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A REPRESENTATION OF H^p-FUNCTIONS WITH $0<p<\infty$

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Let E be an open arc in the unit circle. Let F belong to the Hardy space H^p, $0 < p < \infty$, and let g be the restriction of the boundary distribution of F to E. For each $0 < \lambda < 1$ we construct functions $G_\lambda \in H^p$ from g such that $G_\lambda \to F$ in the topology of H^p as $\lambda \to 1$.

I. Introduction. The purpose of this article is to extend to the case $0 < p < 1$ the following theorem of D. J. Patil.

THEOREM A. [2, Th. I, p. 617]. *Let E be a subset of the unit circle T, of positive Lebesgue measure. Let $1 \leqq p \leqq \infty$, let F be in the Hardy space H^p, and let g be the restriction to E of the boundary-value function of F. Denote the normalized Lebesgue measure on T by m, the open unit disc in the complex plane by U, and define for each $\lambda > 0$*

$$H_\lambda(z) = \exp\left\{-\frac{1}{2}\log(1+\lambda)\int_E \frac{w+z}{w-z}\,dm(w)\right\} \qquad (z \in U),$$

$$G_\lambda(z) = \lambda H_\lambda(z)\int_E \frac{\overline{h_\lambda(w)}g(w)}{1-\bar{w}z}\,dm(w)\,, \qquad (z \in U)\,,$$

where h_λ is the boundary-value function of H_λ.

Then as $\lambda \to \infty$, G_λ approaches F uniformly on compact subset of U. Moreover, if $1 < p < \infty$ then $\|G_\lambda - F\|_{H^p} \to 0$ as $\lambda \to \infty$.

The extension of the above to the case $0 < p < 1$ involves a strengthening of the hypotheses: the set E of positive measure will be replaced by an open arc in T, and instead of the characteristic function of E we will work with an infinitely differentiable function with support in E.

Specifically, let E be an open arc in T, and let ψ be an infinitely differentiable function on T with support in E such that

(i) $0 \leqq \psi(w) \leqq 1 \quad (w \in T)$,

(ii) $J = \{w \in T: \psi(w) = 1\}$ has positive Lebesgue measure.

THEOREM B. *Let $0 < p < \infty$, let F be in H^p, and let g be the restriction to E of the boundary distribution of F on T. Define for each $0 < \lambda < 1$*

$$\chi_\lambda(w) = \frac{\lambda\psi(w)}{1 - \lambda\psi(w)} \qquad (w \in T)\ ,$$

$$H_\lambda(z) = \exp\left\{-\frac{1}{2}\int_E \frac{w+z}{w-z}\log\left[1 + \chi_\lambda(w)\right]dm(w)\right\} \qquad (z \in U)\ ,$$

$$G_\lambda(z) = H_\lambda(z)\langle g, \chi_\lambda h_\lambda C_z\rangle_E \qquad (z \in U)\ ,$$

where h_λ is the boundary-value function of H_λ, $\langle,\rangle_E$ is the pairing between distributions and test functions on E, and C_z is the Cauchy kernel, i.e.,

$$C_z(w) = \frac{1}{1 - w\bar{z}} \qquad (w \in T,\ z \in U)\ .$$

Then $\|G_\lambda - F\|_{H^p} \to 0$ as $\lambda \to 1$. In particular G_λ approaches F uniformly on compact subsets of U.

Our main result, Theorem B (Theorem 4.6 in the text), is proven in § IV. In § II we establish the notation and terminology, and list well-known properties of the Hardy spaces and Toeplitz operators. Our proof of Theorem B closely parallels the method of Patil in [2]; it involves the use of Toeplitz operators associated with infinitely differentiable functions, which, we prove in § III, can be extended to bounded operators on H^p for all $0 < p < \infty$.

II. Preliminaries. In the sequel, U will be the open unit disc in the complex plane and T its boundary, the unit circle. We shall denote the normalized Lebesgue measure on T by m; the corresponding L^p-spaces will be denoted by $L^p(T)$ and the L^p-norm by $\|\ \|_{L^p(T)}$. The phrase "almost everywhere" will always refer to the measure m.

1. Test functions and distributions. Let E be an open arc in T. The space of *test functions* on E will be represented by $C_0^\infty(E)$. The test functions on E, we recall, are infinitely differentiable complex-valued functions on E with compact support. If $E = T$, we write $C^\infty(T)$ instead of $C_0^\infty(T)$. By a *distribution* on E we shall mean a continuous *skewlinear* functional on the topological linear space $C_0^\infty(E)$. The space of distributions on E will be denoted by $D(E)$.

If $\langle\phi, \varphi\rangle_E$ represents the *sesquilinear* pairing between $\phi \in D(E)$ and $\varphi \in C_0^\infty(E)$, we identify a locally integrable function f on E with the distribution f defined by

$$\langle f, \varphi\rangle_E = \int_E f(w)\overline{\varphi(w)}dm(w)\ .$$

The same symbol $\langle,\rangle_E$ shall be used to represent the inner product in $L^2(E)$.

Let $\phi \in D(T)$, and define $e_n \in C^\infty(T)$ by $e_n(w) = w^n$ for each integer n. The *Fourier coefficients* of ϕ are the numbers

$$\hat{\phi}(n) = \langle \phi, e_n \rangle_T .$$

The *Fourier series* of ϕ is the formal series $\sum_{-\infty}^{+\infty} \hat{\phi}(n) w^n$. A straightforward calculation shows that $\sum_{-\infty}^{+\infty} a_n w^n$ is the Fourier series of a test function on T if and only if

$$|a_n| = O(|n|^q)$$

for all integers q. Consequently, a necessary and sufficient condition for $\sum_{-\infty}^{+\infty} a_n w^n$ to be the Fourier series of a distribution on T is that

$$|a_n| = O(|n|^{-q})$$

for some integer q.

If $\phi \in D(T)$ has Fourier series $\sum_{-\infty}^{+\infty} a_n w^n$, we denote by $P\phi$ the distribution of Fourier series $\sum_{n=0}^{\infty} a_n w^n$. We refer to P as the *projection operator*. If $\varphi \in C^\infty(T)$, we define $M_\varphi \phi \in D(T)$, by

$$\langle M_\varphi \phi, \psi \rangle_T = \langle \phi, \bar{\varphi}\psi \rangle_T$$

for all $\psi \in C^\infty(T)$. We call M_φ the *multiplication by* φ.

Finally, we remark that the partial sums of the Fourier series of $\phi \in D(T)$ converge to ϕ in the topology of $D(T)$ and that

$$\langle \phi, \varphi \rangle_T = \sum_{-\infty}^{+\infty} \hat{\phi}(n) \overline{\hat{\varphi}(n)}$$

for $\varphi \in C^\infty(T)$ and $\phi \in D(T)$.

2. Hardy spaces. Let F be a holomorphic function in the open unit disc U. If $0 < r < 1$, and if $w \in T$, we write $F_r(w) = F(rw)$ and define, for $0 < p < \infty$,

$$||F||_{H^p(U)} = \lim_{r \to 1} ||F_r||_{L^p(T)} .$$

The *Hardy space* $H^p(U)$ is the linear space of all holomorphic functions F on U such that $||F||_{H^p(U)} < \infty$. The space $H^\infty(U)$ is the space of bounded holomorphic functions in U, and $|| \; ||_{H^\infty(U)}$ is the supremum norm.

If $p \geqq 1$, then $H^p(U)$ is a Banach space with norm $|| \; ||_{H^p(U)}$. This is no longer true if $0 < p < 1$; in this case, however, we can regard $H^p(U)$ as a complete metric space with the translation-invariant metric

$$d(F, G) = \| F - G \|^p_{H^p(U)} .$$

For all $0 < p < \infty$ the polynomials are dense in $H^p(U)$. If $0 < p < q \leqq \infty$ it can be verified that $\| \|_{H^p(U)} \leqq \| \|_{H^q(U)}$; consequently $H^q(U)$ is a dense subspace of $H^p(U)$. We also remark that the topology of $H^p(U)$, $0 < p \leqq \infty$, is stronger than that of uniform convergence on compact subsets of U.

Let $1 \leqq p \leqq \infty$ and let $F(z) = \sum_{n=0}^\infty a_n z^n$ be in $H^p(U)$; as is well-known, $\sum_{n=0}^\infty a_n w^n$ is the Fourier series of a function $f \in L^p(T)$. Moreover,

$$\lim_{r \to 1} F_r(w) = f(w)$$

for almost all $w \in T$,

$$\| F \|_{H^p(U)} = \| f \|_{L^p(T)} ,$$

and, if $1 \leqq p < \infty$,

$$\lim_{r \to 1} \| F_r - f \|_{L^p(T)} = 0 .$$

Thus, $F \to f$ is an isometry between $H^p(U)$ and a closed linear subspace $H^p(T)$ of $L^p(T)$, which consists of the functions in $L^p(T)$ whose Fourier coefficients corresponding to negative integers are identically zero. We refer to F as the *holomorphic extension of f to U*, and to f as the *boundary-value function of F on T*.

Our main concern, in this article, is with the spaces $H^p(U)$ with $0 < p < 1$. The following theorem is due to Hardy and Littlewood, and will be used in the sequel.

2.1. THEOREM [1, Th. 6.4, p. 98]. *Let $0 < p \leqq 1$, and let $F(z) = \sum_{n=0}^\infty a_n z^n$ be in $H^p(U)$. Then*

$$| a_n | \leqq C(p) n^{1/p-1} \|F\|_{H^p(U)}$$

for $n = 1, 2, \cdots$, where $C(p)$ is a constant which depends only on p.

[Clearly $C(1) = 1$ is best possible.]

If $0 < p < 1$ and if $F(z) = \sum_{n=0}^\infty a_n z^n$, the above implies that $\sum_{n=0}^\infty a_n w^n$ is the Fourier series of a distribution f on T. As with the case $1 \leqq p \leqq \infty$, we refer to F as the *holomorphic extension of f to U*, and to f as the *distributional boundary-value of F on T*. The space of all distributional boundary-values of functions in $H^p(U)$ will be denoted by $H^p(T)$. We endow $H^p(T)$ with a metric structure isometric to that of $H^p(U)$ by setting

$$\|f\|_{H^p(T)} = \|F\|_{H^p(U)}$$

whenever f and F are related as above.

It is known ([1, Th. 7.5, p. 115]) that each $\varphi \in C^\infty(T)$ gives rise to a bounded linear functional Λ_φ on $H^p(U)$, $0 < p < 1$, defined by

$$\Lambda_\varphi F = \langle f, \varphi \rangle_T .$$

This implies that the topology of $H^p(T)$ is stronger than the one inherited from $D(T)$.

Let $0 < p < \infty$, let $F \in H^p(U)$, define $F_r(w) = F(rw)$ for $0 < r < 1$ and $w \in T$, and let f be the distributional boundary-value of F. For $z \in U$ and $0 < r < 1$, Cauchy's formula

$$F(rz) = \int_T \frac{F_r(w)}{1 - \bar{w}z} dm(w)$$

holds. Since in all cases $0 < p < \infty$ the functions F_r converge to f in $H^p(T)$, and hence in the weaker topology of $D(T)$, it follows that

$$F(z) = \langle f, C_z \rangle_T$$

where

$$C_z(w) = \frac{1}{1 - w\bar{z}} ,$$

$z \in U$, and $w \in T$.

3. **Toeplitz operators.** Let P be the orthogonal projection of $L^2(T)$ onto $H^2(T)$. Fix $\varphi \in L^\infty(T)$ and let M_φ be the corresponding multiplication operator on $L^2(T)$. The *Toeplitz operator* S_φ: $H^2(T) \to H^2(T)$ is the composition PM_φ; i.e.,

$$S_\varphi f = P(\varphi f)$$

for $f \in H^2(T)$. It can be immediately verified that

$$G(z) = \int_T \frac{\varphi(w) f(w)}{1 - \bar{w}z} dm(w) = \langle M_\varphi f, C_z \rangle_T$$

is the holomorphic extension of $S_\varphi f$ to U.

The following elementary properties will be used in the sequel:

(a) *$S_{\bar{\varphi}}$ is the adjoint operator of S_φ.*

(b) *If either $\bar{\varphi} \in H^\infty(T)$ or $\psi \in H^\infty(T)$, then $S_{\varphi\psi} = S_\varphi S_\psi$.*

A consequence of (b) ([2, Lemma 1, p. 618]) is:

(c) *If $h \in H^\infty(T)$, if $1/h \in H^\infty(T)$, and if $\varphi = |h|^{-2}$, then S_φ is invertible and $(S_\varphi)^{-1} = S_h S_{\bar{h}}$.*

III. **Toeplitz operators on $H^p(T)$, $0 < p \leqq 1$.** Since the orthogonal projection P of $L^2(T)$ onto $H^2(T)$ extends or restricts to a bounded projection of $L^p(T)$ onto $H^p(T)$, the Toeplitz operator $S_\varphi = PM_\varphi$ is bounded on $H^p(T)$ whenever $1 < p < \infty$ and $\varphi \in L^\infty(T)$. The projection P, however, is not bounded on $L^1(T)$; thus, in general, S_φ will not be a bounded operator on $H^1(T)$, or on $H^p(T)$ with $0 < p < 1$. As was noted earlier, the projection P can be naturally extended to the space $D(T)$ of distributions; namely, by assigning to the distribution $\phi \sim \sum_{-\infty}^{+\infty} a_n w^n$ the "analytic" distribution $P\phi \sim \sum_{n=0}^{\infty} a_n w^n$. If $\varphi \in C^\infty(T)$, the multiplication operator M_φ can also be naturally extended to $D(T)$. Thus, the symbol $PM_\varphi f$ is meaningful for $f \in H^p(T)$, $0 < p < \infty$. *Our goal is to prove that* $S_\varphi = PM_\varphi$, *with* $\varphi \in C^\infty(T)$, *is a bounded operator of* $H^p(T)$ *into itself, even if* $0 < p \leqq 1$.

LEMMA 3.1. *Let* $\varphi \in C^\infty(T)$, *let* $f \in H^2(T)$, *and let* $0 < p \leqq 1$. *Then*

$$\|S_\varphi f\|_{H^p(T)} \leqq K_\varphi(p)\, \|f\|_{H^p(T)}\ ,$$

where $K(p)$ *depends on* p *and* φ *but is independent of* f. *Moreover, if* φ *has Fourier series* $\sum_{-\infty}^{+\infty} c_n w^n$ *and if* $C(p)$ *is the constant of Theorem* 2.1, *then we can choose*

$$K_\varphi(p) = \{\sum_{n=0}^{\infty} |c_n|^p + \sum_{n=1}^{\infty} [2 + C(p)^p (n-1)^{2-p}] |c_{-n}|^p\}^{1/p}\ .$$

Proof. Let G be the holomorphic extension of $S_\varphi f$ to U, i.e.,

$$G(z) = \int_T \frac{\varphi(w) f(w)}{1 - \bar{w}z}\, dm(w)\ ,$$

and let $F(z) = \sum_{j=0}^{\infty} a_j z^j$ be the holomorphic extension of f to U. We proceed to establish

$$\|G\|_{H^p(U)} \leqq K_\varphi(p)\, \|F\|_{H^p(U)}\ ,$$

which is equivalent to the assertion of the lemma. To this effect we write

$$G(z) = \sum_{-\infty}^{+\infty} c_n \int_T \frac{w^n f(w)}{1 - \bar{w}z}\, dm(w)\ , \tag{3.1.1}$$

and define

$$M_n(z) = \int_T \frac{w^n f(w)}{1 - \bar{w}z}\, dm(w)\ ,$$

$$N_n(z) = \int_T \frac{\bar{w}^n f(w)}{1 - \bar{w}z}\, dm(w)\ ,$$

for all nonnegative integers n, and $z \in U$.

Both M_n and N_n are holomorphic in U. Clearly $M_n(z) = z^n F(z)$: hence

$$\|M_n\|_{H^p(U)} = \|F\|_{H^p(U)} . \tag{3.1.2}$$

On the other hand, for $n = 1, 2, \cdots$,

$$N_n(z) = \sum_{j=0}^{\infty} \hat{f}(j+n) z^j = z^{-n}\left\{\sum_{j=0}^{\infty} \hat{f}(j) z^j - \sum_{j=0}^{n-1} \hat{f}(j) z^j\right\},$$

which can be rewritten (since the Fourier coefficients of f are the Taylor coefficients of F)

$$N_n(z) = z^{-n}\left\{F(z) - \sum_{j=0}^{n-1} a_j z^j\right\}.$$

Consequently, for $0 < p \leqq 1$,

$$|N_n(z)|^p \leqq |z|^{-np}\left\{|F(z)|^p + \sum_{j=0}^{n-1} |a_j|^p\right\},$$

and

$$\lim_{r\to 1} \int_T |N_n(rw)|^p \, dm(w) \leqq \|F\|_{H^p(U)}^p + \sum_{j=0}^{n-1} |a_j|^p .$$

Since by Theorem 2.1

$$|a_j| \leqq C(p) j^{1/p-1} \|F\|_{H^p(U)}$$

for $j = 1, 2, \cdots$, and since

$$|a_0| \leqq \|F\|_{H^p(U)},$$

we get

$$\|N_n\|_{H^p(U)}^p \leqq 2\|F\|_{H^p(U)}^p + C(p)^p (n-1)^{2-p} \|F\|_{H^p(U)}^p . \tag{3.1.3}$$

By (3.1.1) we have

$$G(z) = \sum_{n=0}^{\infty} c_n M_n(z) + \sum_{n=1}^{\infty} c_{-n} N_n(z) ;$$

(3.1.2) and (3.1.3) then imply

$$\begin{aligned} \|G\|_{H^p(U)}^p &\leqq \sum_{n=0}^{\infty} |c_n|^p \|M_n\|_{H^p(U)}^p + \sum_{n=1}^{\infty} |c_{-n}|^p \|N_n\|_{H^p(U)}^p \\ &\leqq \|F\|_{H^p(U)}^p \left\{\sum_{n=0}^{\infty} |c_n|^p + \sum_{n=1}^{\infty} [2 + C(p)^p (n-1)^{2-p}] |c_{-n}|^p\right\}. \end{aligned} \tag{3.1.4}$$

This completes the proof. [We recall that $|c_n| = O(n^{-q})$ for all positive integers q; consequently, the right-hand term in (3.1.4) is finite.]

THEOREM 3.2. *If $\varphi \in C^\infty(T)$, the Toeplitz operator $S_\varphi = PM_\varphi$ is a bounded operator on $H^p(T)$ for $0 < p \leqq 1$.*

If φ has Fourier series $\sum_{-\infty}^{+\infty} c_n w^n$, the norm

$$|||S_\varphi|||_{H^p(T)} = \sup\{||S_\varphi f||_{H^p(T)} : ||f||_{H^p(T)} \leqq 1\}$$

satisfies the estimate

$$(1) \qquad |||S_\varphi|||_{H^p(T)} \leqq K_\varphi(p) .$$

Finally, if $f \in H^p(T)$, then $S_\varphi f$ is the distributional boundary-value of the holomorphic function (of the variable z)

$$(2) \qquad \langle M_\varphi f, C_z \rangle_T ,$$

where $C_z(w) = 1/1 - w\bar{z}$, $w \in T$, and $z \in U$.

Proof. Fix $0 < p \leqq 1$. That the operator S_φ: $H^2(T) \to H^2(T)$ can be uniquely extended to a bounded operator L on $H^p(T)$ and that the norm of L satisfies (1) is a direct consequence of Lemma 3.1 and of the fact that $H^2(T)$ is dense in $H^p(T)$.

To establish $L = PM_\varphi$, fix $f \in H^p(T)$ and let $G \in H^p(U)$ be the holomorphic extension of Lf to U. Our immediate goal is to show that

$$G(z) = \langle f, \bar{\varphi} C_z \rangle_T .$$

Let F be the holomorphic extension of f to U, set $F_r(w) = F(rw)$, and denote by G_r the holomorphic extension of $LF_r = S_\varphi F_r$ to U. It is clear that

$$G_r(z) = \int_T \frac{\varphi(w) F_r(w)}{1 - \bar{w} z} \, dm(w) = \langle F_r, \bar{\varphi} C_z \rangle_T .$$

Since the functions F_r converge to the distribution f in the topology of $H^p(T)$ as r tends to 1, it follows that

$$(3.2.1) \qquad \lim_{r \to 1} G_r(z) = \langle f, \bar{\varphi} C_z \rangle_T$$

for each $z \in U$. On the other hand, the continuity of L implies that LF_r approaches Lf in $H^p(T)$; or equivalently for the holomorphic extensions: that G_r converges to G in $H^p(U)$, in particular

$$(3.2.2) \qquad \lim_{r \to 1} G_r(z) = G(z)$$

for $z \in U$. The equalities (3.2.1) and (3.2.2) now establish

$$(3.2.3) \qquad G(z) = \langle f, \bar{\varphi} C_z \rangle_T = \langle M_\varphi f, C_z \rangle_T .$$

By a straightforward calculation it can be shown that the boundary-value of G (the distribution Lf) is the analytic projection of $M_\varphi f$, i.e., $Lf = PM_\varphi = S_\varphi$. This completes the proof.

COROLLARY 3.3. *If $\varphi \in C^\infty(T)$, if $h \in H^\infty(T)$, if $1/h \in H^\infty(T)$, and if $\varphi = |h|^{-2}$ then the Toeplitz operator $S_\varphi: H^p(T) \to H^p(T)$ is invertible, and $S_\varphi^{-1} = S_h S_{\bar{h}}$, for all $0 < p < \infty$.*

Proof. The case $1 < p < \infty$ is dealt with in [2]. To prove the remaining case it suffices to show that $h \in C^\infty(T)$, for then the operators S_h, $S_{\bar{h}}$, S_φ will be bounded operators on $H^p(T)$, $0 < p \leqq 1$, that satisfy $S_\varphi^{-1} = S_h S_{\bar{h}}$ on a dense subset [say $H^2(T)$] of $H^p(T)$. This, however, follows readily. The hypotheses on h imply that $\log|h|$, the real part of $\log h$, is in $C^\infty(T)$, consequently $\log h \in C^\infty(T)$ which implies $h \in C^\infty(T)$.

IV. The representation of functions in $H^p(U)$.

DEFINITIONS 4.1. Let E be an open arc in the unit circle T. Choose $\psi \in C^\infty(T)$ such that

(a) *ψ has compact support in E,*

(b) $0 \leqq \psi(w) \leqq 1 \quad (w \in T)$,

(c) *$J = \{w \in T: \psi(w) = 1\}$ has positive Lebesgue measure.*

For each $0 < \lambda < 1$ define

$$\chi_\lambda(w) = \frac{\lambda\psi(w)}{1 - \lambda\psi(w)} \qquad (w \in T)\,,$$

$$H_\lambda(z) = \exp\left\{-\frac{1}{2}\int_T \frac{w+z}{w-z}\log[1 + \chi_\lambda(w)]dm(w)\right\} \qquad (z \in U)\,.$$

It is immediate that $\chi_\lambda \in C^\infty(T)$, and that $H_\lambda \in H^\infty(U)$. Denote by h_λ the boundary-value of H_λ. The following are verified:

(d) $|h_\lambda(w)|^{-2} = 1 + \chi_\lambda(w) \quad (w \in T)$,

(e) *h_λ and h_λ^{-1} are in $H^\infty(T)$.*

Finally, define for each $0 < \lambda < 1$

$$\varphi_\lambda(w) = 1 + \chi_\lambda(w) \qquad (w \in T)\,.$$

Then

(f) $$\varphi_\lambda(w) = \frac{1}{1 - \lambda\psi(w)} \qquad (w \in T)\,.$$

Our next lemma is an immediate consequence of Corollary 3.3.

LEMMA 4.2. *Each S_{φ_λ} is an invertible operator on $H^p(T)$,*

$0 < p < \infty$, with inverse $S_{\varphi_\lambda}^{-1} = S_{h_\lambda} S_{\bar{h}_\lambda}$.

LEMMA 4.3. *The operators $S_{\varphi_\lambda}^{-1}$, $0<\lambda<1$, are uniformly bounded on $H^p(T)$, $0 < p < \infty$.*

Proof. The case $1 < p < \infty$ is a consequence of the conjugate function theorem of M. Riesz (as in [2, Lemma 5, p. 618]).

Assume $0 < p \leqq 1$, and let $f \in H^p(T)$. Then

$$\begin{aligned} S_{h_\lambda} S_{\bar{h}_\lambda} f &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \hat{h}_\lambda(m(\hat{\bar{h}}_\lambda(n) \hat{f}(q) e_{q-n+m} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=\max(0,n-m)}^{\infty} \hat{h}_\lambda(m) \hat{\bar{h}}_\lambda(n) \hat{f}(q) e_{q-n+m} \\ &\quad - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=\max(0,n-m)}^{n-1} \hat{h}_\lambda(m) \hat{\bar{h}}_\lambda(n) \hat{f}(q) e_{q-n+m} \\ &= S_{\bar{h}_\lambda} S_{h_\lambda} f - \sum_{k=-\infty}^{+\infty} \sum_{m-n=k} \hat{h}_\lambda(m) \hat{\bar{h}}_\lambda(n) \sum_{q=\max(0,n-m)}^{n-1} e_{q-n+m} . \end{aligned}$$

Recalling $|h_\lambda(w)|^2 = 1 - \lambda\psi(w)$, and letting $K_\psi(p)$ be the constant of Lemma 3.1, we verify (using the estimates 2.1):

$$\| S_{h_\lambda} S_{\bar{h}_\lambda} f \|_{H^p(T)}^p \leqq 2[1 + \lambda^p K_\psi^p(p)] \| f \|_{H^p(T)}^p ,$$

which establishes the Lemma.

LEMMA 4.4. *Fix $z \in U$. Then $\lim_{\lambda\to 1} \| S_{\varphi_\lambda}^{-1} C_z \|_{H^p(T)} = 0$, for $0 < p < \infty$. Moreover $S_{\varphi_\lambda}^{-1} C_z = \overline{H_\lambda(z)} h_\lambda C_z$.*

[For $z \in U$ and $w \in T$ we recall the definition $C_z(w) = 1/1 - w\bar{z}$.]

Proof. The same argument used in [2, Lemma 3, p. 618] establishes

$$S_{\bar{h}_\lambda} C_z = \overline{H_\lambda(z)} C_z .$$

Since $S_{\varphi\varphi_\lambda}^{-1} = S_{h_\lambda} S_{\bar{h}_\lambda}$, we have

$$S_{\varphi_\lambda}^{-1} C_z = S_{h_\lambda} S_{\bar{h}_\lambda} C_z = S_{h_\lambda} \overline{H_\lambda(z)} C_z = \overline{H_\lambda(z)} h_\lambda C_z . \tag{4.4.1}$$

From the definition of H_λ it follows that

$$\begin{aligned} |H_\lambda(z)| &= \exp\left\{ -\frac{1}{2} \int_T \frac{1 - |z|^2}{|1 - \bar{w}z|^2} \log[1 + \chi_\lambda(w)] dm(w) \right\} \\ &\leqq \exp\left\{ \frac{1}{2} \int_J \frac{1 - |z|}{1 + |z|} \log(1 - \lambda) dm(w) \right\} = (1 - \lambda)^\alpha , \end{aligned} \tag{4.4.2}$$

where $2\alpha = \{1 - |z|/1 + |z|\} m(J) > 0$.

By (4.4.2) we have

$$
(4.4.3) \qquad \| H_\lambda(z) h_\lambda C_z \|_{H^p(T)} = | H_\lambda(z) | \, \| h_\lambda C_z \|_{H^p(T)} \leqq (1 - \lambda)^\alpha \| h_\lambda \|_{H^\infty(T)} \| C_z \|_{H^p(T)} .
$$

Combining (4.4.1), (4.4.3), and

$$
| h_\lambda(w) | = [1 + \chi_\lambda(w)]^{-1/2} \leqq 1 ,
$$

we get

$$
\lim_{\lambda \to 1} \| S_{\varphi_\lambda}^{-1} C_z \|_{H^p(T)} \leqq \lim_{\lambda \to 1} (1 - \lambda)^\alpha \| C_z \|_{H^p(T)} = 0 .
$$

LEMMA 4.5. *If $0 < p < \infty$ and $f \in H^p(T)$, then*

$$
\lim_{\lambda \to 1} \| f - (I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f \|_{H^p(T)} = 0 .
$$

Proof. Lemma 4.3 and Lemma 4.4, in conjunction with the well-known fact that the linear span of $\{C_z : z \in U\}$ is dense in $H^p(T)$, $0 < p < \infty$, imply

$$
\lim_{\lambda \to 1} \| S_{\varphi_\lambda}^{-1} f \|_{H^p(T)} = 0
$$

for all $f \in H^p(T)$. Since $(I + S_{\chi_\lambda})^{-1} = (S_{\varphi_\lambda})^{-1} = S_{\varphi_\lambda}^{-1}$ by Lemma 4.2, we have

$$
\lim_{\lambda \to 1} \| (I + S_{\chi_\lambda})^{-1} f \|_{H^p(T)} = 0 .
$$

Observing that

$$
(I + S_{\chi_\lambda})^{-1} f = f - (I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f ,
$$

we get

$$
\lim_{\lambda \to 1} \| f - (I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f \|_{H^p(T)} = 0 .
$$

THEOREM 4.6. *Let $F \in H^p(U)$, with $0 < p < \infty$, let $f \in H^p(T)$ be the distributional boundary-value of F on T, and let g be the restriction of f to the open arc E. For $0 < \lambda < 1$ define holomorphic functions G_λ on U by*

$$
G_\lambda(z) = H_\lambda(z) \langle g, \chi_\lambda h_\lambda C_z \rangle_E .
$$

Then as $\lambda \to 1$ we have $\| G_\lambda - F \|_{H^p(U)} \to 0$. In particular G_λ approaches F uniformly on compact subsets of U.

Proof. In view of Lemma 4.5, the proof will be complete if we succeed in showing that G_λ is the holomorphic extension of $(I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f$ to U. The case $1 < p < \infty$ is essentially dealt with in [2]; we restrict ourselves to $0 < p \leqq 1$.

Let $f \in H^2(T)$. Since $(I+S_{\chi_\lambda})^{-1}$ is a self-adjoint operator on $H^2(T)$,

$$\begin{aligned}(4.6.1) \qquad \langle (I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f, C_z \rangle_T &= \langle S_{\chi_\lambda} f, (I + S_{\chi_\lambda})^{-1} C_z \rangle_T \\ &= \langle M_{\chi_\lambda} f, (I + S_{\chi_\lambda})^{-1} C_z \rangle_T .\end{aligned}$$

By Lemma 4.4,

$$(I + S_{\chi_\lambda})^{-1} C_z = S_{\varphi_\lambda}^{-1} C_z = \overline{H_\lambda(z)} h_\lambda C_z ,$$

Consequently

$$\begin{aligned}(4.6.2) \qquad \langle M_{\chi_\lambda} f, (I + S_{\chi_\lambda})^{-1} C_z \rangle_T &= \langle M_{\chi_\lambda} f, \overline{H_\lambda(z)} h_\lambda C_z \rangle_T \\ &= H_\lambda(z) \langle M_{\chi_\lambda} f, h_\lambda C_z \rangle_T \\ &= H_\lambda(z) \langle f, \chi_\lambda h_\lambda C_z \rangle_T .\end{aligned}$$

Since the operators involved are defined and continuous on $H^p(T)$, and since $H^2(T)$ is dense in $H^p(T)$, the relations (4.6.1) and (4.6.2) imply

$$\langle (I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f, C_z \rangle_T = H_\lambda(z) \langle f, \chi_\lambda h_\lambda C_z \rangle_T$$

for all $f \in H^p(T)$. Therefore

$$\begin{aligned}G_\lambda(z) &= H_\lambda(z) \langle g, \chi_\lambda h_\lambda C_z \rangle_E = H_\lambda(z) \langle f, \chi_\lambda h_\lambda C_z \rangle_T \\ &= \langle (I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f, C_z \rangle_T ,\end{aligned}$$

which establishes G_λ as the holomorphic extension (the "Cauchy integral") of $(I + S_{\chi_\lambda})^{-1} S_{\chi_\lambda} f$ to the disc U.

References

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