PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 1, 1978

BOUNDED ANALYTIC FUNCTIONS ON UNBOUNDED COVERING SURFACES

SHIGEO SEGAWA

Let R be an unbounded finite sheeted convering surface over an open Riemann surface with an exhaustion condition. In this paper, the necessary and sufficient condition in order that $H^\infty(R)$ separates the points of R is given in term of branch points, where $H^\infty(R)$ is the algebra of bounded analytic functions on R.

A covering surface R over a Riemann surface G is said to be *unbounded* if for any continuous curve λ; $z = z(t)$ $(0 \leqq t \leqq 1)$ in G and any point p_0 in R with $\pi(p_0) = z(0)$ there exists a continuous curve Λ; $p = p(t)$ $(0 \leqq t \leqq 1)$ in R such that $p(0) = p_0$ and $z(t) = \pi\circ p(t)$ $(0 \leqq t \leqq 1)$, where π is the projection of R onto G. For an unbounded covering surface R over G, the number of points of $\pi^{-1}(z)$ is constant $\leqq \infty$ for every $z \in G$ where branch points are counted repeatedly according to their orders. If such a number n is finite, R is said to be *n-sheeted.*

In [2], Selberg proved the following: Let R be an unbounded n-sheeted covering surface over the unit disk $|z| < 1$ and $\{\zeta_k\}$ the projections of branch points with the order of branching n_k over ζ_k. Let z_0 be a point in the unit disk over which there exist no branch points of R. Then there exists a single valued bounded analytic function f on R such that f takes distinct values at any two points over z_0 if and only if $\Sigma n_k g(\zeta_k, z_0) < \infty$, where $g(\cdot, z_0)$ is the Green's function on $|z| < 1$ with pole at z_0. Yamamura [5] extended the above result to the case where base surfaces are finitely connected plane regions.

On the other hand, Stanton [3] gave another proof of the above Selberg theorem using the Widom results [4]. The purpose of this paper is, by using the Widom-Stanton approach, to establish a result generalizing the Yamamura, and hence the Selberg, theorem to the case where the base surface $|z| < 1$ is replaced by certain surfaces which may be of infinite connectivity and genus.

1. Let R be an open Riemann surface of hyperbolic type and $g_R(\cdot, p)$ the Green's function on R with pole at p. Denote by $H^\infty(R)$ the algebra of single valued bounded analytic functions on R. For any $\alpha > 0$, set $R_\alpha = R(\alpha, p) = \{q \in R; g_R(q, p) > \alpha\}$. It is easily seen that, for each α, R_α is connected and $R - R_\alpha$ has no compact components. Suppose that each R_α is relatively compact in R. The

surface R with this property is referred to as being *regular*. Let $\beta_R(\alpha) = \beta_R(\alpha, p)$ be the first Betti number of R_α. Consider the quantity

$$m(R) = m(R, p) = \exp\left\{ -\int_0^\infty \beta_R(\alpha)d\alpha \right\}.$$

Widom proved that if $m(R) > 0$, then $H^\infty(R)$ separates the points of R, i.e., for any two distinct points p and q in R, there exists an $f \in H^\infty(R)$ such that $f(p) \neq f(q)$; it is also shown that $m(R) > 0$ does not depend on the choice of points p in R. (See [3] and [4].)

2. Hereafter, we suppose that G is an open Riemann surface of hyperbolic type and R is an unbounded n-sheeted covering surface over G. Then R is also hyperbolic. Let $g_G(\cdot, z_0)$ be the Green's function on G with pole at $z_0 \in G$. Suppose that G is regular and satisfies the condition

$$\int_0^\infty \beta_G(\alpha)d\alpha < \infty$$

where $\beta_G(\alpha) = \beta_G(\alpha, z_0)$ is the first Betti number of $G_\alpha = G(\alpha, z_0) = \{z \in G; g_G(z, z_0) > \alpha\}$. Then, R is also regular.

THEOREM. *Under the assumption stated above, the following four conditions are equivalent by pairs:*

(i) $m(R) > 0$;

(ii) *$H^\infty(R)$ separates the points of R;*

(iii) *for any $z_0 \in G - \{\zeta_k\}$, where $\{\zeta_k\}$ is the set of projections of branch points of R, there exists an f in $H^\infty(R)$ such that f takes distinct values at any two points of R over z_0;*

(iv) $\Sigma n_k g_G(\zeta_k, z_0) < \infty$

for $z_0 \in G - \{\zeta_k\}$, where n_k is the order of branching over ζ_k.

Since (i) $\rightarrow$ (ii) has been proved, we only have to show (ii) $\rightarrow$ (iii), (iii) $\rightarrow$ (iv), and (iv) $\rightarrow$ (i).

3. *Proof of* (ii) $\rightarrow$ (iii). Let π be the projection of R onto G and set $\pi^{-1}(z_0) = \{p_1, \cdots, p_n\}$ (distinct points) for $z_0 \in G - \{\zeta_k\}$. Since $H^\infty(R)$ separates the points of R, there exists an f_{ij} in $H^\infty(R)$ such that $f_{ij}(p_i) \neq f_{ij}(p_j)$, for any pair (i, j) with $i \neq j$ and $1 \leqq i, j \leqq n$. We set

$$F_i = \prod_{\substack{1 \leqq j \leqq n \\ j \neq i}} (f_{ij} - f_{ij}(p_j)) \qquad (1 \leqq i \leqq n)$$

and

$$f = \sum_{i=1}^{n} c_i F_i$$

for suitable constants c_i specified below. Observe that $f \in H^\infty(R)$. We can choose constants c_i so as to satisfy $f(p_i) \neq f(p_j)$ for any $i \neq j$.

Proof of (iii) $\rightarrow$ (iv). Let z_0 be an arbitrary point in $G - \{\zeta_k\}$ and f a function in $H^\infty(R)$ such that f takes distinct values at any two points over z_0. Then, by the well known argument of algebroidal functions, it is seen that f satisfies the irreducible equation

$$f^n + g_1(z) f^{n-1} + \cdots + g_n(z) = 0$$

where $g_1(z), \cdots, g_n(z)$ are in $H^\infty(G)$. Let $D(z)$ be the discriminant of this equation. Observe that $D(z)$ is in $H^\infty(G)$, vanishes at every point in $\{\zeta_k\}$, and does not vanish at z_0. Hence, by the Lindelöf principle (cf. [1]), we conclude

$$\Sigma n_k g_G(\zeta_k, z_0) < \infty .$$

Proof of (iv) $\rightarrow$ (i). Let z_0 be a point in $G - \{\zeta_k\}$ and p_0 a point in R with $\pi(p_0) = z_0$. We set

$$R_\alpha = R(\alpha, p_0) = \{p \in R;\, g_R(p, p_0) > \alpha\}$$

and

$$V_\alpha = \{p \in R;\, h(p) > \alpha\}$$

where $h(p) = g_G(\pi(p), z_0)$. Denote by $\beta_R(\alpha)$ and $\gamma(\alpha)$ the first Betti numbers of R_α and V_α, respectively. We fix $\alpha_0 (> 0)$ such that V_{α_0} is connected. Then, V_α is also connected for every $\alpha \leqq \alpha_0$. By the maximum principle, $h(p) \geqq g_R(p, p_0)$, and therefore $V_\alpha \supset R_\alpha$. Also, by the maximum principle, $V_\alpha - R_\alpha$ has no relatively compact components in V_α. Therefore

$$(1) \qquad \gamma(\alpha) \geqq \beta_R(\alpha) .$$

Consider each α with $0 < \alpha \leqq \alpha_0$ such that there exist no branch points of R on ∂V_α and no critical points of $g_G(z, z_0)$ on ∂G_α, where ∂V_α and ∂G_α are the boundaries of V_α and G_α, respectively, and $G_\alpha = \{z \in G;\, g_G(z, z_0) > \alpha\}$. Let $\hat{V}_\alpha$ and $\hat{G}_\alpha$ be the doubles of V_α and G_α, respectively. Then, since $\hat{V}_\alpha$ can be considered as an unbounded n-sheeted covering surface over the compact surface $\hat{G}_\alpha$, by the Riemann-Hurwitz and Euler-Poincaré formulas,

$$2(1 - \gamma(\alpha)) = 2(1 - \beta_\alpha(\alpha))n - 2b(\alpha)$$

where $\beta_G(\alpha)$ is the first Betti number of G_α and $b(\alpha)$ is the total sum of the branching order of branch points over G_α. Thus

$$\gamma(\alpha) = \beta_G(\alpha)n + b(\alpha) - n + 1 . \tag{2}$$

Observe that the set of α such that there exist branch points of R on ∂V_α or critical points of $g_G(z, z_0)$ on ∂G_α is isolated. Hence, from (1) and (2), it follows that

$$\begin{aligned} \int_0^{\alpha_0} \beta_R(\alpha)d\alpha &\leqq \int_0^{\alpha_0} \gamma(\alpha)d\alpha \\ &= n\int_0^{\alpha_0} \beta_G(\alpha)d\alpha + \int_0^{\alpha_0} b(\alpha)d\alpha + O(1) . \end{aligned} \tag{3}$$

Observe that

$$\int_0^{\alpha_0} b(\alpha)d\alpha = \sum_{\zeta_k \in G - G_\alpha} n_k g_G(\zeta_k, z_0) .$$

Therefore, by the assumption,

$$\int_0^{\alpha_0} b(\alpha)d\alpha < \infty \tag{4}$$

and also by the assumption

$$\int_0^{\alpha_0} \beta_G(\alpha)d\alpha < \infty . \tag{5}$$

From (3), (4), and (5), it follows that

$$\int_0^{\infty} \beta_R(\alpha)d\alpha = \int_0^{\alpha_0} \beta_R(\alpha)d\alpha + O(1) < \infty ,$$

i.e.,

$$m(R) = \exp\left\{-\int_0^{\infty} \beta_R(\alpha)d\alpha\right\} > 0 .$$

ACKNOWLEDGMENT. The author wishes to express his thanks for helpful suggestions to Professor M. Nakai.

REFERENCES

1. M. Heins, *Lindelöf principle*, Ann. of Math., (2) **61** (1955), 440-473.
2. H. L. Selberg, *Ein Satz über beschränkte endlichvieldeutige analytische funktionen*, Comm. Math. Helv., **9** (1937), 104-108.
3. C. M. Stanton, *Bounded analytic functions on a class of Riemann surfaces*, Pacific J. Math., **59** (1975), 557-565.
4. H. Widom, *$\mathscr{H}_p$ sections of vector bundles over Riemann surfaces*, Ann. of Math., (2) **94** (1971), 304-324.

5. Y. Yamamura, *On the existence of bounded analytic functions*, Sc. Rep. T. K. D. Sect. A, **10** (1969), 88-102.

Received November 4, 1977.

DAIDO INSTITUTE OF TECHNOLOGY
DAIDO, MINAMI, NAGOYA 457
JAPAN