PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 1, 1978

MULTIPLIERS FOR $|C, 1|$ SUMMABILITY OF FOURIER SERIES

G. S. PANDEY

In the present paper we improve the conditions of all previously known theorems on the absolute $(C, 1)$ summability factors of Fourier series.

1. Let the formal expansion of a function $f(x)$, periodic with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$, in a Fourier-trigonometric series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) . \tag{1.1}$$

We write

$$\phi(u) = f(x+u) + f(x-u) - 2f(x)$$

and throughout this paper A will denote a positive constant, not necessarily the same at each occurrence.

Whittaker [5], in 1930, proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^{\alpha} , \qquad \alpha > 0 ,$$

is summable $|A|$ almost everywhere.

Later, Prasad [4] demonstrated that the series

$$\sum_{n=n_0}^{\infty} A_n(x)/\mu_n ,$$

where

$$\mu_n = \left(\prod_{\nu=1}^{k-1} \log^{\nu} n\right)(\log^k n)^{1+\varepsilon} , \qquad \log^k n_0 > 0 , \quad \varepsilon > 0 ,$$

and

$$\log^k n = \log(\log^{k-1} n), \cdots, \log^2 n = \log\log n ;$$

is summable $|A|$ almost everywhere.

Chow [2], on the other hand, has shown that the series $\sum \lambda_n A_n(x)$ is summable $|C, 1|$ almost everywhere, provided $\{\lambda_n\}$ is a convex sequence satisfying the condition $\sum n^{-1}\cdot\lambda_n < \infty$.

Cheng [1], in 1948, established the following:

THEOREM A. *If*

$$\Phi(t) \equiv \int_0^t |\phi(u)|\,du = O(t)$$

as $t \to 0$, *then the series*

$$\sum_{n=2}^{\infty} A_n(x)/(\log n)^{1+\delta}\,, \qquad \delta > 0\,,$$

is summable $|C, \alpha|$, $\alpha > 1$.

In a recent paper, Hsiang [3] has proved the following theorems:

THEOREM B. *If*

$$\Phi(t) = O(t) \qquad (t \longrightarrow +0)\,, \tag{1.2}$$

then the series $\sum_{n=1}^{\infty} A_n(x)/n^{\alpha}$ *is summable* $|C, 1|$ *for every* $\alpha > 0$.

THEOREM C. *If*

$$\Phi(t) = O\left\{t/\prod_{\nu=1}^{k} \log^{\nu}(1/t)\right\} \tag{1.3}$$

as $t \to +0$, *then the series*

$$\sum_{n=0}^{\infty} A_n(x)\Big/\left(\prod_{\nu=1}^{k-1} \log^{\nu} n\right)(\log^k n)^{1+\varepsilon} \tag{1.4}$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

In the present paper we prove the following theorem, which includes the theorem of Cheng and both the theorems of Hsiang:

THEOREM. *If*

$$\varphi(t) \equiv \int_t^{\delta} \frac{|\phi(u)|}{u} du = O\{(\log^k(1/t))^{\eta}\} \quad as \quad t \longrightarrow +0\,, \tag{1.5}$$

$0 < \delta \leqq \pi$, *then the series* (1.4) *is summable* $|C, 1|$ *for* $0 < \eta < \varepsilon$.

The conditions of our theorem are less stringent than those of Cheng and Hsiang.

2. The proof of the theorem is based on the following lemmas:

LEMMA 1. *Let* $S_n(x)$ *be the* nth *partial sum of the series* (1.1), *then under the condition* (1.5), *we have*

$$\sum_{\nu=0}^{n} |S_{\nu}(x) - f(x)| = O\{n(\log^k n)^{\eta}\}\,. \tag{2.1}$$

Proof. Let $\varepsilon_\nu = \text{sign}\,[S_\nu(x) - f(x)]$, so that $\varepsilon_\nu = \pm 1$ and it depends only upon x and ν, and is independent of t. Also, we write

$$K_n(t) = \sum_{\nu=0}^{n} \varepsilon_\nu \sin \nu t \,.$$

Thus, we have

$$\begin{aligned}\sum_{\nu=0}^{n} |S_\nu(x) - f(x)| &= \frac{2}{\pi}\int_0^\pi \frac{\phi(t)}{t} K_n(t)dt + o(n)\\ &= \frac{2}{\pi}\left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right] + o(n)\\ &= I_1 + I_2 + I_3 + o(n)\,,\end{aligned}$$

say. Now,

$$\begin{aligned}I_1 &\leqq \int_0^{1/n} |\phi(t)| \cdot O(n^2)dt\\ &= O(n^2)\int_0^{1/n} - t\varphi'(t)dt\,, \qquad \varphi'(t) = \frac{d}{dt}\varphi(t)\,.\\ &= O(n^2)[-t\varphi(t)]_0^{1/n} + O(n^2)\int_0^{1/n} \varphi(t)dt\\ &= O\{n(\log^k n)^\eta\}\,.\end{aligned} \tag{2.3}$$

Also, for $nt \geqq 1$, we have

$$\begin{aligned}I_2 &\leqq \int_{1/n}^{\delta} \frac{|\phi(t)|}{t} \cdot ndt\\ &= O\{n(\log^k n)^\eta\}\,.\end{aligned} \tag{2.4}$$

Since, by Riemann-Lebesgue theorem,

$$\int_\delta^\pi \frac{\phi(t)}{t} \sin ntdt = o(1)\,,$$

we have

$$I_3 = O(n)\,. \tag{2.5}$$

Combining (2.1), (2.2), $\cdots$, (2.5), the lemma follows.

LEMMA 2. *Let*

$$t_n(x) = \frac{1}{(n+1)}\sum_{\nu=1}^{n} \nu A_\nu(x)\,.$$

Then

$$T_n(x) \equiv \sum_{\nu=1}^{n} |t_\nu(x)| = O\{n(\log^k n)^\eta\}$$

and

$$\sum_{n=n_0}^{\infty} (\mu_n)^{-1} \cdot n^{-1} |t_\nu(x)| < \infty .$$

Proof. Let

$$\sigma_n(x) = \frac{1}{(n+1)} \sum_{\nu=0}^{n} S_\nu(x) .$$

Thus, we have

$$\sigma_n(x) - f(x) = \frac{1}{(n+1)} \sum_{\nu=0}^{n} \{S_\nu(x) - f(x)\}$$

$$\Longrightarrow |\sigma_n(x) - f(x)| \leqq \frac{1}{(n+1)} \sum_{\nu=0}^{n} |S_\nu(x) - f(x)| \tag{2.6}$$
$$= O\{(\log^k n)^\eta\}$$

by Lemma 1.

Therefore, we find that

$$\begin{aligned} T_n(x) &= \sum_{\nu=1}^{n} |t_\nu(x)| \\ &= \sum_{\nu=1}^{n} |S_\nu(x) - \sigma_\nu(x)| \\ &\leqq \sum_{\nu=1}^{n} |S_\nu(x) - f(x)| + \sum_{\nu=1}^{n} |\sigma_\nu(x) - f(x)| \\ &= O[n(\log^k n)^\eta] \end{aligned} \tag{2.7}$$

by (2.6) and Lemma 1.

Finally, by Abel's transformation, we have

$$\begin{aligned} \sum_{n=m}^{M} (\mu_n)^{-1} \cdot n^{-1} |t_n(x)| &= \sum_{n=m}^{M-1} T_n(x) \Delta\{(\mu_n)^{-1} \cdot n^{-1}\} \\ &\quad - (\mu_{m-1})^{-1}(m-1)^{-1} T_{m-1}(n) + \mu_M^{-1} \cdot M^{-1} T_M(x) \\ &= \sum_{n=m}^{M-1} \Delta\{(\mu_n)^{-1}\} \cdot n^{-1} T_n(x) \\ &\quad + \sum_{m=m}^{M-1} (\mu_{n+1})^{-1} \cdot n^{-1}(n+1)^{-1} T_n(x) + O(1) \\ &= \sum_{n=m}^{M-1} \Delta\{(\mu_n)^{-1}\} \cdot (\log^k n)^\eta \\ &\quad + \sum_{n=m}^{M-1} (\mu_{n+1})^{-1}(n+1)^{-1}(\log^k n)^\eta + O(1) \\ &\leqq \sum_{n=m}^{M-1} \frac{A \cdot (\log^k n)^\eta}{n\left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}} + O(1) \\ &= O(1) , \end{aligned} \tag{2.8}$$

for $m \to \infty$ and $M \to \infty$.

In view of (2.7) and (2.8) the lemma is proved.

3. Proof of the theorem. Let $\tau_n(x)$ denotes the nth Cesàro mean of the sequence $\{n(\mu_n^{-1}) \cdot A_n(x)\}$.

By Abel's transformation, we have

$$\begin{aligned} \tau_n(x) &= \frac{1}{(n+1)} \sum_{\nu=n_0}^{n} \nu(\mu_\nu)^{-1} \cdot A_\nu(x) \\ &= \frac{1}{(n+1)} \sum_{\nu=n_0}^{n-1} \Delta(\mu_\nu)^{-1} \cdot (\nu+1) t_\nu(x) + (\mu_n)^{-1} t_n(x) \qquad (3.1) \\ &= J_1^{(n)}(x) + J_2^{(n)}(x) , \end{aligned}$$

say. Now, by Lemma 2, we find that

$$\begin{aligned} \sum_{n=m_0}^{m} J_1^{(n)}(x)/n &\leqq \sum_{n=m_0}^{m} n^{-1}(n+1)^{-1} \sum_{\nu=n_0}^{n-1} \Delta(\mu_\nu)^{-1}(\nu+1)|t_\nu(x)| , \quad \log^k m_0 > 0 \\ &\leqq A \sum_{\nu=m_0}^{m} \Delta(\mu_\nu)^{-1}(\nu+1)|t_\nu(x)| \sum_{n=\nu+1}^{m} n^{-1}(n+1)^{-1} \\ &\leqq A \sum_{\nu=m_0}^{m} \Delta(\mu_\nu)^{-1}|t_\nu(x)| \qquad (3.2) \\ &= A \sum_{\nu=m_0}^{m-1} \Delta^2[(\mu_\nu)^{-1}] \cdot T_\nu(x) + \Delta(\mu_m^{-1}) T_m(x) + O(1) \\ &= O(1) . \end{aligned}$$

Also, we have

$$\begin{aligned} \sum_{n=m_0}^{m} J_2^{(n)}(x)/n &\leqq \sum_{n=m_0}^{m} (\mu_n)^{-1} \cdot n^{-1} t_n(x)| \qquad (3.3) \\ &= O(1) . \end{aligned}$$

From (3.1), (3.2), and (3.3), we have

$$\sum_{n=m_0}^{m} \frac{|\tau_n(x)|}{n} = O(1) .$$

This completes the proof of the theorem.

REFERENCES

1. M. T. Cheng, *Summability factors of Fourier series*, Duke Math. J., **15** (1948), 17-27.

2. H. C. Chow, *On the summability factors of Fourier series*, J. London Math. Soc., **16** (1941), 215-220.

3. F. C. Hsiang, *On C, I summability factors of Fourier series at a given point*, Pacific J. Math., **33** (1970), 139-147.

4. B. N. Prasad, *On the summability of Fourier series and the bounded variation of power series*, Proc. London Math. Soc., (2), **35** (1933), 407-424.

5. J. M. Whittaker, *The absolute summability of Fourier series*, Proc. Edinburgh Math. Soc., (2), **2** (1930), 1–5.

Received January 16, 1975 and in revised form April 3, 1978.

MOSUL UNIVERSITY
MOSUL, IRAQ