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AN ALGEBRA OF PSEUDO-DIFFERENTIAL OPERATORS WITH NON-SMOOTH SYMBOL

H. O. CORDES AND D. A. WILLIAMS

In [4], [1], [7], and [5] certain algebras of zero-order pseudo-differential operators were discussed which all were generated by closing the operator algebra $\mathring{\mathfrak{A}}$ finitely generated from the elements

$$\{a(M),\ b(D):\ a \in \mathscr{A}^+,\ b \in \mathscr{A}^\#\}\ , \tag{0.1}$$

with multiplication operators $u(x) \rightarrow a(x)u(x)$ denoted by $a(M)$ and convolution operators (or formal Fourier multipliers) $b(D) = F^*a(M)F$, with $F =$ Fourier transform. Various classes $\mathscr{A}^+$ and $\mathscr{A}^\#$, and various operator topologies were used, with the purpose of using the generated topological algebra for proving normal solvability of singular elliptic problems $Lu = f$, $x \in \boldsymbol{R}^n$, with a suitable linear differential operator $L = \sum_{|\alpha| \leqq N} a_\alpha(x) D^\alpha$.

At present let us focus on the algebra $\mathfrak{A}_\infty$ obtained from the classes

$$\mathscr{A}^+ = \{a \in C^\infty(\boldsymbol{R}^n):\ a(x) = O(1),\ a^{(\beta)}(x) = o(1),\ \beta \neq 0\} \tag{0.2}$$

and

$$\mathscr{A}^\# = \{b \in C^\infty(\boldsymbol{R}^n):\ b^{(\beta)} \in C(\boldsymbol{B}^n),\ \beta \in \boldsymbol{Z}_+^n\}\ , \tag{0.3}$$

with the compactification $\boldsymbol{B}^n$ of $\boldsymbol{R}^n$ obtained by continuous extension of the vector-valued function $x \rightarrow x(1 + x^2)^{-1/2}$, where we close under the following operator topology: $\mathring{\mathfrak{A}}$, with $\mathscr{A}^+$ and $\mathscr{A}^\#$ as in (0.2) and (0.3) may be seen to be a subalgebra of $\mathscr{L}(\mathfrak{H}_s)$, the algebra of continuous operators $\mathfrak{H}_s \rightarrow \mathfrak{H}_s$, with the L^2-Sobolev space $\mathfrak{H}_s = \{u:\ u \in \mathscr{S}',\ \|(1 - \Delta)^{s/2}u\|_{L^2} = \|u\|_s < \infty\}$ of $\boldsymbol{R}^n$. This is true for every $s \in \boldsymbol{R}$, and therefore the elements of $\mathring{\mathfrak{A}}$ also take the Frechet space $\mathfrak{H}_\infty$ continuously to itself. A locally convex topology on $\mathring{\mathfrak{A}}$ is generated by all the operator norms $\|A\|_s = \sup\{\|Au\|_s:\ \|u\|_s \leqq 1\}$. In fact this is a Frechet topology, and it suffices to only take the norms $\|A\|_k$, $k \in \boldsymbol{Z}$. All this is discussed in details in [2]. We define $\mathfrak{A}_\infty$ to be the completion of $\mathring{\mathfrak{A}}$ under that topology.

Similarly one may complete $\mathring{\mathfrak{A}}$ as a subalgebra of any given fixed $\mathscr{L}(\mathfrak{H}_s)$ in the norm topology, to obtain a Banach algebra $\mathfrak{A}_s$, which proves to be a C^*-subalgebra of $\mathscr{L}(\mathfrak{H}_s)$, containing the compact ideal $\mathfrak{K}_s = \mathfrak{K}(\mathfrak{H}_s)$ of $\mathscr{L}(\mathfrak{H}_s)$. In fact, $\mathfrak{A}_s/\mathfrak{K}_s$ is commutative, thus we have $\mathfrak{A}_s/\mathfrak{K}_s = C(\boldsymbol{M}_s)$, with a certain compact Hausdorff space $\boldsymbol{M}$, by the Gelfand-Naimark theorem. The space $\boldsymbol{M} = \boldsymbol{M}_s$ proves independent of s, and may be explicitly described as follows.

$$\boldsymbol{M} = \boldsymbol{P}^n \times \boldsymbol{B}^n - \boldsymbol{R}^n \times \boldsymbol{R}^n , \tag{0.4}$$

with the above compactification $\boldsymbol{B}^n$ of $\boldsymbol{R}^n$, and another one, $\boldsymbol{P}^n$, determined as smallest compactification allowing continuous extension of all the functions in $\mathscr{A}^+$.

The homomorphism $\mathfrak{A}_s \to \mathfrak{A}_s/\mathfrak{K}_s \to C(\boldsymbol{M})$ assigns a continuous function σ_A to every $A \in \mathfrak{A}_s$, which is called the *symbol* of $A \in \mathfrak{A}_s$. For $A \in \mathfrak{A}_\infty \subset \mathfrak{A}_s$ the symbol σ_A proves to be independent of s. In fact, for $A \in \mathring{\mathfrak{A}}$, the symbol coincides with the restriction to $\boldsymbol{M}$ of the continuous extension of the Pseudo-differential-operator symbol of A. The symbol σ_A of A proves of fundamental importance for the normal solvability of an equation $Au = f$, $u, f \in \mathfrak{H}_\infty$, $A \in \mathfrak{A}_\infty$: A necessary and sufficient condition for existence of a *Green inverse* of order 0 (that is a Fredholm inverse B such that $\|B\|_s$ is defined for all s, and that 1-AB, 1-BA: $\mathfrak{H}_{-\infty} \to \mathfrak{H}_\infty$ are continuous and have finite rank) is that $\sigma_A \neq 0$ on all of $\boldsymbol{M}$.

All the above facts are discussed in [2]. Moreover, it is shown there that $\mathfrak{A}_\infty$ coincides with an algebra introduced by M. Taylor [7], p. 505, denoted by $PS(0)$, as the class of all $A \in \mathfrak{A}_0$ such that $(1 - \Delta)^s A (1 - \Delta)^{-s} - A \in \mathfrak{K}_0$, for all s.

In the present paper we are going to attack the question of proving existence of operators in $\mathfrak{A}_\infty$, having a given symbol $\sigma_A = a \in C(\boldsymbol{M})$. For the C^*-algebras $\mathfrak{A}_s$ this question is easily answered: For *every* continuous function $a \in C(\boldsymbol{M})$ there always exist operators $A \in \mathfrak{A}_s$ with $\sigma_A = A$, because for the C^*-algebra with compact commutator and with unit we get $\mathfrak{A}_s/\mathfrak{K}_s$ *equal to* $C(\boldsymbol{M})$. One may expect that differentiability of a, in a manner to be specified, will guarantee existence also of an $A \in \mathfrak{A}_\infty$ with $\sigma_A = a$. Our first result will indeed make that point.

THEOREM 0.1. *Let $C_p^\infty(\boldsymbol{M})$ denote the subalgebra of $C(\boldsymbol{M})$ of all functions a such that $D_x^\alpha a(x, \xi)$ exists for all α on the subset*

$$\boldsymbol{W} = \{(x, \xi): x \in \boldsymbol{R}^n \subset \boldsymbol{P}^n, \xi \in \partial \boldsymbol{B}^n = \boldsymbol{B}^n - \boldsymbol{R}^n\} \tag{0.5}$$

of $\boldsymbol{M}$, and that the functions $a_{(\alpha)} = i^{|\alpha|} D_x^\alpha a$ on $\boldsymbol{W}$, $=0$ on $\boldsymbol{M} - \boldsymbol{W}$, is in $C(\boldsymbol{M})$. Then, if $a \in C_p^\infty(\boldsymbol{M})$, there exists an operator $A \in \mathfrak{A}_\infty$ with $\sigma_A = a$.

Perhaps it is remarkable that no differentiability with respect to the ξ-variables is required at all. In fact at the entire portion $|x| = \infty$ of the 'symbol space' $\boldsymbol{M}$ no differentiability with respect to either x or ξ is required. On the other hand there are indications that even the above differentiability requirement is much too strong for the assertion of Theorem 0.1.

Our proof of Theorem 0.1 is entirely independent of conventional Pseudo-differential operator calculus. The basic idea is the invariance of $\mathfrak{A}_\infty$ and $\mathfrak{A}_0$ under the 'translation group'

$$\{e^{it.D}: t \in \boldsymbol{R}^n\} . \tag{0.6}$$

In fact, conjugation with the unitary operators (0.6) defines a strongly continuous group of linear operators: $\mathfrak{A}_0 \to \mathfrak{A}_0$. The condition $a \in C_p^\infty(\boldsymbol{M})$ in effect means that a is in the joint domain of the projection of the infinitesimal generators

$$(adD)^\alpha = \prod_{j=1}^n (adD_j)^{\alpha_j},\ (adD_j)A = [D_j, A] = D_jA - AD_j,\ A \in \mathfrak{A}_0 \tag{0.7}$$

to $\mathfrak{A}_0/\mathfrak{K}_0$. In other words, for $a \in C_p^\infty(\boldsymbol{M})$ there will exist operators $A \in \mathfrak{A}_0$ such that all the commutators $(ad\,D)^\alpha A$ will be well defined elements of $\mathfrak{A}_0$ again. Such operators A can be shown to satisfy Taylor's condition, of belonging to $PS(0) = \mathfrak{A}_\infty$. This idea of proof will be discussed in details in §§1 and 3.

Let $\mathfrak{A}_{ad}$ denote the algebra of all $A \in \mathfrak{A}_0$ with the property that $(ad\,D)^\alpha A \in \mathfrak{A}_0$ for all α. We shall see that $\mathring{\mathfrak{A}} \subset \mathfrak{A}_{ad} \subset \mathfrak{A}_\infty$. However, examples are easily given which show that not all operators in $\mathfrak{A}_{ad}$ are pseudo-differential operators in the sense of [3], for example. Accordingly one will not expect the calculus of pseudo-differential operotors, like that in [3], §7, for example, to be valid in $\mathfrak{A}_{ad}$. Our second result is a 'Leibnitz formula' of the asymptotic kind, showing that part of that calculus can be saved.

THEOREM 0.2. *Let $\mathscr{M} \subset C^\infty(\boldsymbol{R}^n)$ denote the algebra of all functions u such that for $k = 0, 1, 2, \cdots$ there exists $N(k)$ with*

$$x^\alpha u^{(\beta)} = O(1) \text{ for all } |\alpha| \leqq k \text{ and all } |\beta| \geqq N(k) .$$

For $A \in \mathfrak{A}_{ad}$ let $A_{(\alpha)} = i^{|\alpha|}(ad\,D)^{(\alpha)}A$.

Then we have the asymptotic expansions

$$\begin{aligned} b(D)A &= \sum_\theta i^{-|\theta|}/\theta!\ A_{(\theta)}b^{(\theta)}(D) \quad (\mathrm{mod}\ \mathscr{O}(-\infty)) \\ Ab(D) &= \sum_\theta i^{|\theta|}/\theta!\ b^{(\theta)}(D)A_{(\theta)} \quad (\mathrm{mod}\ \mathscr{O}(-\infty)) \end{aligned} \tag{0.8}$$

for all $A \in \mathfrak{A}_{ad}$ and $b \in \mathscr{M}$.

In details, we have the order ρ_N of the remainder

$$b(D)A - \sum_{|\theta| \leqq N} i^{-|\theta|}/\theta!\ A_{(\theta)}b^{(\theta)}(D) = R_N \tag{0.9}$$

going to $-\infty$, as $N \to \infty$. Similarly for the second expansion (0.9). Here we say that an operator $B: \mathfrak{H}_\infty \to \mathfrak{H}_\infty$ has order r if it induces a continuous map $\mathfrak{H}_s \to \mathfrak{H}_{s-r}$, for all $s \in \boldsymbol{R}$.

Theorem 0.2 will be discussed in § 2. We will begin our proofs in § 1 with showing that $\mathfrak{A}_\infty = PS(0)$ (Lemma 1.1), and then prove that $\sigma_{\mathfrak{A}_{ad}} = C_p(\boldsymbol{M})$, or rather, a slightly more general result (Theorem 1.4). In § 3 we will use the Leibnitz formulas of Theorem 0.2 for a proof of the inclusion $\mathfrak{A}_{ad} \subset PS(0)$, which will establish Theorem 0.1. Most proofs are also discussed in [2], in much broader details.

1. Taylor's algebra $PS(0)$, and the algebras $\mathfrak{A}_\infty$, $\mathfrak{A}_{ad}$. The fact that our algebra $\mathfrak{A}_\infty$ is identical with Taylor's algebra $PS(0)$ in [7] is a consequence of the lemma, below.

LEMMA 1.1. *The algebra $\mathfrak{A}_\infty$ is identical with the class of all $A \in \mathfrak{A}_0$ such that, for every $s \in \boldsymbol{R}$, (or only for every integer $s = k \in \boldsymbol{Z}$), the unbounded operator product $\Lambda^{-s} A \Lambda^s$, with $\Lambda^s = (1-\Delta)^{-s/2} = (1+D^2)^{-s/2}$ has dense domain and extends continuously to a bounded operator on $\mathfrak{A}_0$ satisfying*

$$(1.1) \qquad \Lambda^{-s} A \Lambda^s - A \in \mathfrak{R}_0\,.$$

Proof. The inclusion $\mathfrak{A}_\infty \subset PS(0)$, where $PS(0)$ denotes the algebra of operators satisfying (1.1) for all $s \in \boldsymbol{R}$, is a consequence of Lemma 6 in [4] (for more details c.f. [2], Chapter IV, 3). In particular (1.1) is immediate for $A \in \mathring{\mathfrak{A}}$, and it then follows for $A \in \mathfrak{A}_\infty$, for which we have a sequence $A_k \in \mathring{\mathfrak{A}}$ with $\|A - A_k\|_s \to 0$ for all s due to $\|\Lambda^{-s} A \Lambda^s - \Lambda^{-s} A_k \Lambda^s\|_0 = \|A - A_k\|_s$.

Now let $A \in \mathfrak{A}_0$ satisfy (1.1). It follows that A extends continuously to an operator $A_s: \mathfrak{H}_s \to \mathfrak{H}_s$, for $s < 0$, and that, for $s > 0$ the restriction $A_s = A \,|\, \mathfrak{H}_s$ is bounded from $\mathfrak{H}_s$ to $\mathfrak{H}_s$, by a calculation. In fact, we have $\|A\|_s = \|A_s\|_s = \|\Lambda^{-s} A \Lambda^s\|_0 < \infty$. To complete the proof we will construct a sequence of operators in $\mathfrak{A}_\infty$ which is Cauchy, and converges to A above in $\mathfrak{A}_0$, and in every $\mathfrak{A}_s$. By Lemma 1.2 below, which we quote without proof, it suffices to only consider the norms $\|\cdot\|_k$ with integers $s = k$.

LEMMA 1.2. *Suppose an operator $A: \mathcal{S} \to \mathcal{S}'$ maps into $\mathfrak{H}_s \cap \mathfrak{H}_t$ for some pair of reals $s < t$, and that*

$$(1.2) \qquad \|Au\|_s \leqq c_s \|u\|_s, \quad \|Au\|_t \leqq c_t \|u\|_t, \quad u \in \mathcal{S}\,.$$

Then we have

$$(1.3) \qquad \|Au\|_r \leqq c_r \|u\|_r \quad u \in \mathcal{S}\,,$$

for all $s \leqq r \leqq t$, *where the constant* c_r *may be chosen as*

$$c_r = c_s^{(t-r)/(t-s)} c_t^{(r-s)/(t-s)} . \tag{1.4}$$

For the proof of Lemma 1.2 we refer to Seeley [6] (or [2]). This interpolation lemma shows in effect that the topology of $\mathfrak{A}_\infty$ is Frechet.

For $A \in \mathfrak{A}_0$ satisfying (1.1) first choose a sequence $A_j \in \mathfrak{A}_0$ with $\|A - A_j\|_0 \to 0$, as $j \to \infty$. Let $\chi \in C_0(\boldsymbol{R}^n)$, $\chi = 1$ near 0, $0 \leqq \chi \leqq 1$, and let $\chi_m(x) = \chi(x/m)$, $m = 1, 2, \cdots$, and let $X_m = \chi_m(D)$. Conclude that $X_m A X_m \in \mathfrak{A}_\infty$, because we get

$$\begin{aligned} \|X_m A X_m - X_m A_j X_m\|_l &= \|\Lambda^{-l} X_m (A - A_j) X_m \Lambda^l\|_0 \\ &\longrightarrow 0, \text{ as } j \longrightarrow \infty , \end{aligned} \tag{1.5}$$

for fixed m, l, since $\Lambda^l X_m = (\lambda^l \chi_m)(D)$ is bounded in $\mathfrak{H}_0$, $\lambda(x) = (1 + x^2)^{-1/2}$. Introduce the operator $A_{mj} \in \mathfrak{A}_\infty$ by

$$A_{mj} = X_m A X_m + (1 - X_m) A_j X_m + A_j (1 - X_m) . \tag{1.6}$$

(Observe that $X_m \in \mathfrak{A}_\infty$, in that respect.) We write

$$\begin{aligned} \Lambda^{-l}(A - A_{mj})\Lambda^l &= \Lambda^{-l}(1 - X_m)(A - A_j) X_m \Lambda^l \\ &+ \Lambda^{-l}(A - A_j)(1 - X_m)\Lambda^l = (1 - X_m)(A - A_j) X_m \\ &+ (A - A_j)(1 - X_m) + C_{jl} - X_m C_{jl} X_m , \end{aligned} \tag{1.7}$$

with

$$C_{jl} = \Lambda^{-l}(A - A_j)\Lambda^l - (A - A_j) \in \mathfrak{K}_0 , \tag{1.8}$$

because A satisfies (1.1), and $A_j \in \mathfrak{A}_\infty$ also satisfies (1.1). For $N = 1, 2, \cdots$ first choose $k = k_N$ such that $\|A - A_k\|_0 \leqq 1/4N$. Then keep k fixed and choose $m = m_N$ large to insure that

$$\|C_{k_N l} - X_{m_N} C_{k_N l} X_{m_N}\|_0 \leqq 1/4N,\ l = 0, \pm 1, \cdots, \pm N , \tag{1.9}$$

as follows because $X_m \to 1$, as $m \to \infty$, in strong operator convergence, while the operators C_{kl} are compact.

$$\|A - A_{\eta_N m_N}\|_1 \leqq 1/N,\quad |1| \leqq N , \tag{1.10}$$

so that indeed $\lim_{N\to\infty} A_{k_N m_N} = A$ in $\mathfrak{A}_\infty$.

In the remainder of this section we shall be concerned with the group (0.6) of unitary operators, referred to as the translation group. In the introduction we have sketched the intended use for this group.

LEMMA 1.3. *For* $A \in \mathfrak{A}_0$ *with symbol* $\sigma_A = a$ *let us define* $B_j = E_j A$ *by*

$$B_j = \int_0^\infty e^{-iD_jt}Ae^{iD_jt}e^{-t}dt, \qquad j = 1, \cdots, n . \tag{1.11}$$

Assertion. (i) The integrals (1.11) converge as improper Riemann integrals in norm convergence of $\mathscr{L}(\mathfrak{H}_0)$, and $B_j \in \mathfrak{A}_0$.

(ii) The symbol $b_j = \sigma_{B_j}$ has its first derivative $\partial b_j/\partial x_j$ continuous over $\boldsymbol{W}$, and zero at $|x| = \infty$, and is explicitly given by

$$b_j = \int_0^\infty a(x - te_j, \xi)e^{-t}dt \tag{1.12}$$

with $e_j = (\delta_{1j}, \cdots, \delta_{nj})$, and $x - h = x$, as $|x| = \infty$, $|h| < \infty$.

(iii) The function b_j is uniquely determined as the solution of

$$\partial v/\partial x_j + v = a \quad \text{on} \quad \boldsymbol{M},\ v,\ \partial v/\partial x_j \in C(\boldsymbol{M}) , \tag{1.13}$$

where we interpret $\partial v/\partial x_j = \lim_{h\to 0} (v(x + he_j, \xi) - v(x, \xi))/h = 0$, as $|x| = \infty$, (because $x + he_j = x$ implies the difference quotient to be zero).

(iv) The commutator $[D_j, B_j] = D_jB_j - B_jD_j$ between the unbounded operator D_j of $\mathfrak{H}_0$ with domain $\mathfrak{H}_1$, and the bounded operator B_j has dense domain and extends to an operator in $\mathscr{L}(\mathfrak{H}_0)$, which in fact is in $\mathfrak{A}_0$ and is explicitly given by the equation

$$A = i[D_j, B_j] + B_j . \tag{1.14}$$

Proof. It is known that e^{iDr}, $r \in \boldsymbol{R}^n$ is the translation operator $u(x) \to u(x + r)$, so that $e^{-iDr}a(M)e^{iDr} = a(M + r)$. Also $e^{-iDr}b(D)e^{irD} = b(D)$, so that the function $\varphi(r) = e^{-iDr}Ae^{iDr}$ is norm continuous for $r \in \boldsymbol{R}^n$ whenever A is a generator of $\mathfrak{A}_0$. This also holds for the general $A \in \mathfrak{A}_0$, which is uniform limit of finitely generated elements. Also, the integrand in (1.11) is $O(1)$, in the norm $\|\cdot\|_0$, which implies (i). We may take symbols under the integral sign, by (i) and because the projection $\mathfrak{A}_0 \to \mathfrak{A}_0/\mathfrak{K}_0$ is continuous. Also, since we know the action of the automorphism $A \to e^{-iDr}Ae^{iDr}$ on the generators of $\mathfrak{A}_0$ we can easily calculate the action on the symbols, using techniques involving the dual map, as in [4]. This will serve to confirm (1.12), and thus (ii). Now (iii) follows by methods involving the Greens function of the ordinary differential operator $\tilde{D}_j = \partial/\partial x_j + 1$ on $\boldsymbol{R}$, or by simple differentiation. Regarding (iv) consider the commutator $[D_j, B_j]$ as an operator in $\mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_{-1})$, and then write

$$\begin{aligned}
[D_j, B_j] &= \int_0^\infty (D_je^{-iD_jt}Ae^{iD_jt} - e^{-iD_jt}Ae^{iD_jt}D_j)e^{-t}dt \\
&= i\int_0^\infty d/dt(e^{-iD_jt}Ae^{iD_jt})e^{-t}dt = ie^{iD_jt}Ae^{-iD_jt}e^{-t}\Big|_0^\infty \\
&\quad + i\int_0^\infty e^{-iD_jt}Ae^{iD_jt}e^{-t}dt = -iA + iB_j ,
\end{aligned} \tag{1.15}$$

where all operations may be seen to be legitimate. This proves (iv) and establishes Lemma 1.3.

THEOREM 1.4. *Let* $\mathfrak{A}_{ad,k}$, *for* $k = 0, 1, \cdots, \infty$, *denote the algebra of all operators* $A \in \mathfrak{A}_0$ *with the property that*

$$(1.16) \qquad A_{(\alpha)} = i^{|\alpha|}(ad\, D)^\alpha A \in \mathfrak{A}_0, \quad [\alpha] = \operatorname*{Max}_{j=1}^{n} \alpha_j \leqq k\ ,$$

with $(ad\, D)^\alpha$ *defined as operators in* $\mathscr{L}(\mathfrak{H}_{|\alpha|}, \mathfrak{H}_{-|\alpha|})$ *by* (0.7). *Let* $C_p^k(\boldsymbol{M})$, *for* $k = 0, 1, \cdots, \infty$, *be the class of functions in* $C(\boldsymbol{M})$, *with the property that* $a_{(\alpha)} = i^{|\alpha|} D_x^\alpha a(x, \xi)$ *exist and are in* $C(\boldsymbol{W})$, *and vanish, as* $|x| \to \infty$, *for all* $[\alpha] \leqq k$, *as in Theorem* 0.1.

Assertion. For every function $a \in C_p^k(\boldsymbol{M})$ there exist operators $A \in \mathfrak{A}_{ad,k}$ such that $\sigma_A = a$, and, more generally, $\sigma_{A_{(\alpha)}} = a_{(\alpha)}$, for all $[\alpha] \leqq k$.

Proof. For finite k Theorem 1.4 is an almost immediate consequence of Lemma 1.3: For $a \in C_p^k$, with finite k let $\alpha^k = (k, k \cdots, k, k)$ be the unique 'largest multi-index', and just pick any operator $P \in \mathfrak{A}_0$ with symbol $a_{(\alpha^k)}$. Notice that the operators $E_j \in \mathscr{L}(\mathfrak{A}_0)$ defined by Lemma 1.3 all commute. Then define $A_{\{\alpha\}} = E^{\alpha^k - \alpha}P$ with $E^\beta = \prod_{j=1}^n E_j^{\beta_j}$. Then notice that $A_{\{\alpha\}}$ has symbol $\tilde{D}^\alpha a$, with $\tilde{D}_j = \partial/\partial x_j + 1$, $\tilde{D}^\beta = \prod_{j=1}^n D_j^{\beta_j}$. This suggests defining operators $A_{(\alpha)}$ by properly combining the $A_{\{\alpha\}}$: $A_{(0)} = A_{\{0\}}$, $A_{(e_j)} = A_{\{e_j\}} - A_{\{0\}}$, etc. This choice may be seen to also satisfy the commutator relations (1.16), by (1.14), establishing the result for finite k.

Next let $k = \infty$. Then we can make the above selection of A and $A_{(\alpha)}$, $[\alpha] \leqq k$ for every $k = 1, 2, \cdots$. Let the corresponding operators be denoted by A_k and $A_{(\alpha),k}$, for a moment, and note that

$$(1.17) \qquad A_k - A_l = C_{kl} \in \mathfrak{R}_0, \quad A_{(\alpha),k} - A_{(\alpha),l} \in \mathfrak{R}_0\ ,$$

for all k, l such that the terms are defined. Specifically,

$$(1.18) \quad A_{k+1} - \chi_k(D) C_{k+1,k} \chi_k(D) = A_k + \{C_{k+1,k} - \chi_k(D) C_{k+1,k} \chi_k(D)\}\ ,$$

with a suitable function $\chi_k \in C_0^\infty(\boldsymbol{R}^n)$, $0 \leqq \chi_k \leqq 1$, and $\chi_k = 1$ near 0. Introducing the notation F_{k+1} and G_{k+1} for the compact operator at left and at right in (1.18), respectively, we find that

$$(1.19) \qquad \tilde{A}_k = A_k - \sum_{l=1}^{k} F_l = A_m + \sum_{l=m+1}^{k} G_l - \sum_{l=1}^{m} F_l\ ,$$

where we have used induction. The operators F_l may carry as many commutations $(ad\, D)^\alpha$ as desired, since the factor $\chi(D)$ neutralizes the unboundedness of arbitrary powers D^r, from left or right. On the other hand a proper choice of χ_j, for example as $\chi_j(x)=\chi(x/\tau_j)$ with sufficiently large τ_j, will insure

$$\|(ad\,D)^\alpha G_j\|_0 \leqq 2^{-j}, \quad [\alpha] \leqq j\ . \tag{1.20}$$

Now the right hand side in (1.19) will converge, as $k \to \infty$, supplying an operator $A=\lim \tilde{A}_k$. Moreover, it follows that also $(ad\,D)^\alpha A_k$ converges to $(ad\,D)^\alpha A$, using (1.20). Then it follows that $A = \lim \tilde{A}_k$ is the desired operator satisfying the assertion for $k = \infty$. This proves Theorem 1.4.

It is clear that $\mathfrak{A}_{ad} = \mathfrak{A}_{ad,\infty}$. Accordingly Theorem 1.4 shows that all functions in $C_p^\infty(\boldsymbol{M})$ can be obtained as symbols of operators in $\mathfrak{A}_{ad}$. For the proof of Theorem 0.1 we therefore must show that $\mathfrak{A}_{ad} \subset \mathfrak{A}_\infty$, using the Leibnitz formulas of Theorem 0.2.

2. An asymptotic formula.

An operator $A \in \mathscr{L}(\mathfrak{H}_\infty)$ is said to have order r if for every $s \in \boldsymbol{R}$ there exists a continuous extension $A_s\colon \mathfrak{H}_s \to \mathfrak{H}_{s-r}$. Let $\mathscr{O}(r)$ denote the class of all operators of order r. Suppose $A_j \in \mathscr{O}(\rho_j)$, with $\rho_j \searrow -\infty$. Then we shall say that $A \in \mathscr{L}(\mathfrak{H}_\infty)$ allows an asymptotic expansion

$$A = \sum_{j=0}^{\infty} A_j \,(\mathrm{mod}\ \mathscr{O}(-\infty)) \tag{2.1}$$

if for every $N = 0, 1, 2, \cdots,$ we have

$$A - \sum_{j=0}^{N} A_j \in (\mathscr{O}\rho_{N+1})\ . \tag{2.2}$$

With this notation we now will discuss the proof of Theorem 0.2. In that respect it is sufficient to establish the first formula (0.8), because the second formula follows by taking adjoints.

It is convenient to introduce a concept called *Fourier kernel product* (for details c.f. [2], IV, 4). For an operator $Q \in \mathscr{L}(\mathfrak{H}_\infty, \mathfrak{H}_{-\infty})$ let $\mathscr{S}'(\boldsymbol{R}^{2n})$ denote the *Fourier distribution kernel*, defined as the distribution q such that

$$\langle FQF^{-1}u, v\rangle = \langle q, u \otimes v\rangle, \quad u, v \in \mathscr{S}\ . \tag{2.3}$$

If P, Q have the kernels p and q, and if $p = \varphi, q$, with a function $\varphi \in C^\infty(\boldsymbol{R}^{2n})$ then we shall call P the *kernel product* of φ and Q. This relation shall be written as

$$P = \varphi \triangle Q\ . \tag{2.3}$$

For example, if $\varphi = \varphi(\xi, \eta) = \varphi(\xi)$ depends only on ξ, then $\varphi \triangle Q = \varphi(D)Q$, and $\varphi \triangle Q = Q\varphi(D)$, if φ depends only on η. Also, if $\xi_j =$

$\xi_j - \eta_j$, $\xi^\alpha = \prod_j \xi_j^{\alpha_j}$, then $(ad\, D)^\alpha Q = \xi^\alpha \triangle Q$.

For $A \in \mathfrak{A}_{ad}$, and a function $b \in \mathscr{M}$ let us apply Taylor's formula with integral remainder, in conjunction with the above kernel product:

$$b(D)A = \Big(\sum_{|\theta| \leqq N} b^{(\theta)}(\eta)/\theta!(\xi - \eta)^\theta + r_N(\xi, \eta)\Big) \triangle A$$
$$= \sum_{|\theta| \leqq N} i^{-|\theta|}/\theta!\, A_{(\theta)} b^{(\theta)}(D) + r_N \triangle A \,, \tag{2.4}$$

where

$$r_N(\xi, \eta) = \int_0^1 \tau^N \sum_{|\theta| = N+1} (N+1)/\theta!\, b^{(\theta)}(t\xi + \tau\eta)d\tau, \quad t + \tau = 1 \,. \tag{2.5}$$

Comparing (2.4) and (0.8) it is found that for Theorem 0.2 we must show that the order of the remainder $r_N \triangle A$ decreases to $-\infty$, as $N \to \infty$. This will be accomplished, evidently, if Lemma 2.1, below, is proven.

LEMMA 2.1. *For every integer* $1 = 0, 1, 2, \cdots$, *there exists* $N_0(l)$ *such that for* $N \geqq N_0(l)$ *we get*

$$R_N = r_N \triangle A = \Lambda^{2l} Q_{N,1} \Lambda^{2l}, \quad \textit{with } Q_{N,l} \in \mathfrak{A}_0 \,. \tag{2.6}$$

Proof. We note that explicitly

$$R_N = \int_0^1 \tau^N d\tau \sum_{|\theta| = N+1} (N+1)/\theta!\; i^{-|\theta|} b^{(\theta)}(t\xi + \tau\eta) \triangle A_{(\theta)} \,, \tag{2.7}$$

where the discussion of limit interchanges is postponed. In order to control the kernel product we write

$$b^{(\theta)}(t\xi + \tau\eta) = \int d'\kappa\, b^{(\theta)\vee}(\kappa) e^{i\kappa\xi t} e^{i\kappa\xi\tau} \,, \tag{2.8}$$

with $d'\kappa = (2\pi)^{-n/2} d\kappa$, and with the inverse Fourier transform $\vee$. Accordingly,

$$b^{(\theta)}(t\xi + \tau\eta) \triangle A_{(\theta)} = \int d'\kappa\, b^{(\theta)\vee}(\kappa) e^{i\kappa D} A_{(\theta), -\kappa\tau} = J \,, \tag{2.9}$$

with the 'translated operator' $P_\kappa = e^{iD\kappa} P e^{-iD\kappa}$. In that respect, the Fourier transform $b^{(\theta)\vee} = F^{-1} b^{(\theta)}$ proves to be a function in $L^1(\boldsymbol{R}^n)$, making the integral (2.8) meaningful as an improper Riemann integral in norm convergence of $\mathfrak{A}_0$, for large $|\theta|$. In details we note Lemma 2.2, below.

LEMMA 2.2. *Let* $\mathscr{S}^0_{ps}$ *denote the class of distributions* $u \in \mathscr{S}'$ *with singular support at* 0 *only, such that* (i) *u equals a function* *in* $\mathcal{S}$ *for* $|x| \geqq 1$, (ii) $x^\alpha u \in C^k(\boldsymbol{R}^n)$ *for all* $|\alpha| \geqq N(k)$, *with suitable* $N(k)$, *and for every* $k=0, 1, \cdots$. *Let* $\mathcal{M} \subset C^\infty(\boldsymbol{R}^n)$ *be defined as in the introduction. Then we have* $u \in \mathcal{M}$ *if and only if* $u^\wedge = Fu \in \mathcal{S}^0_{ps}$ (*or if and only if* $u^\vee \in \mathcal{S}^0_{ps}$).

The proof of Lemma 2.2 will be omitted (c.f. [2], I, Thm. 6.3). Continuing with Lemma 2.1 note that $c(\kappa) = b^{(\theta)\vee}(\kappa) = (-\kappa)^\theta b^\vee(\kappa) \in C^k(\boldsymbol{R}^n)$, as θ gets large enough, by Lemma 2.2. Also $c \in \mathcal{S}$ for $|x| \geqq 1$. Using the identity $e^{i\xi\kappa} = (1+\xi^2)^{-l}(1-\Delta_\kappa)^l e^{i\xi\kappa}$, $1=0, 1, 2, \cdots$, we formally get

$$J = \Lambda^{-21}\int d'\kappa e^{i\kappa D}(1-\Delta_\kappa)^1(c(\kappa)A_{(\theta),-\kappa\tau}) . \tag{2.10}$$

There will be no trouble justifying (2.10), as an improper Riemann integral in norm convergence of $\mathfrak{A}_0$, after Lemma 2.3, below.

LEMMA 2.3. *For* $A \in \mathfrak{A}_{ad}$ *we have* $A_t = e^{iDt}Ae^{-iDt} \in C^\infty(\boldsymbol{R}^n, \mathfrak{A}_0)$, *and the derivatives are explicitly given as*

$$i^{|\alpha|}D_t^\alpha A_t = A_{(\alpha),t} = e^{iDt}A_{(\alpha)}e^{-iDt} . \tag{2.11}$$

The proof of Lemma 2.3 is an immediate consequence of Lemma 1.3. Applying Lemma 2.3 it is found that J of (2.10) is a linear combination (with complex constants as coefficients) of the expressions

$$\Lambda^{2l}\tau^{|\beta|}\int d'\kappa e^{i\kappa D}c^{(\alpha)}(\kappa)A_{(\theta+\beta),-\kappa\tau}, \quad |\alpha|+|\beta| \leqq 2l . \tag{2.12}$$

Again $c^{(\alpha)} \in L^1(\boldsymbol{R}^n)$ for sufficiently large N (and these functions are continuous). Thus all the terms (2.12) are of the form $\Lambda^{2l}P$, with some $P \in \mathfrak{A}_0$.

This process may be repeated to create a power Λ^{2l} at right: Write the operator $P = \int d'\kappa e^{i\kappa D}c^{(\alpha)}(\kappa)A_{(\theta+\beta),-\kappa\tau}$ in the form

$$\int d'\kappa c^{(\alpha)}(\kappa)A_{(\theta+\beta),+\kappa t}e^{i\kappa D} . \tag{2.13}$$

Here again we may use the exponential identity, and integrate by parts to arrive at expressions of the form

$$\left(\int d'\kappa c^{(\alpha+\gamma)}(\kappa)A_{(\theta+\beta+\xi),+\kappa t}\right)\Lambda^{2l} . \tag{2.14}$$

The integrals (2.14) exist, as before, and supply operators $Q(t) \in \mathfrak{A}_0$ depending continuously on t, in the norm of $\mathfrak{A}_0$. Hence the integrals $\int_0^1 \tau^N d\tau Q(t)$ exist in norm convergence, and give operators in

$\mathfrak{A}_0$ again. Finally it will be necessary to justify the limit exchanges leading to (2.7) and (2.9). But this simply is a consequence of the fact that the integrals (2.5) and (2.8) converge as (improper) Riemann integrals, in the Frechet topology of $\mathscr{S}$. This proves Lemma 2.1 and thus establishes Theorem 0.2.

COROLLARY 2.4. *For the operator* $Q_{N,l} \in \mathfrak{A}_0$ *of Lemma* 2.1 *we have*

$$\sigma_{Q_{N,l}} = 0 \quad for \quad (x, \xi) \in \boldsymbol{M} - \boldsymbol{W},\ N \geqq 1\ . \tag{2.15}$$

Proof. We simply must observe that all the operators $A_{(\theta+\beta+\xi),\kappa t}$ occurring in (2.14) have symbols vanishing at $\boldsymbol{M} - \boldsymbol{W}$. Also, it already was found that all the integrals leading to the construction of $Q_{N,l}$ converge in the sense of $\mathfrak{A}_0$, so that symbols may be calculated by integrating the symbols. This implies the corollary.

3. The inclusion $\mathfrak{A}_{ad} \subset PS(0)$. It is clear now that for Theorem 0.1 we now only are left with proving that $\mathfrak{A}_{ad} \subset PS(0)$. For then, if $a \in C_p(\boldsymbol{M})$ we apply Theorem 1.4, with $k = \infty$, to construct an operator $A \in \mathfrak{A}_{ad,\infty} = \mathfrak{A}_{ad}$ with $\sigma_A = a$. We have $A \in \mathfrak{A}_{ad} \subset PS(0) = \mathfrak{A}_\infty$, by Lemma 1.1 which proves Theorem 0.1. For the inclusion $\mathfrak{A}_{ad} \subset PS(0)$ we only must prove Lemma 3.1, below.

LEMMA 3.1. *Let* $A \in \mathfrak{A}_{ad}$. *Then we have* $\Lambda^s A \Lambda^{-s} - A \in \mathfrak{K}_0$ *for all* $s \in \boldsymbol{R}$.

Proof. Let us repeat the discussion of § 2 for the special function $b(\xi) \equiv \lambda^s(\xi) = (1 + \xi^2)^{s/2}$. From Lemma 2.1 we get

$$\Lambda^s A \Lambda^{-s} - A = \sum_{1 \leqq |\theta| \leqq N} i^{-|\theta|}/\theta!\ A_{(\theta)}\mu_\theta(D) + \Lambda^{2l} Q_{N,l} \Lambda^{2l-s}\ , \tag{3.1}$$

with the functions $\mu_\theta = (\lambda^s)^{(\theta)}/\lambda^s$. (3.1) is valid for large N only, depending on the choice of l. For a given s we choose l according to $2l > s$, and then $N \geqq 1$ large enough to insure (3.1). Then it is observed that the entire right hand side of (3.1) is in $\mathfrak{K}_0$, by Corollary 2.4, because the symbol is calculated to be zero on all of $\boldsymbol{M}$. In particular we notice that $\mu_\theta \in \mathscr{S}^\#$, $\theta \neq 0$, thus $\mu_\theta(D) \in \mathfrak{A}_0$, and that $\mu_\theta = 0$ as $|\xi| = \infty$, thus $\sigma_{\mu_\theta(D)} = 0$ on $\boldsymbol{W}$. This proves Lemma 3.1.

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UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720