B}$ containing ζ_0 of the maximal modular Hilbert algebra $\mathscr{A}_0$ of the generalized Hilbert algebra $M\zeta_0$ such that*

(i) *$\mathscr{B}$ in invariant;*

(ii) *$\mathscr{L}^*(\mathscr{B})$ is a separable C^*-algebra;*

(iii) *$\mathscr{L}^*(\mathscr{B})''$ is invariant under each θ_t $(t \in \boldsymbol{R})$;*

(iv) *$\mathscr{L}^*(\mathscr{B})''$ is of type II_1 (resp. type II_∞);*

(v) *the restriction of ψ to $\mathscr{L}^*(\mathscr{B})''$ is semi-finite; and*

(vi) *the complement of the set of all χ in X such that the restriction of τ_χ to $\mathscr{L}^*(\mathscr{B})$ is not a primary state of $\mathscr{L}^*(\mathscr{B})$ is nowhere dense in X.*

Proof. Since M is semi-finite there is a self-adjoint operator a affiliated with M such that $\sigma_t(x) = \exp(ita)\, x \exp(-ita)$ for t in $\boldsymbol{R}$ and x in M [34, Theorem 14.2]. Let $\{g(\lambda)\}$ be the spectral resolution of a. We may write $g_n = g(n) - g(-n)$ as

$$g_n = \sum \{u^*u \mid u \in I(n, m)\}$$

where the u are partial isometries in M with $uu^* = vv^*$ for every u, v in $I(n, m)$ and the cardinality of $I(n, m)$ is 2^m. If M is properly infinite, let I_0 be a set of partial isometries in M of cardinality equal to that of the natural numbers such that

$$\sum \{u^*u \mid u \in I_0\} = 1$$

and such that $uu^* = vv^*$ for every u, v in I_0. If M is a finite algebra, let I_0 be the empty set. Let $\{p_n\}$ be a monotonely increasing sequence of projections in M of least upper bound 1 such that $\psi(p_n)$ is finite for every n. Let $\mathscr{S}$ be a countable subset of $\mathscr{A}_0$ such that the weak closure of $\pi(\mathscr{S})$ contains union of all the sets

$$I(n, m),\ I_0,\ \{g(\lambda) \mid \lambda \text{ rational}\},\ \{p_n\}\ .$$

There is a subalgebra $\mathscr{B}$ containing 1 of $\mathscr{A}_0$ containing $\mathscr{S}$ and possessing properties (i)-(iii) and (vi) due to Lemma 9. We verify that $\mathscr{B}$ satisfies (iv) and (v).

First we consider (v). A proof is necessary only when M is of type II_∞. The restriction ϕ of ψ to $\mathscr{L}^*(\mathscr{B})''$ is a faithful normal trace. If x is an element in $\mathscr{L}^*(\mathscr{B})''$, then the sequence $\{xp_n\}$ in $N_\phi^* N_\phi$ converges weakly to x. This proves that ϕ is a faithful normal semi-finite trace on $\mathscr{L}^*(\mathscr{B})''$.

Now if M is of type II_∞, the algebra $\mathscr{L}^*(\mathscr{B})''$ is properly infinite since 1 is the least upper bound of the infinite set $\{u^*u \mid u \in I_0\}$ of equivalent orthogonal projections. If M is finite, it is clear that $\mathscr{L}^*(\mathscr{B})''$ is finite.

Finally, we show that $\mathscr{L}^*(\mathscr{B})''$ is a continuous algebra. Let p be the largest central projection of $\mathscr{L}^*(\mathscr{B})''$ such that $\mathscr{L}^*(\mathscr{B})''_p$ is discrete. The modular automorphism group of $\mathscr{L}^*(\mathscr{B})''$ associated with the faithful normal state ω_{ζ_0} restricted to $\mathscr{L}^*(\mathscr{B})''$ is given by $x \to \exp(ita)\, x \exp(-ita)$ (Lemma 3). Because the spectral projections $g(\lambda)$ (λ rational) for a are in $\mathscr{L}^*(\mathscr{B})''$, the operator a is affiliated with $\mathscr{L}^*(\mathscr{B})''$. For every projection $g = g_n$, we can show that the functional

$$x \longrightarrow (\exp(-ag) gx\zeta_0, \zeta_0)$$

is a faithful normal trace on $\mathscr{L}^*(\mathscr{B})''_g$. Indeed, given x and y in $\mathscr{L}^*(\mathscr{B})''_g$, there is a bounded continuous function $h(\lambda)$ on the strip $0 \leqq \operatorname{Im} \lambda \leqq 1$, which is holomorphic on $0 < \operatorname{Im} \lambda < 1$, and which satisfies the conditions

$$h(t) = (\sigma_t(x) y\zeta_0, \zeta_0)$$

and

$$h(t + i) = (y\sigma_t(x)\zeta_0, \zeta_0)$$

for t in $\boldsymbol{R}$. Since the function

$$\lambda \longrightarrow (\exp(i\lambda ag) gx \exp(-i\lambda ag) gy\zeta_0, \zeta_0)$$

is an entire function which coincides with h on the real axis, it agrees with h on the other boundary of the strip due to Schwarz's reflection principle, and in particular at the point i, so that

$$(yx\zeta_0, \zeta_0) = (\exp(-ag) x \exp(ag) y\zeta_0, \zeta_0) .$$

Replacing y by $\exp(-ag)y$, we see that

$$(\exp(-ag) xy\zeta_0, \zeta_0) = (\exp(-ag) yx\zeta_0, \zeta_0)$$

for every x, y in $\mathscr{L}^*(\mathscr{B})''_g$. Now, if p is nonzero, there is a $g = g_n$ such that pg is nonzero. There is a central projection q in $\mathscr{L}^*(\mathscr{B})''$ such that $\mathscr{L}^*(\mathscr{B})''_{qg}$ is of type I_m for some $1 \leqq m < \infty$. However, the projection g and consequently gq must be the sum of the 2^m

equivalent orthogonal projections $\{u^*uq \mid u \in I(n, m)\}$. This is a contradiction. Thus, the algebra $\mathscr{L}^*(\mathscr{B})''$ has no type I part.

COROLLARY 12. *Every σ-finite von Neumann algebra N of type I_n (resp. type II_1, II_∞) has a faithful, σ-weakly continuous projection of norm one onto a countably generated von Neumann subalgebra of the same type whose center is contained in that of N.*

Proof. First let N be of type I_n. Then N is isomorphic to the von Neumann algebra $A \otimes \mathscr{B}(H_n)$, where A is an abelian von Neumann algebra and $\mathscr{B}(H_n)$ is the algebra of all bounded operators on the Hilbert space H_n of dimension n. The algebra A is σ-finite and is isomorphic to an algebra with a separating and cyclic vector. We may assume A has a cyclic and separating vector ζ. There is a faithful σ-weakly continuous projection ε of norm one of A onto a countably generated von Neumann subalgebra B (cf. [35]). Then the map $\varepsilon \otimes$ (identity) is a faithful σ-weakly continuous projection of norm one of $A \otimes \mathscr{B}(H_n)$ onto $B \otimes \mathscr{B}(H_n)$ [41, Theorem 2].

Now let N be of type II_1. Since N is σ-finite, it is isomorphic to a finite Neumann algebra with cyclic and separating vector. So we may assume that N has a cyclic and separating vector ξ. There is an invariant subalgebra $\mathscr{B}$ of the maximal modular Hilbert algebra of the generalized Hilbert algebra $N\xi$ such that $\mathscr{L}^*(\mathscr{B})''$ is a countably generated type II_1 subalgebra of N whose center is contained in that of N (Propositions 10 and 11). The algebra $\mathscr{L}^*(\mathscr{B})''$ is invariant under the modular automorphism group of $N\xi$ (Lemma 3) and so there is a faithful σ-weakly continuous projection of norm one of N onto $\mathscr{L}^*(\mathscr{B})''$ [35].

If N is of type II_∞, one can use the existence of a projection in the type II_1 case to present an argument similar to the first paragraph showing the required projection exists.

REMARK 13. If N is a finite type I (resp. properly infinite type I, type II_1, type II_∞), then there is a set S of orthogonal central projections of sum 1 so that each p in S can be written as the sum of a set $S(p)$ of orthogonal equivalent σ-finite projections. Each algebra Np (p in S) is isomorphic to an algebra of the form $N_g \otimes \mathscr{B}(H_p)$ where g is in $S(p)$ and H_p is a Hilbert space with dimension equal to the cardinality of $S(p)$. If ϕ is a normal state of $\mathscr{B}(H_p)$ while ε is a projection of norm one for N_g of the type described in Proposition 11, then $\varepsilon \otimes \phi$ is σ-weakly continuous projection of norm one of $N_g \otimes \mathscr{B}(H_p)$ onto a countably generated subalgebra of the same type [41]. This induces a σ-weakly continuous projection of norm one of Np onto a countably generated subalgebra of the same type.

We now consider the projections of norm one of a type III algebra onto its countably generated subalgebra. For this we need the crossed product of a von Neumann algebra with its one parameter automorphism group. We briefly review some of the necessary notation and results of Takesaki [36] with appropriate reference citations. Let N be a von Neumann algebra on the Hilbert space K and let $\{\theta_t\}$ be a strongly continuous one parameter group of automorphisms of N. Let $L^2(K; \boldsymbol{R})$ be the completion of the pre-Hilbert space $\mathscr{K}(K; \boldsymbol{R})$ of all continuous functions of compact support of $\boldsymbol{R}$ into K in the inner product

$$\langle \zeta, \xi \rangle = \int (\zeta(t), \xi(t)) dt .$$

There is a canonical normal isomorphism π_θ of N onto a von Neumann algebra on $L^2(K; \boldsymbol{R})$ given by

$$(\pi_0(x)\zeta)(t) = \theta_t^{-1}(x)\zeta(t) , \tag{10}$$

and there is a strongly continuous unitary representation λ_θ of $\boldsymbol{R}$ on $L^2(K; \boldsymbol{R})$ given by

$$(\lambda_\theta(t)\zeta)(s) = \zeta(s - t) . \tag{11}$$

[36, (3.1), (3.2).] The von Neumann algebra on $L^2(K; \boldsymbol{R})$ generated by $\pi_\theta(N)$ and $\lambda_\theta(\boldsymbol{R})$ is called the *crossed product* of N by the action of θ on $\boldsymbol{R}$ and is denoted by $\mathscr{R}((N, K); \theta) = \mathscr{R}(N; \theta)$. For every x in N and t in $\boldsymbol{R}$, the relation

$$\lambda_\theta(t)\pi_\theta(x)\lambda_\theta(-t) = \pi_\theta(\theta_t(x))$$

holds [36, (3.2)]. Furthermore, the von Neumann algebra $\mathscr{R}((N, K); \theta)$ does not depend on K in the following sense: if there is an isomorphism Φ of N onto a von Neumann algebra N_1 on the Hilbert space K_1 with a strongly continuous one parameter automorphism group ϕ such that $\Phi \cdot \theta_t = \phi_t \cdot \Phi$ for every $t \in \boldsymbol{R}$, then there is an isomorphism $\tilde{\Phi}$ of $\mathscr{R}((N, K); \theta)$ onto $\mathscr{R}((N_1, K_1); \phi)$ such that $\pi_\phi \cdot \Phi = \tilde{\Phi} \cdot \pi_\theta$ and $\lambda_\phi = \tilde{\Phi} \cdot \lambda_\theta$ [36, Proposition 3.4].

Let $\boldsymbol{R}$ be identified with its dual group under the action $\langle t, s \rangle$. Let $\mu(s)$ be the unitary operator on $L^2(K; \boldsymbol{R})$ defined by $\mu(s)\zeta(t) = \langle t, s \rangle^- \zeta(t)$; then $\mu(s)$ satisfies the relations

$$\mu(s)\pi_\theta(x)\mu(-s) = \pi_\theta(x)$$

and

$$\mu(s)\lambda_\theta(t)\mu(-s) = \langle t, s \rangle^- \lambda(t)$$

so that

$$\hat{\theta}_s(x) = \mu(s)x\mu(-s) \qquad (s \in \boldsymbol{R})$$

is a strongly continuous one parameter automorphism of $\mathscr{R}(N;\theta)$ called the *dual action* of (the dual group) $\boldsymbol{R}$ on $\mathscr{R}(N;\theta)$ [36, (4.1)-(4.5)]. The algebra $\mathscr{R}(\mathscr{R}(N;\theta);\hat{\theta})$ is isomorphic to $N\otimes\mathscr{B}(L^2(\boldsymbol{R}))$ [36, Theorem 4.5].

Let ψ be a faithful normal semi-finite weight on N such that $\psi\cdot\theta_t = e^{-t}\psi$ and let $\mathscr{A}_0$ be the maximal modular Hilbert algebra of the generalized Hilbert algebra associated with ψ supplied with structure of a locally convex topological vector space induced by the semi-norms

$$\|x\|_S = \text{lub}\{\|x\| + \|\pi(\Delta^\lambda x)\| + \|\pi'(\Delta^\lambda x)\| \mid \lambda\in S\} \tag{12}$$

where S runs through the compact subsets of $\boldsymbol{C}$. The algebra $\mathscr{K}(\mathscr{A}_0;\boldsymbol{R})$ of all continuous functions of compact support of $\boldsymbol{R}$ into $\mathscr{A}_0$ is a modular Hilbert algebra with suitably defined multiplication and modular automorphisms whose completion and left algebra are $L^2(H(\psi);\boldsymbol{R})$ and $\mathscr{R}((N, H(\psi));\theta)$ respectively. The canonical weight $\tilde{\psi}$ on $\mathscr{R}(N;\theta)$ associated with $\mathscr{K}(\mathscr{A}_0;\boldsymbol{R})$ given by

$$\tilde{\psi}(x) = \begin{cases} \|\zeta\|^2\,, & \text{if } x = \pi(\zeta)^*\pi(\zeta) \quad \text{for} \quad \zeta\in\mathscr{K}(\mathscr{A}_0;\boldsymbol{R})'' \\ \infty\,, & \text{otherwise} \end{cases}$$

is called the *dual weight* of ψ, and it is faithful, normal, and semi-finite [36, Theorem 5.12, Definition 5.14].

Now let M be a type III von Neumann algebra on the Hilbert space H. Let ζ_0 be a cyclic and separating vector for M, and let $\{\sigma_t\}$ be the modular automorphism group of the generalized Hilbert algebra $M\zeta_0$. The algebra $\mathscr{R}(M;\sigma)$ is of type II_∞ [36, Theorem 8.11] and admits a faithful normal semi-finite trace ψ such that $\psi\cdot\hat{\sigma}_t = e^{-t}\psi$ [36, Lemma 8.2]. The modular automorphism group of the dual weight $\tilde{\psi}$ on $\mathscr{R}(\mathscr{R}(N;\sigma);\hat{\sigma})$ is $\hat{\sigma}\hat{}$ [36, Theorem 8.3].

We now show that a σ-finite type III algebra has a σ-weakly continuous projection of norm one onto a countably generated type III subalgeba, whose center is contained in the original algebra. (The proof of the latter appears in Theorem 15.)

PROPOSITION 14. *Every σ-finite type III von Neumann algebra has a faithful σ-weakly continuous projection of norm one onto a countably generated von Neumann subalgebra of type III.*

Proof. Every σ-finite von Neumann algebra is isomorphic to a Neumann algebra with cyclic and separating vector. Thus, it von is sufficient to show that the canonical algebra M has a projection of the specified kind whenever M is type III.

The von Neumann algebra $M \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ is isomorphic to both M and $\mathscr{R}(\mathscr{R}(M;\sigma);\hat{\sigma})$, and the algebra $N = \mathscr{R}(M;\sigma)$ is an algebra of type II_∞ that admits a faithful normal trace ψ satisfying the relation $\psi \cdot \hat{\sigma}_t = e^{-t}\psi$ for $t \in \boldsymbol{R}$. The algebra N is σ-finite since it is isomorphic under the canonical embedding to a weakly closed subalgebra of the σ-finite algebra $\mathscr{R}(N;\hat{\sigma})$. There is an isomorphism Φ of N onto a von Neumann algebra with cyclic and separating vector ξ_0. There is a subalgebra $\mathscr{B}$ containing ξ_0 of the maximal modular Hilbert algebra of $\Phi(N)\xi_0$ satisfying the properties:

(i) $\mathscr{B}$ is invariant;

(ii) $\mathscr{L}^*(\mathscr{B})$ is a separable C^*-algebra;

(iii) $\mathscr{L}^*(\mathscr{B})''$ is invariant under each automorphism $\Phi \cdot \hat{\sigma}_t \cdot \Phi^{-1}$;

(iv) $\mathscr{L}^*(\mathscr{B})''$ is of type II_∞;

(v) the restriction of $\psi \cdot \Phi^{-1}$ to $\mathscr{L}^*(\mathscr{B})''$ is semi-finite; and

(vi) the complement of the set of all χ in the spectrum of the center D of $\Phi(N)$ such that $(\tau_g)_\chi$ is not a primary state of $\mathscr{L}^*(\mathscr{B})$ is nowhere dense (Proposition 11).

Here g is the maximal abelian projection in D' corresponding to the subspace closure $D\xi_0$. The center of $\mathscr{L}^*(\mathscr{B})''$ is contained in the center of $\Phi(N)$ (Proposition 10). Therefore, the algebra $P = \Phi^{-1}(\mathscr{L}^*(\mathscr{B})'')$ is a countably generated type II_∞ von Neumann subalgebra of N, whose center E is contained in that of N, on which ψ is semi-finite, and which is invariant under $\hat{\sigma}_t$.

The von Neumann algebra B generated by the image of the canonical embedding of P and $\boldsymbol{R}$ in $\mathscr{R}(\mathscr{R}(M;\sigma);\hat{\sigma})$ is countably generated, semi-finite under the dual weight $\tilde{\psi}$ of ψ, and invariant under the modular automorphisms associated with $\tilde{\psi}$ [36, §§5, 8]. There is a faithful σ-weakly continuous projection of norm one of $\mathscr{R}(\mathscr{R}(M;\sigma);\hat{\sigma})$ onto B [35]. We complete the proof by obtaining a contradiction to the assumption that B has a nonzero semi-finite part. Since the algebra $\mathscr{R}(P;(\hat{\sigma}|P))$ is isomorphic to B, we may assume that $\mathscr{R}(P;(\hat{\sigma}|P))$ has a nonzero semi-finite part. There would be a nonzero projection p in the center E of P which is fixed under $\{\hat{\sigma}_t\}$ and a strongly continuous one parameter unitary group $\{u_t\}$ in Ep such that

$$\hat{\sigma}_s(u_t) = e^{ist}u_t$$

[36, Corollary 8.7]. Since the algebra E is contained in the center of N, the algebra $\mathscr{R}((M;\sigma);\hat{\sigma})$ would also have a nonzero semi-finite part [36, Corollary 8.7]. This is a contradiction. Hence, the algebra B is purely infinite.

We now obtain the information on the field $\{M_\chi\}$ when M is of type III.

THEOREM 15. *Let M be a type III von Neumann algebra with center C, let ζ_0 be a cyclic and separating vector, let e be the abelian projection of C' corresponding to the subspace generated by $C\zeta_0$, and τ be the C-module homomorphism of M into C defined by $\tau(x)e = exe$. Then there is an open dense subset X_0 of the spectrum X of C such that, for every χ in X_0, the canonical representation ρ_χ of the state τ_χ of M given by $\tau_\chi(x) = \tau(x)\hat{}(\chi)$ is of type III. In particular, there is an invariant subalgebra $\mathscr{B}$ of the maximal modular Hilbert algebra of the generalized Hilbert algebra $M\zeta_0$ with the following property: there is an open dense subset X_0 of X such that $(\rho_\chi(\mathscr{L}^*(\mathscr{B}))'')_{q_\chi}$ is a type III factor and there is a faithful σ-weakly continuous projection ε_χ of norm one of $(\rho_\chi(M)'')_{q_\chi}$ onto $(\rho_\chi(\mathscr{L}^*(\mathscr{B}))'')_{q_\chi}$ with $\omega_\chi \cdot \varepsilon_\chi = \omega_\chi$. Here ω_χ is the normal functional on $\rho_\chi(M)''$ with $\omega_\chi \cdot \rho_\chi = \tau_\chi$ and q_χ is the support of ω_χ on $\rho_\chi(M)''$.*

Proof. There is a faithful σ-weakly continuous projection ε of norm one of M onto a countably generated type III von Neumann subalgebra N (Proposition 14). There is a countable subset $\mathscr{S}$ of the maximal modular Hilbert algebra $\mathscr{A}_0$ of the generalized Hilbert algebra $M\zeta_0$ such that the weak closure of $\pi(\mathscr{S})$ contains N and there is an invariant subalgebra $\mathscr{B}$ of $\mathscr{A}_0$ containing $\mathscr{S} \cup \{\zeta_0\}$ and an open dense subset Υ_0 of X such that

(1) $\mathscr{L}^*(\mathscr{B})$ is separable, and

(2) the restriction of τ_χ to $\mathscr{L}^*(\mathscr{B})$ is a primary state for χ in Υ_0 (Lemma 9). The restriction of ε to $\mathscr{L}^*(\mathscr{B})''$ is a faithful σ-weakly continuous projection of norm one onto N. This means that $\mathscr{L}^*(\mathscr{B})''$ is a type III algebra ([40, Theorem 3], [29, 2.6.5]) whose center is contained in the center of M (Proposition 10).

We now show that $\rho_\chi(\mathscr{L}^*(\mathscr{B}))'' = \mathscr{L}^*(\mathscr{B})_\chi$ is a type III factor for every χ in an open dense subset X_0 for Υ_0. We could use the results of Lance [19] to show that, for any given Borel measure on X, the exceptional set of all χ such that $\mathscr{L}^*(\mathscr{B})_\chi$ is not of type III is of measure 0. Here it is more appropriate to show that the exceptional set is nowhere dense. However, we work with fields of inner automorphisms as does Lance.

The state $\omega = \omega_{\zeta_0}$ of $\mathscr{L}^*(\mathscr{B})''$ is faithful and normal, and the modular automorphism group of $\mathscr{L}^*(\mathscr{B})''$ associated with ω is the restriction of $\{\sigma_t\}$ to $\mathscr{L}^*(\mathscr{B})''$ (Lemma 3). There is a sequence $\{p_i\}$ of nonzero central projections of $\mathscr{L}^*(\mathscr{B})''$ of sum 1 and a sequence $\{s_i\}$ of positive real numbers such that σ_{s_i} is not an inner automorphism on $\mathscr{L}^*(\mathscr{B})''_p$ for any nonzero projection p center of $\mathscr{L}^*(\mathscr{B})''_{p_i}$; otherwise, there is a nonzero projection p in the center of $\mathscr{L}^*(\mathscr{B})''$ such that σ_s is inner on $\mathscr{L}^*(\mathscr{B})''_p$ for every $s > 0$, and this is impossible due to the fact that the restriction of $\{\sigma_t\}$ to

$\mathscr{L}^*(\mathscr{B})''_p$ is the modular automorphism group of $\mathscr{L}^*(\mathscr{B})''_p$ associated with the faithful normal functional $\omega_{p\zeta_0}$ ([3, Lemma 4.1a], [34, Corollary 14.3]).

Now there is no loss of generality in assuming $p = p_i$ is equal to 1. In fact, the vector $\xi = p\zeta_0$ is a cyclic and separating vector for M_p, and the set $p\mathscr{A}_0$ is the maximal modular Hilbert algebra for the algebra $M_p\xi$ due to the fact that modular operator and the unitary involution of $Mp\xi$ are Δp and Jp. Also the algebra $p\mathscr{B}$ is an invariant subalgebra of $p\mathscr{A}_0$; $\mathscr{L}^*(p\mathscr{B})$ is equal to $\mathscr{L}^*(\mathscr{B})p$, and the von Neumann algebra generated by $\mathscr{L}^*(p\mathscr{B})$ on pH is equal to $\mathscr{L}^*(\mathscr{B})''_p$. Thus M_p has a faithful σ-weakly continuous projection of norm one onto $\mathscr{L}^*(\mathscr{B})''_p$ (Lemma 3 and [35]). Furthermore, the projection ep of pH onto clos $C\xi$ is an abelian projection of the commutant of the center C_p of M_p and under the identification of the spectrum of C_p with the set X_i of all χ in X with $p\hat{}(\chi) = 1$, the Hilbert space of the canonical representation as well as the image under the canonical representation of $(\tau_{ep})_\chi$ on $\mathscr{L}^*(p\mathscr{B})$ or M_p is exactly the Hilbert space of the canonical representation and the image of the canonical representation of τ_χ on $\mathscr{L}^*(\mathscr{B})$ or M respectively for χ in X_i. Finally, the union of open dense sets in each X_i will be an open dense subset of X. So we may assume that $p = 1$. For simplicity we let $s = s_i$.

Let π_χ denote the canonical representation of $\mathscr{L}^*(\mathscr{B})$ on the Hilbert space K_χ induced by τ_χ, let $\pi_\chi(\mathscr{L}^*(\mathscr{B}))'' = A_\chi$ and let σ^χ be the modular automorphism group of the generalized Hilbert algebra $A_\chi e(\chi)$ (Proposition 4). Let Π_m be the subset of all pairs (χ, x) of the set

$$\Pi_0 = \{(\chi, x) \mid \chi \in X, x \in A_\chi, \|x\| \leqq 1\}$$

such that

(i) $x\sigma^\chi_{s(m)}(y) = yx$ for all $y \in A_\chi$, and

(ii) $2^{-1} \leqq (xe(\chi), e(\chi)) + (x^*e(\chi), e(\chi))$.

Here $s(m)$ denotes the number $s(m) = s/2^m$. Let ι denote the map of a pair in Π_0 onto its first coordinate. We can see that the set $\{\chi \in X \mid A_\chi$ is a semi-finite factor$\}$ is contained in $\iota(\Pi_m)$. Indeed, if A_χ is a semi-finite factor, there is a strongly continuous one parameter group of unitary operators $\{u_t\}$ in A_χ such that

$$\sigma^\chi_t(x) = u_t x u_{-t}$$

for every x in A_χ [34, Theorem 14.2] and thus $(\chi, u_{s(m)})$ is in Π_m for some sufficiently large m.

Now the space Π_0 is compact in the weakest topology induced by the functions ι and $\{\tau_{y,z} \mid y, z \in \mathscr{L}^*(\mathscr{B})\}$ where

$$\tau_{y,z}(\chi, x) = (xye(\chi), ze(\chi))$$

for (χ, x) in Π_0. We use ideas from Strătilă and Zsidó [32, 1.1.1] in verifying this. There is a homeomophism Φ of Π_0 onto the compact product space

$$\Pi = X \times \Pi\{C(y, z) \mid (y, z) \in \mathscr{L}^*(\mathscr{B}) \times \mathscr{L}^*(\mathscr{B})\} ,$$

where

$$C(y, z) = \{\lambda \in C \mid |\lambda| \leqq \|y\| \|z\|\}$$

given by

$$\Phi(\chi, x) = (\chi, \{\tau_{y,z}(\chi, x)\}_{y,z}) .$$

If $\{(\chi_n, x_n)\}$ is a net in Π_0 whose image $\{\Phi(\chi_n, x_n)\}$ converges to $(\chi, \{\lambda_{y,z}\})$ in Π, the numbers $\lambda_{y,z}$ satisfy the relation

$$\begin{aligned} |\lambda_{y,z}| &= \lim |(x_n y e(\chi_n), z e(\chi_n))| \\ &\leqq \limsup \|x_n\| \|y e(\chi_n)\| \|z e(\chi_n)\| \\ &\leqq \|y e(\chi)\| \|z e(\chi)\| \end{aligned}$$

[8, Lemma 9]. There is a bilinear form

$$\langle y e(\chi), z e(\chi) \rangle = \lambda_{y,z}$$

on the dense linear manifold $\mathscr{L}^*(\mathscr{B}) e(\chi)$ of K_χ. There is a unique bounded linear operator x on K_χ such that

$$\lambda_{y,z} = (x y e(\chi), z e(\chi))$$

for all y, z in $\mathscr{L}^*(\mathscr{B})$. We show that x is in A_χ. Let ζ, η, ξ be in $\mathscr{B}$ and let p_χ be the projection of $C' e(\chi)$ onto K_χ; then we have that

$$\begin{aligned} &(x \pi'(\xi)(\chi) p_\chi \pi(\zeta) e(\chi), \pi(\eta) e(\chi)) \\ &\quad = (x \pi(\zeta\xi) e(\chi), \pi(\eta) e(\chi)) \\ &\quad = \lim (x_n \pi(\zeta\xi) e(\chi_n), \pi(\eta) e(\chi_n)) \\ &\quad = \lim (x_n \pi(\zeta) e(\chi_n), \pi(\eta(J\Delta^{-1/2}\xi)) e(\chi_n)) \\ &\quad = (\pi'(\xi)(\chi) p_\chi x \pi(\zeta) e(\chi), \pi(\eta) e(\chi)) . \end{aligned}$$

Since the commutant of A_χ is generated by $\mathscr{R}^*(\mathscr{B})(\chi) p_\chi$ (Proposition 4), the operator x is in A_χ. This means that Π_0 is homeomorphic to a closed subset of the compact product space Π, and thus Π_0 is compact.

Now we show that Π_m is a closed subset of Π_0. If $\{(\chi_n, x_n)\}$ is a net in Π_m converging to (χ, x) in Π_0, then it is clear that (χ, x) satisfies property (ii) of the definition of Π_m. If ζ, ξ, η are in $\mathscr{B}$ and $s = s(m)$, then we have

$$\begin{aligned} &(x \sigma_s^\chi(\pi_\chi(\pi(\xi))) \pi(\zeta) e(\chi), \pi(\eta) e(\chi)) \\ &\quad = (x \pi((\Delta^{is}\xi)\zeta) e(\chi), \pi(\eta) e(\chi)) \end{aligned}$$

$$= \lim (x_n \pi((\Delta^{is}\xi)\zeta)e(\chi_n),\ \pi(\eta)e(\chi_n))$$
$$= \lim (x_n \pi(\zeta)e(\chi_n),\ \pi((J\Delta^{1/2}\xi)\eta)e(\chi_n))$$
$$= ((\pi_\chi(\pi(\xi)))x\pi(\zeta)e(\chi),\ \pi(\eta)e(\chi))\ .$$

Hence, the pair (χ, x) satisfies property (i) and the set Π_m is closed in Π_0

We show that $\iota(\Pi_m)$ is nowhere dense in X. Since ι is a continuous map, the set $\iota(\Pi_m)$ is closed. We obtain a contradiction from the assumption that $\iota(\Pi_m)$ contains a nonvoid open subset and consequently a nonvoid open and closed subset X_m. There is projection in C whose Gelfand transform is the characteristic function of X_m. As already proved, there is no loss of generality in the assumption that this projection is 1. We now find a nonzero y_0 in $\mathscr{L}^*(\mathscr{B})''$ such that $y_0\sigma_{s(m)}(y) = yy_0$ for all y in $\mathscr{L}^*(\mathscr{B})''$. There is a continuous function ϕ of X into Π_m such that $\iota \cdot \phi(\chi) = \chi$ ([7], [12], cf. [32, § 1]). Let x_χ be the element in the unit sphere of A_χ such that

$$\phi(\chi) = (\chi, x_\chi)$$

for χ in X. The function

$$\chi \longrightarrow (x_\chi ye(\chi),\ ze(\chi))$$

is a continuous complex-valued function of X for y, z in $\mathscr{L}^*(\mathscr{B})$ due to the definition of the topology on Π_0. For y and z in $\mathscr{L}^*(\mathscr{B})$, let $\langle ye, ze\rangle$ denote the element of C satisfying

$$\langle ye, ze\rangle^\wedge(\chi) = (x_\chi ye(\chi),\ ze(\chi))$$

for every χ in X. We have that

$$\begin{aligned}
|(\langle ye, ze\rangle\zeta_0, \zeta_0)| &= \left|\int (x_\chi ye(\chi),\ ze(\chi))\,d\nu(\chi)\right| \\
&\leqq \int \|ye(\chi)\|\,\|ze(\chi)\|\,d\nu(\chi) \\
&\leqq \int \tau(y^*y)^{1/2\wedge}(\chi)\tau(z^*z)^{1/2\wedge}(\chi)\,d\nu(\chi) \\
&\leqq (\tau(y^*y)^{1/2}\tau(z^*z)^{1/2}\zeta_0, \zeta_0) \\
&\leqq \|\tau(y^*y)^{1/2}\zeta_0\|\,\|\tau(z^*z)^{1/2}\zeta_0\| \\
&\leqq \|y\zeta_0\|\,\|z\zeta_0\|\ ,
\end{aligned}$$

where ν is the spectral measure on X such that

$$\int w^\wedge(\chi)\,d\nu(\chi) = (w\zeta_0, \zeta_0)$$

for every w in C. Hence, there is a bounded linear operator x_0 on the subspace K generated by $\mathscr{L}^*(\mathscr{B})\zeta_0$ such that

$$(x_0 y\zeta_0, z\zeta_0) = (\langle ye, ze\rangle \zeta_0, \zeta_0)$$

for all y, z in $\mathscr{L}^*(\mathscr{B})$. If ζ, ξ, η are in $\mathscr{B}$, we have that

$$\begin{aligned}(x_0\pi'(\xi)\zeta, \eta) &= (x_0\pi(\zeta\xi)\zeta_0, \pi(\eta)\zeta_0)\\ &= \int (x_\chi \pi(\zeta\xi)e(\chi), \pi(\eta)e(\chi))d\nu(\chi)\\ &= \int (x_\chi \pi_\chi(\pi(\zeta))\pi'(\xi)(\chi)p_\chi e(\chi), \pi(\eta)e(\chi))d\nu(\chi)\\ &= \int (x_\chi \pi_\chi(\pi(\zeta))e(\chi), \pi(\eta(J\Delta^{-1/2}\xi))e(\chi))d\nu(\chi)\\ &= (\pi'(\xi)x_0\zeta, \eta)\ ,\end{aligned}$$

and consequently, that x_0 is in $\mathscr{R}^*(\mathscr{B})'_q$. Here q is the projection of H onto K. We recall that $\mathscr{B}$ is a modular algebra such that

$$\mathscr{L}^*(\mathscr{B})''_q = \mathscr{L}(\mathscr{B}) \quad \text{and} \quad \mathscr{R}^*(\mathscr{B})''_q = \mathscr{R}(\mathscr{B})$$

(Lemma 3) so that

$$\mathscr{R}^*(\mathscr{B})'_q = \mathscr{L}^*(\mathscr{B})''_q\ .$$

Therefore, the element x_0 is in $\mathscr{L}^*(\mathscr{B})''_q$. The projection q is in $\mathscr{L}^*(\mathscr{B})'$ and has central support 1 in $\mathscr{L}^*(\mathscr{B})'$ since $\mathscr{L}^*(\mathscr{B})'\zeta_0$ contains the dense subset $M'\zeta_0$ of H. So there is a unique y_0 in $\mathscr{L}^*(\mathscr{B})''$ with $y_0 q = x_0$. For ζ, ξ, η in $\mathscr{B}$ and $s = s(m)$, we have (from relation (2)) that

$$\begin{aligned}(y_0\sigma_s(\pi(\xi))\zeta, \eta) &= (y_0\pi((\Delta^{is}\xi)\zeta)\zeta_0, \pi(\eta)\zeta_0)\\ &= \int (x_\chi\pi_\chi(\pi((\Delta^{is}\xi)\zeta))e(\chi), \pi(\eta)e(\chi))d\nu\\ &= \int (x_\chi\sigma_s^\chi(\pi_\chi(\pi(\xi)))\pi(\zeta)e(\chi), \pi(\eta)e(\chi))d\nu\\ &= \int (\pi_\chi(\pi(\xi))x_\chi\pi(\zeta)e(\chi), \pi(\eta)e(\chi))d\nu\\ &= \int (x_\chi\pi(\zeta)e(\chi), \pi((J\Delta^{1/2}\xi)\eta)e(\chi))d\nu\\ &= (y_0\pi(\zeta)\zeta_0, \pi((J\Delta^{1/2}\xi)\eta)\zeta_0)\\ &= (\pi(\xi)y_0\zeta, \eta)\ .\end{aligned}$$

Thus, we have that

$$qy_0\sigma_s(\pi(\xi))q = q\pi(\xi)y_0q$$

for all $\xi \in \mathscr{B}$ and so

$$y_0\sigma_s(z) = zy_0$$

for all z in $\mathscr{L}^*(\mathscr{B})''$. We also have that

$$(y_0\zeta_0, \zeta_0) + (y_0^*\zeta_0, \zeta_0) = \int((x_\chi e(\chi), e(\chi)) + (x_\chi^* e(\chi), e(\chi)))d\nu$$
$$\geqq 2^{-1}\int d\nu = 2^{-1}||\zeta_0||^2$$

implies y_0 is nonzero. Now this means that there is a nonzero direct summand of $\mathscr{L}^*(\mathscr{B})''$ (viz. $\mathscr{L}^*(\mathscr{B})''$ reduced modulo the central support of y_0) on which σ_s is inner [17, proof, Theorem 1.1]. This is incompatible with the earlier choice of s. Thus we see that the set $\iota(\Pi_m)$ and consequently, $\cup\iota(\Pi_m)$ is nowhere dense in X. Thus, the set

$$\{\chi \in X | A_\chi \text{ is not a semi-finite factor}\}$$

is nowhere dense in X. Since the set

$$\{\chi \in X | A_\chi \text{ is not a factor}\}$$

is also nowhere dense in X, the set

$$\{\chi \in X | A_\chi \text{ is a type III factor}\}$$

contains an open dense subset X_0 of X. The projection q_χ in $M_\chi = \rho_\chi(M)''$ of H_χ onto clos $M'_\chi e(\chi)$ has central support 1. There is a faithful σ-weakly continuous projection ε_χ of norm one of $(M_\chi)_{q_\chi}$ onto its von Neumann subalgebra $(\mathscr{L}^*(\mathscr{B})_\chi)_{q_\chi}$ such that $\omega_\chi \cdot \varepsilon_\chi = \omega_\chi$ (Corollary 5). The projection q_χ is in $\mathscr{L}^*(\mathscr{B})'_\chi$ and has the same central support in $\mathscr{L}^*(\mathscr{B})'_\chi$ as the projection p'_χ of H_χ onto K_χ (cf. proof, Corollary 5). Thus, the algebras $(\mathscr{L}^*(\mathscr{B})_\chi)q_\chi$ and $A_\chi = (\mathscr{L}^*(\mathscr{B})_\chi)p'_\chi$ are isomorphic for every χ in X, and the algebras $(\mathscr{L}^*(\mathscr{B})_\chi q_\chi$ are type III factors for every χ in X_0. This means that the $(M_\chi)_{q_\chi}$ are type III algebras for every χ in X_0; and therefore, that the algebras M_χ are type III algebras for every χ in X_0 due to the fact that the central support of each q_χ is 1.

REMARKS 16 (i). For every χ in X_0, the von Neumann algebra D_χ generated by $(\mathscr{L}^*(\mathscr{B})_\chi)_{q_\chi}$ and its relative commutant in $(M_\chi)_{q_\chi}$ is isomorphic to the tensor product of $(\mathscr{L}^*(\mathscr{B})_\chi)_{q_\chi}$ and its relative commutant [35, Corollary 1]. There is a unique σ-weakly continuous projection of norm one of $(M_\chi)_{q_\chi}$ onto D_χ ([35, Corollary 1] and [4, Theorem 1.5.5]) and this projection necessarily leaves ω_χ invariant.

(ii) For every χ in X_0, the algebra B_χ generated by M_χ and the projection p'_χ of H_χ onto K_χ is a type III factor. In fact, the algebra $p'_\chi M_\chi p'_\chi$ is equal to $\mathscr{L}^*(\mathscr{B})_\chi p'_\chi$ (Corollary 5), and therefore, the algebra

$$p'_\chi B_\chi p'_\chi = \mathscr{L}^*(\mathscr{B})_\chi p'_\chi$$

is a type III factor on K_χ (Theorem 15). If x is in the center of

B_χ, then the element $xp'_\chi = p'_\chi x p'_\chi$ is in the center of $\mathscr{L}^*(\mathscr{B})_\chi p'_\chi$ and so is a scalar multiple $\lambda p'_\chi$ of p'_χ. This means x is equal to λ because

$$xye(\chi) = \lambda ye(\chi)$$

for every y in M_χ. Thus the algebra B_χ is a type III factor.

(iii) For any χ in X, the algebra M_χ is a factor if and only if p'_χ is in the commutant of the center of M_χ. Indeed, if p'_χ is in the commutant, the map $x \to xp'_\chi$ of the center of M_χ is an isomorphism onto the scalar multiples of p'_χ.

For the next corollary, it is convenient to introduce the following terminology. A von Neumann algebra N is said to be the *simple product* of type III algebras if there is a set $\{p_n\}$ of central projections of N of sum 1, a set $\{N_n\}$ of isomorphic type III algebras, and a set $\{H_n\}$ of Hilbert spaces such that Np_n is isomorphic to $N_n \otimes \mathscr{B}(H_n)$ for every n.

If ρ is a representation of a von Neumann algebra P such that $\rho(P)''$ is compatible with a simple product of type III algebras so is every representation of P quasi-equivalent to ρ (cf. [6, § 5]).

We no longer assume that M has a cyclic and separating vector.

COROLLARY 17. *Let M be a type III von Neumann algebra with center C. There is a one-one map $\chi \to (\chi)$ of the spectrum X of C into the set of quasi-equivalence classes of representations of M compatible with simple products of type III algebras with the following properties: if ϕ is a normal functional on M, then there are states $\{\phi_\chi \mid \chi \in X\}$ of M and a measure ν on X such that*

(i) *ν is the spectral measure obtained by restricting ϕ to C;*

(ii) *the canonical representation of M induced by ϕ_χ is in class (χ) except perhaps for a nowhere dense set of X;*

(iii) *for every χ in X except for perhaps a nowhere dense set, there is a convex w^*-compact set F_χ of type III states of M whose extreme points are type III factor states of M, and a Borel measure μ_χ on F_χ quasi-supported by the extreme points of F_χ such that $\phi_\chi(x) = \int \omega(x) d\mu_\chi(\omega)$ for every x in M;*

(iv) *$\phi_\chi(x) = x^\wedge(\chi)$ for every x in C and χ in X;*

(v) *$\chi \to \phi_\chi(x)$ is continuous on X for fixed x in M; and*

(vi) *$\phi(x) = \int \phi_\chi(x) d\nu(\chi)$ for every x in M.*

Proof. Let $\{g_i\}$ be a set of nonzero σ-finite projections of M whose respective central supports $\{p_i\}$ form a set of orthogonal projections of sum 1 [5; III, 1, Lemma 7]. There is no loss of generality in assuming that p_i is the sum of an infinite family of orthogonal equivalent projections one of which is g_i [5; III, 8, Theorem 1, Corollary 2]. Each algebra $M_{g_i} = M_i$ acts (i.e., is isomorphic to a von Neumann algebra that acts) on a Hilbert space H_i with separating and cyclic vector ζ_i. Let e_i be the maximal abelian projection of the commutant of the center $C_i = Cg_i$ of M_i corresponding to the subspace of H_i generated by $C_i\zeta_i$ and let τ_i be the map of M into Cp_i satisfying

$$\tau_i(x)g_i = \tau_{e_i}(g_i x g_i) .$$

The existence of τ_i follows from the fact that Cp_i is isomorphic to C_i under the map $x \to xg_i$ [5; I, 2, Proposition 2]. The map ϕ of M into C defined by

$$\phi(x) = \sum \tau_i(x)$$

is a positive σ-weakly continuous C-module homomorphism of M with $\phi(1) = 1$. Let X_0 be the dense open subset of X equal to

$$X_0 = \{\chi \in X \mid p_i\hat{}(\chi) = 1 \text{ for some } p_i\}$$

and let χ be in X_0 with $p_i\hat{}(\chi) = 1$. We have that

$$\phi_\chi(x) = \phi(x)\hat{}(\chi) = \tau_i(x)\hat{}(\chi)$$

for x in M. Let π_χ be the canonical representation of M on the Hilbert space H_χ induced by ϕ_χ and let ζ_χ be the cyclic vector such the $\omega_{\zeta_\chi} \cdot \pi_\chi = \phi_\chi$. The central support of $\pi_\chi(g_i)$ in $\pi_\chi(M)''$ is 1 since

$$1 = \phi_\chi(g_i) = \|\pi_\chi(g_i)\zeta_\chi\| .$$

The subspace $\pi_\chi(g_i)H_\chi$ of H_χ is generated by $\pi_\chi(g_i)\pi_\chi(M)\zeta_\chi$, or equivalently by $\pi_\chi(g_i M g_i)\zeta_\chi$, and thus is identified with the Hilbert space of the canonical representation of the state $\tau_{i\chi}(x) = \tau_{e_i}(x)\hat{}(\chi)$ of M_i. The von Neumann algebra $(M_i)_\chi = M_{i\chi}$ generated by $\pi_\chi(g_i M g_i)$ on $\pi_\chi(g_i)H_\chi$ is the von Neumann algebra generated by the image of M_i under the canonical representation of $\tau_{i\chi}$. Let q_χ be the support of $\omega_\chi = \omega_{\zeta_\chi}$ on $M_{i\chi}$. For every χ in an open dense subset of $\{\chi \in X \mid p_i\hat{}(\chi) = 1\}$, there exists a faithful σ-weakly continuous projection ε_χ of norm one of $(M_{i\chi})_{q_\chi}$ onto its type III factor von Neumann subalgebra P_χ as in Theorem 15. The projection q_χ is in the commutant P'_χ of P_χ (Corollary 5). By replacing X_0 by a smaller open dense set, we may assume such a projection ε_χ of norm one exists for all χ in X_0.

If χ is in X_0, then we show that the algebra $\pi_\chi(M)''$ is compatible with a simple product of type III algebras. For χ in X_0, there is a $p_i\hat{}$ containing χ in its support. The projection $\pi_\chi(g_i)q_\chi$ is a purely infinite σ-finite projection of $\pi_\chi(M)''$ of central support 1, and thus the algebra $\pi_\chi(M)''$ is purely infinite. Furthermore, the projection

$\pi_\chi(g_i)q_\chi$ can be embedded in an infinite set of equivalent orthogonal projections. Thus, it is sufficient to show that a type III algebra N is compatible with a simple product of type III algebras whenever it satisfies the following conditions: there is a σ-finite projection r of central support one in N, an infinite set of orthogonal equivalent projections in N summing to r, and a type III factor subalgebra Q of N containing r in its commutant such that N_r has a faithful σ-weakly continuous projection θ of norm one onto Q. Now we can find a family $\{z_j \mid j \in J\}$ of nonzero orthogonal central projections of sum 1 in N and corresponding infinite families $\{r_{jk} \mid k \in K_j\}$ of orthogonal equivalent projections of sum r_j such that $r_{j0} = rz_j$. Here we are assuming that 0 is in each index set K_j. There is no loss of generality in the assumption that the cardinalities $|K_j|$ of the sets K_j are distinct. The algebras N_{z_j} are isomorphic to $N_{z_j} \otimes \mathscr{B}(H_j)$, where H_j is a Hilbert space of dimension $|K_j|$. The projection r_{j0} is in Q' and thus $\theta(r_{j0})$ is a strictly positive scalar. The map

$$\theta_j(x) = \theta(r_{j0})^{-1}\theta(x)r_{j0}$$

is a faithful ρ-weakly continuous projection of norm one of $N_{r_{j0}}$ onto $Q_{r_{j0}}$ and thus $\theta_h \otimes$ (identity) is a faithful σ-weakly continuous projection of norm 1 of an isomorphic image of $N_{r_{j0}}$ onto $Q_{r_{j0}} \otimes \mathscr{B}(H_j)$ [41, Theorem 2]. Therefore the map

$$x \longrightarrow \sum_j (\theta_j \otimes \text{id.})(xz_j)$$

is a faithful σ-weakly continuous projection of norm one of an isomorphic image of N onto $\sum Q_{r_{j0}} \otimes \mathscr{B}(H_j)$. The projections r_{j0} have central support 1 in Q' since Q' is a factor. Thus, the algebras $Q_{r_{j0}}$ are all isomorphic to Q and the algebra N is compatible with a simple product of type III algebras. Thus, we have proved that $\pi_\chi(M)''$ is compatible with a simple product of type III algebras whenever χ is in the open dense set X_0 of X.

For χ in X_0, let (χ) be the quasi-equivalence class of π_χ. For each χ not in X_0, let (χ) be an arbitrary class of the III factor representations of $M/[\chi]$ [28].

Now let ϕ be a normal functional on M. There is a state Φ in M_*^+ with $(\phi \mid C) \cdot \Phi = \phi$ and a spectral measure ν on X such that

$$\phi(x) = \int x\hat{}(\chi) d\nu(\chi)$$

for every x in C so that the field $\{\phi_\chi \mid \chi \in X\}$ given by $\phi_\chi(x) = \Phi(x)\hat{}(\chi)$ satisfies properties (i)–(vi) with the possible exception of (iii) ([11], cf. §1). We verify $\{\phi_\chi\}$ satisfies (iii).

There is no loss of generality in the assumption that the center C of M is σ-finite and has a separating vector ζ since every commutative von Neumann algebra is the product of such algebras [5; I, 2, Proposition 3, Corollary]. The support f of the normal functional $\omega_\zeta \cdot \Phi$ on M is σ-finite and is equal to the support of Φ. The central support of f is 1 since the C-linearity of Φ implies $1-f$ cannot majorize a nonzero central projection. The functional $\omega_\zeta \cdot \Phi$ restricted to M_f is a faithful normal functional and so there is a cyclic and separating vector ξ for M_f such that $\omega_\xi(x) = \omega_\zeta \cdot \Phi(x)$ for all $x \in M_f$ [5; III, 1, Theorem 3]. The functional ω_ξ may also be written as $\omega_\xi = \omega_\xi \cdot \Phi$ on M_f due to the fact Φ is C-linear.

Now the projection e corresponding to the subspace generated by $C\xi$ or equivalently, by $Cf\xi$ is a maximal abelian projection in the commutant of the center Cf of M_f such that $\omega_\xi \cdot \tau = \omega_\xi$. Here $\tau = \tau_e$. Since f has central support 1, the algebra C is isomorphic to Cf under the map $c \to cf$ and the spectrum X of C is homeomorphic to the spectrum Υ of Cf under the map $\chi \to \chi f$. We have that

$$\tau(fxf)\hat{}(\chi f) = (\Phi(fxf)f)\hat{}(\chi f) = \phi_\chi(x)$$

for every χ in X and x in M.

Let $\mathscr{B}$ be an invariant subalgebra with 1 of the maximal modular algebra of the generalized Hilbert algebra $M_f\xi$ such that (1) $\mathscr{L}^*(\mathscr{B})$ is separable, and (2) τ_υ is a type III primary state of M_f on an open dense subset Υ_0 of Υ (Theorem 15). Let $\chi f = \upsilon$ be in Υ_0. We show that τ_υ can be decomposed as required in part (iii) on Υ_0. Let ρ be the canonical representation induced by τ_υ and let p' be the projection of $H(\tau_\upsilon)$ onto the subspace generated by $\mathscr{L}^*(\mathscr{B})e(\upsilon)$. The map ε_0 of $\rho(M_f)$ into $B = p'\rho(\mathscr{L}^*(\mathscr{B}))''p'$ given by

$$\varepsilon_0(x) = p'xp'$$

(Corollary 5) has the properties: (a) $\varepsilon_0(x^*) = \varepsilon_0(x)^*$; (b) $\varepsilon_0(x) \geqq 0$ if $x \geqq 0$; (c) $\varepsilon_0(yxz) = \varepsilon_0(y)\varepsilon_0(x)\varepsilon_0(z)$ if y, z are in $\rho(\mathscr{L}^*(\mathscr{B}))$; and (d) $\varepsilon_0(\rho(\mathscr{L}^*(\mathscr{B})))$ is σ-weakly dense in B. The set E of linear maps ε of $\rho(M_f)$ into B satisfying properties (a), (b), (c) and $\varepsilon(x) = \varepsilon_0(x)$ for x in $\rho(\mathscr{L}^*(\mathscr{B}))$ is a convex set which is compact in the topology of pointwise σ-weak convergence of the space $\mathscr{B}(\rho(M_f), B)$ of bounded linear operators of $\rho(M_f)$ into B. The canonical representation induced by $\omega_{e(\upsilon)} \cdot \varepsilon \cdot \rho$ is a type III (resp. a type III factor) representation of M_f if ε is a point (resp. extreme point) of E. A suitable adaptation of the proof given by S. Sakai ([28], cf. [29,4.6.9 and 4.6.10]) for extreme points can be used in the present case. By the Choquet-Bishop-de Leeuw theorem (cf. [23, §4]), there is a Borel measure μ on E quasi-supported by the extreme points of E such that

$$\psi(\varepsilon_0) = \int \psi(\varepsilon)d\mu(\varepsilon)$$

for every continuous linear functional ψ on the space $\mathscr{B}(\rho(M_f), B)$ with the specified topology. In particular, we have that

$$\tau_\upsilon(x) = \omega_{e(\upsilon)}(\rho(x)) = \omega_{e(\upsilon)}(\varepsilon_0(\rho(x))) = \int \omega_{e(\upsilon)}(\varepsilon(\rho(x)))d\mu(\varepsilon)$$

for all x in M_f. Now the map

$$\varepsilon \longrightarrow \omega_{e(\upsilon)} \cdot \varepsilon \cdot \rho(f \cdot f)$$

is an affine homeomorphism of E onto a convex w^*-compact set F_χ of type III states of M whose extreme points are primary type III states. Identifying the measure μ on E with the measure induced on F_χ by the homeomorphism, we get a Borel measure $\mu = \mu_\chi$ on F_χ that is quasi-supported by the extreme points such that

$$\phi_\chi(x) = \int_{F_\chi} \omega(x)d\mu_\chi(\omega)$$

for every x in M. Since the set $\{\chi' | \chi' f \in \Gamma_0\}$ is an open dense subset of X, we have proved that conclusion (iii) holds true.

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