PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 2, 1978

MULTIFUNCTIONS AND GRAPHS

JAMES E. JOSEPH

In this paper we introduce the notions of multifunctions with strongly-closed graphs and θ-closed graphs. We then extend these notions as well as the notion of multifunctions with closed graphs and use these extensions to generalize and extend various known results on multifunctions. These generalizations and extensions include a number of sufficient conditions for multifunctions to be upper-semi-continuous, several generalizations and extensions of the well-known Uniform Boundedness Principle from analysis, and several "fixed set" theorems.

A *multifunction* $\Phi: X \to Y$ is a correspondence from X to Y with $\Phi(x)$ a nonempty subset of Y for each $x \in X$. We will denote the graph of Φ, i.e., $\{(x, y): x \in X$ and $y \in \Phi(x)\}$, by $G(\Phi)$. As usual, if X and Y are topological spaces (hereafter referred to as "spaces") and $\Phi: X \to Y$ is a multifunction we will say that Φ has a *closed graph* if $G(\Phi)$ is a closed subset of the product space $X \times Y$. If X and Y are spaces a multifunction $\Phi: X \to Y$ is said to be *upper-semi-continuous* (*u.s.c.*) *at* $x \in X$ if for each W open about $\Phi(x)$ in Y there is a V open about x in X with $\Phi(V) \subset W$; Φ is said to be *upper-semi-continuous* (*u.s.c.*) if Φ is u.s.c. at each $x \in X$. It is not difficult to establish that Φ is u.s.c. if and only if $\Phi^{-1}(K)$ is closed in X whenever K is closed in Y. Smithson [20] has given a survey of some of the principal results on multifunctions. See [20] and [8] for definitions not given here.

2. Some preliminary definitions and theorems. We will denote the closure of a subset K of a space by $\mathrm{cl}\,(K)$, the adherence of a filterbase Ω on the space by $\mathrm{ad}\,\Omega$, and the family of open subsets which contain K by $\Sigma(K)$. A point x is in the *θ-closure of a subset* K of a space ($x \in \mathrm{cl}_\theta(K)$) if each $V \in \Sigma(x)$ satisfies $K \cap \mathrm{cl}\,(V) \neq \emptyset$; K is *θ-closed* if $\mathrm{cl}_\theta(K) \subset K$; x is in the *θ-adherence of a filterbase* Ω on the space ($x \in \mathrm{ad}_\theta\,\Omega$) if $x \in \mathrm{cl}_\theta(F)$ for each $F \in \Omega$; a filterbase Ω on a space *θ-converges* to a point x in the space ($\Omega \to_\theta x$) if for each $V \in \Sigma(x)$ there is an $F \in \Omega$ with $F \subset \mathrm{cl}\,(V)$ [25]. In [23], a space is called an *$H(i)$ space* if each open filterbase on the space has a nonempty adherence. Hausdorff $H(i)$ spaces are called *H-closed*. A subset A of a space X is defined to be $H(i)$ if and only if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_\theta\,\Omega \neq \emptyset$ [8].

We will say that a multifunction $\Phi: X \to Y$ has *closed* (*θ-closed*) [*compact*] *point images* if $\Phi(x)$ is closed (θ-closed) [compact] in Y for each $x \in X$. Some of the statements in our first theorem are analogues for multifunctions to some of the known characterizations of functions with closed graphs. Statements (e) and (g) are new for multifunctions and functions.

THEOREM 2.1. *The following statements are equivalent for spaces X, Y, and multifunction $\Phi: X \to Y$:*

(a) *The multifunction Φ has a closed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\Phi)$, there are sets $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y satisfying $\Phi(V) \cap W = \varnothing$ $(V \cap \Phi^{-1}(M) = \varnothing)$.*

(c) *If Ω is a filterbase on X with $\Omega \to x$ in X, then* $\mathrm{ad}\,\Phi(\Omega) \subset \Phi(x)$.

(d) *If Ω is a filterbase on X with $\Omega \to x$ in X, then $y \in \Phi(x)$ whenever $\Omega^* \to y$ and Ω^* is finer than $\Phi(\Omega)$.*

(e) *The multifunction Φ has closed point images and* $\mathrm{ad}\,\Phi(\Omega) \subset \Phi(x)$ *for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$.*

(f) *If $\{x_n\}$ and $\{y_n\}$ are nets in X and Y, respectively, with $x_n \to x$ in X, $y_n \to y$ in Y, and $y_n \in \Phi(x_n)$ for each n, then $y \in \Phi(x)$.*

(g) *The multifunction Φ has closed point images and for each $x \in X$ and net $\{x_n\}$ in $X - \{x\}$ with $x_n \to x$ and net $\{y_n\}$ in Y with $y_n \in \Phi(x_n)$ for each n and $y_n \to y$, we have $y \in \Phi(x)$.*

Proof. The proof that statements (a) and (b) are equivalent is similar to the proof of the lemma in [15]. Statement (b) is easily seen to be equivalent to $\mathrm{cl}\,(G(\Phi)) \subset G(\Phi)$. We establish that statements (a), (c), (d), (e), and (g) are equivalent. Assume (a), let Ω be a filterbase on X with $\Omega \to x$ in X and let $y \in \mathrm{ad}\,\Phi(\Omega)$. If $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y there is an $F \in \Omega$ with $F \subset V$ and $\Phi(F) \cap W \neq \varnothing$. This gives $(V \times W) \cap G(\Phi) \neq \varnothing$. Thus $y \in \Phi(x)$ and (c) holds. It is immediate that (d) follows from (c). Now assume (d). If Ω is a filterbase on $X - \{x\}$ with $\Omega \to x$ and $y \in \mathrm{ad}\,\Phi(\Omega)$, there is a filterbase Ω^* on Y finer than $\Phi(\Omega)$ with $\Omega^* \to y$. This gives $y \in \Phi(x)$ and one part of (e) is proved. If $x \in X$ and $y \in \mathrm{cl}\,(\Phi(x))$, then $\Omega = \{\{x\}\}$ is a filterbase on X with $\Omega \to x$ and $\Omega^* = \{\Phi(x) \cap V: V \in \Sigma(y)\}$ is a filterbase on Y finer than $\Phi(\Omega)$ with $\Omega^* \to y$. Thus $y \in \Phi(x)$ and (e) holds. We can see readily that with the hypothesis of (g) the filterbase Ω induced by the net $\{x_n\}$, and the point x satisfy the hypothesis of (e), and $y \in \mathrm{ad}\,\Phi(\Omega)$. Thus $y \in \Phi(x)$. To see that (a) follows from (g), assume (g) and let $(x, y) \in \mathrm{cl}\,(G(\Phi))$. There is a net $(x_n, y_n) \in G(\Phi)$ with $(x_n, y_n) \to (x, y)$. Then $x_n \to x$ and $y_n \to y$. If $\{x_n\}$ is frequently x, then $y \in \mathrm{cl}\,(\Phi(x))$ so $y \in \Phi(x)$. If not, then, without loss, we may assume that $\{x_n\}$ is a net in $X - \{x\}$ and by (g) we have $y \in \Phi(x)$.

The proof is complete.

Proposition 3.2 of [21] shows that (b) of Theorem 2.1 is satisfied by subcontinuous multifunctions with closed graphs. Theorem 2.1 shows that subcontinuity is superfluous.

In [9], a function $\Phi: X \to Y$ is said to have a strongly-closed graph if for each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y with $(V \times \mathrm{cl}(W)) \cap G(\Phi) = \varnothing$. This notion is used in [9], [10], [11], and [12] to obtain characterizations of H-closed and minimal Hausdorff spaces. If K is a subset of the product space $X \times Y$, a point $(x, y) \in X \times Y$ is in the *(2)θ-closure of* K ($(x, y) \in (2)\,\mathrm{cl}_\theta(K)$) if $(V \times \mathrm{cl}(W)) \cap K \neq \varnothing$ whenever $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y. K is *(2)θ-closed* if $(2)\,\mathrm{cl}_\theta(K) = K$. Evidently, a function $\Phi: X \to Y$ has a strongly-closed graph if and only if $G(\Phi)$ is a *(2)θ-closed* subset of $X \times Y$. Using this equivalence as a model we say here that a multifunction $\Phi: X \to Y$ has a *strongly-closed graph* if $G(\Phi)$ is a (2)θ-closed subset of $X \times Y$. A net $\{x_n\}$ in a space X *θ-accumulates* (*θ-converges*) *to* x in X (x is a θ-accumulation point of $\{x_n\}$) ($(x_n \to_\theta x)$) if x_n is frequently (eventually) in $\mathrm{cl}(V)$ for every $V \in \Sigma(x)$[25]. Our final two theorems in this section are analogous to Theorem 2.1 and are stated without proof.

THEOREM 2.2. *The following statements are equivalent for spaces X, Y, and multifunction $\Phi: X \to Y$.*

(a) *The multifunction Φ has a strongly-closed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y with $(V \times \mathrm{cl}(W)) \cap G(\Phi) = \varnothing$ $(V \cap \Phi^{-1}(\mathrm{cl}(W)) = \varnothing)$ $[\Phi(V) \cap \mathrm{cl}(W) = \varnothing]$.*

(c) *If Ω is a filterbase on X with $\Omega \to x$ in X then $\mathrm{ad}_\theta\, \Phi(\Omega) \subset \Phi(x)$.*

(d) *If Ω is a filterbase on X with $\Omega \to x$ in X then $y \in \Phi(x)$ whenever $\Omega^* \to_\theta y$ and Ω^* is finer than $\Phi(\Omega)$.*

(e) *If $\{x_n\}$ and $\{y_n\}$ are nets on X and Y, respectively, with $x_n \to x$ in X, $y_n \to_\theta y$ in Y and $y_n \in \Phi(x_n)$ for each n, then $y \in \Phi(x)$.*

(f) *The multifunction Φ has θ-closed point images and $\mathrm{ad}_\theta\, \Phi(\Omega) \subset \Phi(x)$ for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$.*

(g) *The multifunction Φ has θ-closed point images and for each $x \in X$ and net $\{x_n\}$ in $X - \{x\}$ with $x_n \to x$ and net $\{y_n\}$ in Y with $y_n \in \Phi(x_n)$ for all n and $y_n \to_\theta y$, we have $y \in \Phi(x)$.*

THEOREM 2.3. *The following statements are equivalent for spaces X, Y, and multifunction $\Phi: X \to Y$:*

(a) *The multifunction Φ has a θ-closed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y with $\Phi(\mathrm{cl}(V)) \cap \mathrm{cl}(W) = \varnothing$ $(\mathrm{cl}(V) \cap \Phi^{-1}(\mathrm{cl}(W)) = \varnothing)$.*

(c) *If Ω is a filterbase on X with $Q \to_\theta x$ in X then* $\mathrm{ad}_\theta \Omega \subset \Phi(x)$.

(d) *If Ω is a filterbase on X with $\Omega \to_\theta x$ in X then $y \in \Phi(x)$ whenever $\Omega^* \to_\theta y$ and Ω^* is finer than $\Phi(\Omega)$.*

(e) *If $\{x_n\}$ and $\{y_n\}$ are nets on X and Y, respectively, with $x_n \to_\theta x$ in X, $y_n \to_\theta y$ in Y and $y_n \in \Phi(x_n)$ for each n, then $y \in \Phi(x)$.*

(f) *The multifunction Φ has θ-closed point images and* $\mathrm{ad}_\theta \Omega \subset \Phi(x)$ *for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to_\theta x$.*

(g) *The multifunction Φ has θ-closed point images and for each $x \in X$ and net $\{x_n\}$ in $X - \{x\}$ with $x_n \to_\theta x$ and net $\{y_n\}$ in y with $y_n \in \Phi(x_n)$ for all n and $y_n \to_\theta y$ in Y, we have $y \in \Phi(x)$.*

3. **Multifunctions with subclosed graphs.** We will say that a multifunction $\Phi: X \to Y$ has a *subclosed graph* if for each $x \in X$ and net $\{x_n\}$ in $X - \{x\}$ with $x_n \to x$ and net $\{y_n\}$ in Y with $y_n \in \Phi(x_n)$ for each n and $y_n \to y$ in Y, we have $y \in \Phi(x)$. Functions with subclosed graphs have been studied in [4] and [13]. Evidently, a multifunction with a closed graph has a subclosed graph and a multifunction with a subclosed graph and closed point images has a closed graph (see Theorem 2.1(g)). In this section, we will use the notion of subclosed graph to extend and generalize a number of known results on multifunctions and/or functions including a well-known uniform boundedness principle. Our first theorem in this section states without proof several characterizations of multifunctions with subclosed graphs.

THEOREM 3.1. *The following statements are equivalent for spaces X, Y, and multifunction $\Phi: X \to Y$.*

(a) *The multifunction Φ has a subclosed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \in \Sigma(x)$ in X and $W \in \Sigma(y)$ in Y with $((V - \{x\}) \times W) \cap G(\Phi) = \varnothing((V \times (W - \Phi(x))) \cap G(\Phi) = \varnothing)[\Phi(V - \{x\}) \cap W = \varnothing]\langle V \cap \Phi^{-1}(W - \Phi(x)) = \varnothing\rangle$.*

(c) *If Ω is a filterbase on $X - \{x\}$ with $\Omega \to x$ in X then* $\mathrm{ad}\, \Phi(\Omega) \subset \Phi(x)$.

A multifunction $\Phi: X \to Y$ is *subcontinuous* if whenever $\{x_n\}$ is a convergent net in X and $\{y_n\}$ is a net in Y with $y_n \in \Phi(x_n)$ for each n, then $\{y_n\}$ has a convergent subnet [21]. Theorem 3.2 below offers an extension of Theorem 3.1 of [21] to multifunctions with subclosed graphs.

THEOREM 3.2. *Let $\Phi: X \to Y$ be a subcontinuous multifunction with a subclosed graph. Then Φ is u.s.c.*

Proof. Let $K \subset Y$ be closed and let v be a limit point of $\Phi^{-1}(K)$. There is a net $\{x_n\}$ in $\Phi^{-1}(K) - \{v\}$ with $x_n \to v$. Let $\{y_n\}$ be a net in K with $y_n \in \Phi(x_n)$ for each n. Since Φ is subcontinuous and K is closed some subnet $\{y_{n_m}\}$ of $\{y_n\}$ converges to some $y \in K$. Since $\{x_{n_m}\}$ is a net in $X - \{x\}$ and $x_{n_m} \to v$ and Φ has a subclosed graph we have $y \in \Phi(v)$ and $v \in \Phi^{-1}(K)$. The proof is complete.

COROLLARY 3.3 [21]. *Let $\Phi: X \to Y$ be a subcontinuous multifunction with a closed graph. Then Φ is u.s.c.*

It is proved in [19] that a multifunction with a closed graph on a locally compact space which maps compact subsets onto compact subsets must be u.s.c. Theorem 3.4 below and Theorem 3.2 above lead us to an improvement of this result in Corollary 3.5 below.

THEOREM 3.4. *A multifunction on a locally compact space which maps compact subsets onto compact subsets is subcontinuous.*

Proof. Let X be locally compact, Y a space, and $\Phi: X \to Y$ be a multifunction which maps compact subsets onto compact subsets. Let $x \in X$ and $\{x_n\}$, $\{y_n\}$ be nets in X, Y, respectively, with $x_n \to x$ and $y_n \in \Phi(x_n)$ for each n. Let K be a compact neighborhood of x. Then $\{x_n\}$ is eventually in K, so $\{y_n\}$ is eventually in $\Phi(K)$ which is compact. Thus $\{y_n\}$ has a convergent subnet and the proof is complete.

COROLLARY 3.5. *A multifunction with a subclosed graph on a locally compact space which maps compact subsets onto compact subsets is u.s.c.*

It is a well-known and useful fact that if Φ, $\alpha: X \to Y$ are continuous functions and Y is Hausdorff then $\{x \in X: \Phi(x) = \alpha(x)\}$ is a closed subset of X. Our next two theorems represent both improvements of this result and extensions of this result to multifunctions. If Φ, $\alpha: X \to Y$ are multifunctions we denote $\{x \in X: \Phi(x) \cap \alpha(x) \neq \emptyset\}$ by $E(X, Y, \Phi, \alpha)$.

THEOREM 3.6. *Let $\Phi: X \to Y$ be an u.s.c. multifunction with compact point images and let $\alpha: X \to Y$ be a multifunction with a subclosed graph. Then $E(X, Y, \Phi, \alpha)$ is closed in X.*

Proof. Let v be a limit point of $E(X, Y, \Phi, \alpha)$. There is a net $\{x_n\}$ in $E(X, Y, \Phi, \alpha) - \{v\}$ with $x_n \to v$. For each $W \in \Sigma(\Phi(v))$ there is a $V \in \Sigma(v)$ in X with $\Phi(V) \subset W$, so $\Phi(x_n) \subset W$ eventually. Choose

$y_n \in \Phi(x_n) \cap \alpha(x_n)$, then $\{y_n\}$ is eventually in each $W \in \Sigma(\Phi(v))$. Since $\Phi(v)$ is compact, some subnet $\{y_{n_m}\}$ of $\{y_n\}$ converges to some $y \in \Phi(v)$. Since α has a subclosed graph we have $y \in \alpha(v)$. Thus $y \in \Phi(v) \cap \alpha(v)$ and $v \in E(X, Y, \Phi, \alpha)$. The proof is complete.

We state the following corollaries which may be readily established.

COROLLARY 3.7. *Let Y be any space. If $\Phi: X \to Y$ is a continuous function and $\alpha: X \to Y$ is a function with a subclosed graph then $\{x \in X: \Phi(x) = \alpha(x)\}$ is a closed subset of X.*

COROLLARY 3.8. *Let $\Phi: X \to Y$ be an u.s.c. multifunction with compact point images and let $\alpha: X \to Y$ be a multifunction with a subclosed graph. If $E(X, Y, \Phi, \alpha)$ is dense in X, then $E(X, Y, \Phi, \alpha) = X$.*

COROLLARY 3.9. *Let Y be any space. If $\Phi: X \to Y$ is a continuous function and $\alpha: X \to Y$ is a function with a subclosed graph then $\{x \in X: \Phi(x) = \alpha(x)\} = X$ if it is dense in X.*

THEOREM 3.10. *Let $\Phi: X \to Y$ be a subcontinuous multifunction with a subclosed graph and let $\alpha: X \to Y$ be a multifunction with a subclosed graph. Then $E(X, Y, \Phi, \alpha)$ is closed in X.*

Proof. Let v be a limit point of $E(X, Y, \Phi, \alpha)$. There is a net $\{x_n\}$ in $E(X, Y, \Phi, \alpha) - \{v\}$ with $x_n \to v$. For each n choose $y_n \in \Phi(x_n) \cap \alpha(x_n)$. There is a subnet $\{y_{n_m}\}$ in Y and $y \in Y$ with $y_{n_m} \to y$. We must have $y \in \Phi(v) \cap \alpha(v)$ and the proof is complete.

COROLLARY 3.11. *If X is a space and the multifunction $\Phi: X \to X$ has a subclosed graph then $\{x \in X: x \in \Phi(x)\}$ is closed in X.*

We may also establish the following generalization of the result that a continuous function into a Hausdorff space has a strongly-closed graph [10].

THEOREM 3.12. *Let $\Phi: X \to Y$ be an u.s.c. multifunction with compact point images. If Y is Hausdorff then Φ has a strongly-closed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(\Phi)$. Then $y \notin \Phi(x)$ and, by a standard argument, since $\Phi(x)$ is compact and Y is Hausdorff there are sets $W \in \Sigma(\Phi(x))$ and $V \in \Sigma(y)$ in Y with $\mathrm{cl}(V) \cap W = \varnothing$. Since Ω is u.s.c. there is an $A \in \Sigma(x)$ in X with $\Phi(A) \subset W$. We now have

$(x, y) \in A \times V$ and $(A \times \mathrm{cl}(V)) \cap G(\Phi) = \emptyset$. The proof is complete.

Our next result is a generalization of a theorem which was proved in [1] for the case when Φ is u.s.c., has compact point images and X is compact Hausdorff rather than with our hypothesis that Φ maps closed sets onto closed sets.

THEOREM 3.13. *If X is compact and $\Phi: X \to Y$ is a multifunction which maps closed sets onto closed sets there is a $K_0 \subset X$ with $K_0 \neq \emptyset$, K_0 closed and $\Phi(K_0) = K_0$.*

Proof. Let $\Delta = \{K \subset X: K \neq \emptyset, K \text{ closed and } \Phi(K) \subset K\}$. Then $X \in \Delta$, so $\Delta \neq \emptyset$. Order Δ by inclusion and let Δ^* be a chain in Δ. Let $Q = \bigcap_{\Delta^*} K$. Then Q is clearly closed and $Q \neq \emptyset$ since Δ^* has the finite intersection property and X is compact. We have $\Phi(Q) = \Phi(\bigcap_{\Delta^*} K) \subset \bigcap_{\Delta^*} K = Q$. Thus $Q \in \Delta$ and $Q \subset K$ for each $K \in \Delta^*$. By Zorn's lemma Δ has a minimal element K_0. $K_0 \neq \emptyset$, K_0 is closed and $\Phi(K_0) \subset K_0$. Also, $\Phi(K_0)$ is closed in X and $\Phi(\Phi(K_0)) \subset \Phi(K_0)$; so $\Phi(K_0) \in \Delta$. Thus $\Phi(K_0) = K_0$ and the proof is complete.

We now give two preliminary theorems to our next main results of this section.

THEOREM 3.14. *If a multifunction $\Phi: X \to Y$ has a subclosed graph and $K \subset X$ is nonempty then the restriction Φ_K of Φ to K has a subclosed graph.*

Proof. Let $x \in K$ and let $\{x_n\}$ be a net in $K - \{x\}$ with $x_n \to x$ in K relative to K. Let $\{y_n\}$ be a net in Y with $y_n \in \Phi_K(x_n)$ for each n and let $y \in Y$ with $y_n \to y$. Then $x_n \to x$ in X and $y_n \in \Phi(x_n)$ for each n; since Φ has a subclosed graph we have $y \in \Phi(x) = \Phi_K(x)$. The proof is complete.

THEOREM 3.15. *If a multifunction $\Phi: X \to Y$ has a subclosed graph and $K \subset Y$ is compact then $\Phi^{-1}(K)$ is closed in X.*

Proof. Let v be a limit point of $\Phi^{-1}(K)$. There are nets $\{x_n\}$, $\{y_n\}$ in $X - \{v\}$ and K, respectively, with $x_n \to x$ and $y_n \in \Phi(x_n)$ for each n. Some subnet $\{y_{n_m}\}$ of $\{y_n\}$ converges to some $y \in K$. This gives $y \in \Phi(v)$; so $v \in \Phi^{-1}(K)$ and the proof is complete.

Our next results in this section are extensions of well-known uniform boundedness principle to multifunctions (see [26], p. 189).

THEOREM 3.16. *Let Y be a countable union of the compact subspaces $\{Y(n)\}_{n=1}^{\infty}$ and let X be a Baire space. Let F be a family of multifunctions from X to Y with subclosed graphs and the property* *that for each* $x \in X$ *there is a integer* $j(x)$ *with* $\Phi(x) \cap Y(j(x) \neq \varnothing$ *for any* $\Phi \in F$. *Then there is an integer* m *and a nonempty open* $V \subset X$ *such that* $\Phi(x) \cap Y(m) \neq \varnothing$ *for each* $x \in V$ and $\Phi \in F$.

Proof. For each integer n let $Q(n) = \bigcap_F \Phi^{-1}(Y(n))$. Then $Q(n)$ is closed as the intersection of subsets which are closed from Theorem 3.15. Furthermore, if $x \in X$ there is an integer $j(x)$ with $\Phi(x) \cap Y(j(x)) \neq \varnothing$ for each $\Phi \in F$. So, $x \in Q(j(x))$ and $\{Q(n)\}_{n=1}^{\infty}$ is a covering of X. Since X is Baire there is an integer m and a nonempty open subset V of X such that $V \subset Q(m)$. $Y(m)$ satisfies $\Phi(x) \cap Y(m) \neq \varnothing$ for each $\Phi \in F$ and $x \in V$. The proof is complete.

THEOREM 3.19. *With the same hypothesis as in Theorem* 3.16 *along with the additional condition that* $\Phi(\Phi^{-1}(Y(n))) \subset Y(n)$ *for each* n *and* Φ *the following additional statements hold*:

(a) *Each* $\Phi \in F$ *is u.s.c. at each point of the open set* V *in the conclusion of Theorem* 3.16.

(b) *There is an open set* W *of* X *such that each* $\Phi \in F$ *is u.s.c. at each point of* W *and* $\mathrm{cl}(W) = X$.

Proof of (a). From the additional condition if $\Phi \in F$ we have $\Phi(V) \subset Y(m)$ and from Theorem 3.14 we have that the restriction Φ_V: $V \to Y(m)$ has a subclosed graph. Thus, Φ_V is subcontinuous and consequently u.s.c. from Theorem 3.2. Since V is open Φ is u.s.c. at each point of V.

Proof of (b). Let W be the union of all open subsets V of X such that each $\Phi \in F$ is u.s.c. at each $x \in V$. Now let A be an open and nonempty subset of X. Then A is Baire and $\{\Phi_A : \Phi \in F\}$ is a family of multifunctions from A to Y satisfying the same conditions relative to A as F satisfies relative to X. So, there is a nonempty open subset B of A with Φ_A u.s.c. at each point of B for each $\Phi \in F$. Since A is open each $\Phi \in F$ is u.s.c. at each point of B. Thus $B \subset W$ and this gives $A \cap W \neq \varnothing$.

The proof is complete.

Employing arguments similar to those used in the proof of Theorem 3.17 we prove the following theorem.

THEOREM 3.18. *Let* X *and* Y *be spaces and let* Φ: $X \to Y$ *be a multifunction with a subclosed graph and the property that* $\Phi(x)$ *has a compact neighborhood for each* $x \in X$. *Then the set of points of* X *at which* Φ *is not u.s.c. is a closed subset of* X.

Proof. Let $D(\Phi)$ be the set of points in X at which Φ fails to be u.s.c. and let $x \in X - D(\Phi)$. Let W be a compact neighborhood of $\Phi(x)$. Since Φ is u.s.c. at x, there is a V open about x with $\Phi(V) \subset W$. By arguments similar to those above Φ is u.s.c. at each point of V. Thus $X - D(\Phi)$ is open, $D(\Phi)$ is closed and the proof is complete.

The following two corollaries offer extensions of the results of Theorem 2 of [2] and Theorem 2 of [24] to multifunctions.

COROLLARY 3.19. *Let X be a Baire space and let Y be the union of a countable family $\{Y(n)\}_{n=1}^{\infty}$ of compact subspaces. Then $D(\Phi)$ is nowhere dense in X for any multifunction $\Phi: X \to Y$ satisfying the following conditions*:

(a) *The multifunction Φ has a subclosed graph.*

(b) *For each n, $\Phi(\Phi^{-1}(Y(n))) \subset Y(n)$.*

Proof. $F = \{\Phi\}$ satisfies the conditions of Theorem 3.17. Let W be an open set satisfying the conditions of Theorem 3.17(b). Then $D(\Phi) \subset X - W$ so $D(\Phi)$ is nowhere dense in X since $X - W$ is nowhere dense in X. The proof is complete.

COROLLARY 3.20. *Let X, Y, and Φ satisfy the conditions of Corollary 3.19. Then $D(\Phi)$ is nowhere dense and closed in X if for each $x \in X$, $\Phi(x)$ has a compact neighborhood.*

Proof. Theorem 3.18 and Corollary 3.19.

Our final theorem in this section is one on common fixed points of a family of multifunctions with subclosed graphs. A multifunction $\Phi: X \to X$ has a fixed point if there is an $x \in X$ with $x \in \Phi(x)$ [20].

THEOREM 3.21. *Let X be a compact space and let F be a family of multifunctions from X to X with subclosed graphs. If for each finite $F^* \subset F$, there is an $x \in X$ satisfying $x \in \Phi(x)$ for all $\Phi \in F^*$, then there is an $x \in X$ satisfying $x \in \Phi(x)$ for all $\Phi \in F$.*

Proof. If $\lambda: X \to X$ is the identity function then $\Omega = \{E(\Phi, \lambda, X, X): \Phi \in F\}$ is a family of closed subsets of X from Corollary 3.11. If $F^* \subset F$ is finite there is an $x \in X$ satisfying $x \in \Phi(x)$ for each $\Phi \in F^*$. Thus Ω has the finite intersection property. Since X is compact then ad $\Omega \neq \emptyset$. So there is an $x \in X$ satisfying $x \in E(\Phi, \lambda, X, X)$ for all $\Phi \in F$. The proof is complete.

4. Multifunctions with strongly-subclosed graphs. We will say that a multifunction $\Phi: X \to Y$ has a *strongly-subclosed graph* if for each $x \in X$ and net $\{x_n\}$ in $X - \{x\}$ with $x_n \to x$ and net $\{y_n\}$

in Y with $y_n \in \Phi(x_n)$ for each n and $y_n \to_\theta y$ in Y, we have $y \in \Phi(x)$. From Theorem 2.2 (g) we see that a multifunction has a strongly-closed graph if and only if it has θ-closed point images and a strongly-subclosed graph. It this section we will extend a number of known results for functions with strongly-closed graphs to multifunctions with strongly-subclosed graphs. Several of these results will be stated without proof as the proofs parallel proofs in §3. Our first theorem in this section gives several characterizations of multifunctions with strongly-subclosed graphs.

THEOREM 4.1. *The following statements are equivalent for spaces X, Y, and multifunction $\Phi: X \to Y$:*

(a) *The multifunction Φ has a strongly-subclosed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \in \Sigma(x)$ in x, $W \in \Sigma(y)$ in Y, with $((V - \{x\}) \times \mathrm{cl}(W)) \cap G(\Phi) = \varnothing$ $((V \times (\mathrm{cl}(W) - \Phi(x))) \cap G(\Phi) = \varnothing)[\Phi(V - \{x\}) \cap \mathrm{cl}(W) = \varnothing]\langle V \cap \Phi^{-1}(\mathrm{cl}(W) - \Phi(x)) = \varnothing\rangle$.*

(c) *If Ω is a filterbase on $X - \{x\}$ with $\Omega \to x$ in X then $\mathrm{ad}_\theta \Phi(\Omega) \subset \Phi(x)$.*

Using the definition of r-subcontinuous function from [8] as a model we obtain the following definition. A multifunction $\Phi: X \to Y$ is *r-subcontinuous* if whenever $\{x_n\}$ is a convergent net in X and $\{y_n\}$ is a net in Y with $y_n \in \Phi(x_n)$ for each n, then $\{y_n\}$ has a θ-convergent subset. Utilizing the notion of weakly-continuous function from [16], Smithson [22] has defined a multifunction $\Phi: X \to Y$ to be *weakly upper-semi-continuous (w.u.s.c.) at x* in X if for each $W \in \Sigma(\Phi(x))$ in Y there is a $V \in \Sigma(x)$ in X satisfying $\Phi(V) \subset \mathrm{cl}(W)$. A subset K of a space X is *rigid* provided whenever Ω is a filterbase on X such that $K \cap \mathrm{ad}_\theta \Omega = \varnothing$ there is an $F \in \Omega$ and an $N \in \Sigma(K)$ satisfying $F \cap \mathrm{cl}(N) = \varnothing$ [5]. Several characterizations of rigid subsets are given in [5]. Theorem 4.2 gives a characterization in terms of nets.

THEOREM 4.2. *A subset K of a space is rigid if and only if each net which is frequently in* $\mathrm{cl}(W)$ *for each $W \in \Sigma(K)$ has a θ-accumulation point in K.*

Proof. Necessity. Let $\{x_n\}$ be a net which is frequently in $\mathrm{cl}(W)$ for each $W \in \Sigma(K)$. Then the filterbase Ω induced by $\{x_n\}$ satisfies $F \cap \mathrm{cl}(W) \neq \varnothing$ for each $F \in \Omega$ and $W \in \Sigma(K)$. Thus, $K \cap \mathrm{ad}_\theta \Omega \neq \varnothing$ since K is rigid. Each $x \in K \cap \mathrm{ad}_\theta \Omega$ is a θ-accumulation point of $\{x_n\}$.

Sufficiency. If K satisfies the net condition and Ω is a filterbase on X such that $\Omega^* = \{F \cap \mathrm{cl}(W): F \in \Omega, W \in \Sigma(K)\}$ is a filterbase on X then the net induced by Ω^* is frequently in $\mathrm{cl}(W)$ for each $W \in \Sigma(K)$. So, this net has a θ-accumulation point in K. Thus, $K \cap \mathrm{ad}_\theta \Omega \neq \varnothing$.

The proof is complete.

Theorems 4.3 and 4.4 extend results from [8] to multifunctions with strongly-subclosed graphs.

THEOREM 4.3. *Let $\Phi: X \to Y$ be a multifunction with rigid point images. If Φ is w.u.s.c. then Φ is r-subcontinuous.*

Proof. Let $x \in X$ and let $\{x_n\}$ and $\{y_n\}$ be nets on X and Y, respectively, with $x_n \to x$ and $y_n \in \Phi(x_n)$ for each n. If $W \in \Sigma(\Phi(x))$ there is a $V \in \Sigma(x)$ in X with $\Phi(V) \subset \mathrm{cl}(W)$ since Φ is w.u.s.c. Since $\{x_n\}$ is eventually in V we have $\{y_n\}$ eventually in $\mathrm{cl}(W)$. So, by Lemma 4.2, some subnet of $\{y_n\}$ θ-converges to a point in $\Phi(x)$. The proof is complete.

THEOREM 4.4. *An r-subcontinuous multifunction $\Phi: X \to Y$ with a strongly-subclosed graph is w.u.s.c.*

Proof. Let $x \in X$ and suppose that Φ is not w.u.s.c. at x. Then, there is a $W \in \Sigma(\Phi(x))$ such that $\Phi(N - \{x\}) \cap (Y - \mathrm{cl}(W)) \neq \varnothing$ for any $N \in \Sigma(x)$. For each $N \in \Sigma(x)$ choose $y_n \in \Phi(N - \{x\}) \cap (Y - \mathrm{cl}(W))$ and $x_n \in N - \{x\}$ with $y_n \in \Phi(x_n)$. With the usual ordering on $\Sigma(x)$, $\{x_n\}$ and $\{y_n\}$ are nets in $X - \{x\}$ and Y, respectively, with $x_n \to x$. Some subnet of $\{y_n\}$ θ-converges to some $y \in Y$ by the r-subcontinuity of Φ. However, $y \notin \Phi(x)$ since $y_n \in Y - \mathrm{cl}(W)$ for every n. This is a contradiction of the fact that Φ has a strongly-subclosed graph. The proof is complete.

Theorem 4.5 below extends Theorem 2.2 of [8] to multifunctions. The proof is similar to that of Theorem 2.2 of [21] and is omitted.

THEOREM 4.5. *Let $\Phi: X \to Y$ be a multifunction. Then Φ maps compact subsets K of X onto $H(i)$ subsets $\Phi(K) \subset Y$ if and only if the restriction $\Phi_K: K \to Y$ is r-subcontinuous with respect to $\Phi(K)$ for each compact $K \subset X$.*

The following corollaries are analogous to Theorem 3.4 and Corollary 3.5, respectively.

COROLLARY 4.6. *A multifunction on a locally compact space which maps compact subsets onto $H(i)$ subsets is r-subcontinuous.*

COROLLARY 4.7. *A multifunction with a strongly-subclosed* *graph on a locally compact space which maps compact subsets onto $H(i)$ subsets is w.u.s.c.*

Theorem 4.8 parallels Theorem 3.6 above and strengthens Theorem 17 of [22] since a point compact w.u.s.c. multifunction into a Urysohn space has a strongly-closed graph.

THEOREM 4.8. *Let $\Phi: X \to Y$ be a w.u.s.c. multifunction with rigid point images and let $\alpha: X \to Y$ be a multifunction with a strongly-subclosed graph. Then $E(X, Y, \Phi, \alpha)$ is closed in X.*

Proof. Similar to the proof of Theorem 3.6.

Corollaries 4.9, 4.10, and 4.11 are similar to Corollaries 3.7, 3.8, and 3.9.

COROLLARY 4.9. *Let Y be any space. If $\Phi: X \to Y$ is a weakly-continuous function and $\alpha: X \to Y$ is a function with a strongly-subclosed graph then $\{x \in X: \Phi(x) = \alpha(x)\}$ is a closed subset of X.*

COROLLARY 4.10. *Let $\Phi: X \to Y$ be a w.u.s.c. multifunction with rigid point images and let $\alpha: X \to Y$ be a multifunction with a strongly-subclosed graph. If $E(X, Y, \Phi, \alpha)$ is dense in X then $E(X, Y, \Phi, \alpha) = X$.*

COROLLARY 4.11. *Let Y be any space. If $\Phi: X \to Y$ is a weakly-continuous function and $\alpha: X \to Y$ is a function with a strongly-subclosed graph then $\{x \in X: \Phi(x) = \alpha(x)\} = X$ if it is dense in X.*

THEOREM 4.12. *Let X and Y be any spaces, $\Phi: X \to Y$ be an r-subcontinuous multifunction with a strongly-subclosed graph, and let $\alpha: X \to Y$ be a multifunction with a strongly-subclosed graph. Then $E(X, Y, \Phi, \alpha)$ is closed in X.*

Proof. Similar to the proof of Theorem 3.10.

Theorem 4.12 is a generalization and extension of the well-known result that if Y is Hausdorff and Φ and α are continuous functions into Y from a space X then $E(X, Y, \Phi, \alpha)$ is closed in X.

We omit the proof of Corollary 4.13.

COROLLARY 4.13. *If X is a space and the multifunction $\Phi: X \to X$ has a strongly-subclosed graph then $\{x \in X: x \in \Phi(x)\}$ is closed in X.*

The proofs of Theorems 4.14 and 4.15 parallel those of Theorem 3.14 and 3.15, respectively, and are not given here. In connection with Theorem 4.15 we note that a subset K of a space is $H(i)$ if and only if each net in K has a θ-accumulation point in K.

THEOREM 4.14. *If a multifunction $\Phi: X \to Y$ has a strongly-subclosed graph and $K \subset X$ is nonempty then the restriction Φ_K of Φ to K has a strongly-subclosed graph.*

THEOREM 4.15. *If a multifunction $\Phi: X \to Y$ has a strongly-subclosed graph and $K \subset Y$ is an $H(i)$ subset then $\Phi^{-1}(K)$ is closed in X.*

Theorems 4.16 and 4.17 are generalizations of the Uniform Boundedness Principle from analysis.

THEOREM 4.16. *Let Y be a countable union of the $H(i)$ subsets $\{Y(n)\}_{n=1}^{\infty}$ and let X be a Baire space. Let F be a family of multifunctions from X to Y with strongly-subclosed graphs and with the property that for each $x \in X$ there is an integer $j(x)$ satisfying $\Phi(x) \cap Y(j(x)) \neq \emptyset$ for any $\Phi \in F$. Then there is an integer m and a nonempty open $V \subset X$ such that $\Phi(x) \cap Y(m) \neq \emptyset$ for each $x \in V$ and $\Phi \in F$.*

Proof. Parallels that of Theorem 3.16 since $\Phi^{-1}(Y(n))$ is closed for each Φ and n from Theorem 4.15.

THEOREM 4.17. *With the same hypothesis as in Theorem* 4.16 *along with the condition that $\Phi(\Phi^{-1}(Y(n))) \subset Y(n)$ for each n and Φ the following additional statements hold*:

(a) *Each $\Phi \in F$ is w.u.s.c. at each point of the open set V in the conclusion of Theorem* 4.16.

(b) *There is an open set W of X such that each $\Phi \in F$ is w.u.s.c. at each point of W and* $\mathrm{cl}(W) = X$.

Proof. With the use of Theorems 4.14, 4.16, and 4.4 the proof parallels that of Theorem 3.17.

Next in this section we list a theorem and corollaries for multifunctions with strongly-subclosed graphs and corollaries which are similar to Theorems 3.18, 3.19, and 3.20 above.

THEOREM 4.18. *Let X and Y be spaces and let $\Phi: X \to Y$ be a multifunction with a strongly-subclosed graph and the property that $\Phi(x)$ has an $H(i)$ subset as a neighborhood for each $x \in X$. Then* *the set of points at which Φ is not w.u.s.c. is a closed subset of X.*

COROLLARY 4.19. *Let X be a Baire space and let Y be the union of a countable family $\{Y(n)\}_{n=1}^{\infty}$ of $H(i)$ subsets. Then the set of points at which a multifunction $\Phi: X \to Y$ satisfying the following properties fails to be w.u.s.c. is nowhere dense in X.*

(a) *The multifunction Φ has a strongly-subclosed graph.*

(b) *For each n, $\Phi(\Phi^{-1}(Y(n))) \subset Y(n)$.*

COROLLARY 4.20. *Let X, Y, and Φ satisfy the conditions of Corollary 4.19 and the condition that for each $x \in X$, $\Phi(x)$ has an $H(i)$ subsets as a neighborhood. Then the set of points at which Φ fails to be w.u.s.c. is a closed and nowhere dense subset of X.*

It is known that a space is Hausdorff if and only if each point in the space is θ-closed [5]. It is well-known that compact subsets of a Hausdorff space are closed and that there are non-Hausdorff spaces with this property [17]. Also known is that rigid subsets of a Hausdorff space are θ-closed and that compact subsets are rigid [5]. These results may be utilized to yield several interesting characterizations of Hausdorff spaces.

THEOREM 4.21. *The following statements are equivalent:*

(a) *The space X is Hausdorff.*

(b) *Each rigid subset of X is θ-closed.*

(c) *Each compact subset of X is θ-closed.*

(d) *Each continuous function into X maps compact subsets onto θ-closed subsets.*

(e) *Each continuous bijection onto X maps compact subsets onto θ-closed subsets.*

Proof. We prove only that (a) is implied by (e). Let $x \in X$ and let X^* be X with the topology which is the simple extension of the topology of X through the set $\{x\}$ [18]. The identity function $i: X^* \to X$ is continuous and $\{x\}$ is compact in X^* so $\{i(x)\} = \{x\}$ is θ-closed in X. The proof is complete.

Finally in this section we extend Theorem 10 of [22] to multifunctions.

THEOREM 4.23. *A w.u.s.c. multifunction with rigid point images into a Hausdorff space has a closed graph.*

Proof. Let Y be Hausdorff and let $\Phi: X \to Y$ be w.u.s.c. with rigid point images. If $(x, y) \in (X \times Y) - G(\Phi)$ then $y \notin \Phi(x)$ and since Y is Hausdorff, from Theorem 4.22, we have $V \in \Sigma(y)$ and $W \in \Sigma(\Phi(x))$

in Y such that $V \cap \mathrm{cl}(W) = \varnothing$. There is an $A \in \Sigma(x)$ in X with $\Phi(A) \subset \mathrm{cl}(W)$; so $(x, y) \in A \times V$ and $(A \times V) \cap G(\Phi) = \varnothing$. The proof is complete.

We point out in closing this section that the hypothesis "compact point images" may be replaced by "rigid point images" in Theorem 3.12 above.

5. **Multifunctions with θ-subclosed graphs.** We will say that a multifunction $\Phi: X \to Y$ has a *θ-subclosed graph* if for each $x \in X$ and net $\{x_n\}$ in $X - \{x\}$ with $x_n \to_\theta x$ and net $\{y_n\}$ in Y with $y_n \in \Phi(x_n)$ for each n and $y_n \to_\theta y$ in Y, we have $y \in \Phi(x)$. Functions with θ-subclosed graphs have been studied in [14]. From Theorem 2.6 (i) we note that a multifunction has a θ-closed graph if and only if it has θ-closed point images and a θ-subclosed graph. In this section we will extend a number of known results on functions with θ-subclosed graphs to multifunctions with θ-subclosed graphs. Several of these results are stated without proof.

THEOREM 5.1. *The following statements are equivalent for spaces X, Y, and multifunction $\Phi: X \to Y$:*

(a) *The multifunction Φ has a θ-closed graph.*

(b) *For each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \in \Sigma(x)$ in X, $W \in \Sigma(y)$ in Y, with $((\mathrm{cl}(V) - \{x\}) \times \mathrm{cl}(W)) \cap G(\Phi) = \varnothing((\mathrm{cl}(V) \times (\mathrm{cl}(W) - \Phi(x))) \cap G(\Phi) = \varnothing)[\Phi(\mathrm{cl}(V) - \{x\}) \cap \mathrm{cl}(W) = \varnothing]\langle \mathrm{cl}(V) \cap \Phi^{-1}(\mathrm{cl}(W) - \Phi(x)) = \varnothing\rangle$.*

(c) *If Ω is a filterbase on $X - \{x\}$ with $\Omega \to_\theta x$ in X then $\mathrm{ad}_\theta\, \Phi(\Omega) \subset \Phi(x)$.*

A function $\Phi: X \to Y$ is *θ-continuous at* $x \in X$ if for each $W \in \Sigma(\Phi(x))$ in Y there is a $V \in \Sigma(x)$ in X with $\Phi(\mathrm{cl}(V)) \subset \mathrm{cl}(W)$. If Φ is θ-continuous at each $x \in X$ we say simply that Φ is *θ-continuous* [7]. This notion has been studied and utilized extensively (see [25], [5]). A number of equivalent statements to the statement that a function is θ-continuous are given in [5] and it is proved there that a function $\Phi: X \to Y$ which is θ-continuous must satisfy the condition $\Phi(\mathrm{cl}_\theta(K)) \subset \mathrm{cl}_\theta(\Phi(K))$ for each $K \subset X$. We show in Theorem 5.2 that this condition is also a sufficient condition for Φ to be θ-continuous.

THEOREM 5.2. *A function $\Phi: X \to Y$ is θ-continuous if and only if $\Phi(\mathrm{cl}_\theta(K)) \subset \mathrm{cl}_\theta(\Phi(K))$ for each $K \subset X$.*

Proof. Only the sufficiency requires proof. Let $x \in X$ and let $W \in \Sigma(\Phi(x))$ in Y. Then $\Phi(x) \notin \mathrm{cl}_\theta(\Phi(X - \Phi^{-1}(\mathrm{cl}(W))))$ so $\Phi(x) \notin \Phi(\mathrm{cl}_\theta(X - \Phi^{-1}(\mathrm{cl}(W))))$. Thus, $x \notin \mathrm{cl}_\theta(X - \Phi^{-1}(\mathrm{cl}(W)))$ and there is a

$V \in \Sigma(x)$ in X with $\mathrm{cl}(V) \cap (X - \Phi^{-1}(\mathrm{cl}(W))) = \varnothing$. This gives $\Phi(\mathrm{cl}(V)) \subset \mathrm{cl}(W)$ and the proof is complete.

We will define a multifunction $\Phi: X \to Y$ to be *θ-upper-semicontinuous (θ-u.s.c.) at* $x \in X$ if for each $W \in \Sigma(\Phi(x))$ in Y there is a $V \in \Sigma(x)$ in X satisfying $\Phi(\mathrm{cl}(V)) \subset \mathrm{cl}(W)$. We say that Φ is *θ-upper-semicontinuous (θ-u.s.c.)* if Φ is θ-u.s.c. at each point of X.

Our next result parallels Theorem 9 of [22] which was given for w.u.s.c. multifunctions. If $\Phi: X \to Y$ is a multifunction we define the *graph map of* Φ to be the multifunction $G_\Phi: X \to X \times Y$ defined by $G_\Phi(x) = \{(x, y): y \in \Phi(x)\}$.

THEOREM 5.3. *A multifunction $\Phi: X \to Y$ with compact point images is θ-u.s.c. if and only if G_Φ is θ-u.s.c.*

Proof. Necessity. Let $A \times B \in \Sigma(G_\Phi(x))$ in $X \times Y$. Then $B \in \Sigma(\Phi(x))$ in Y and since Φ is θ-u.s.c. there is a $Q \in \Sigma(x)$ in X with $Q \subset A$ and $\Phi(\mathrm{cl}(Q)) \subset \mathrm{cl}(B)$. So, $G_\Phi(\mathrm{cl}(Q)) \subset \mathrm{cl}(Q) \times \mathrm{cl}(B) = \mathrm{cl}(Q \times B) \subset \mathrm{cl}(A \times B)$.

Sufficiency. Let $x \in X$ and $W \in \Sigma(\Phi(x))$ in Y. Then $\pi_y^{-1}(W) \cap G(\Phi)$ is open in $G(\Phi)$, where $\pi_y: X \times Y \to Y$ is the projection. So, there is a $V \in \Sigma(x)$ in X with $G_\Phi(\mathrm{cl}(V)) \subset \mathrm{cl}(\pi_y^{-1}(W))$. If $y \in \Phi(\mathrm{cl}(V))$ there is an $x \in \mathrm{cl}(V)$ with $y \in \Phi(x)$. So $(x, y) \in G_\Phi(\mathrm{cl}(V))$ and consequently, $(x, y) \in \mathrm{cl}(\pi_y^{-1}(W))$. This gives $y \in \pi_y(\mathrm{cl}(\pi_y^{-1}(W))) \subset \mathrm{cl}(W)$. The proof is complete.

Using the definition of θ-subcontinuous function from [14] as a model we obtain the following definition. A multifunction $\Phi: X \to Y$ is *θ-subcontinuous* if whenever $\{x_n\}$ is a θ-convergent net in X and $\{y_n\}$ is a net in Y with $y_n \in \Phi(x_n)$ for each n then $\{y_n\}$ has a θ-convergent subnet.

Theorems 5.4 and 5.5 extend the results of Theorem 2.6 of [14] to multifunctions. The proofs are similar to the proofs of Theorems 4.3 and 4.4, respectively, and are omitted.

THEOREM 5.4. *Let $\Phi: X \to Y$ be a multifunction with rigid point images. If Φ is θ-u.s.c. then Φ is θ-subcontinuous.*

THEOREM 5.5. *A θ-subcontinuous multifunction with a θ-subclosed graph is θ-u.s.c.*

Theorem 5.6 below extends Theorem 2.7 of [14] to multifunctions. The proof is similar to that of Theorem 2.2 of [21].

THEOREM 5.6. *Let $\Phi: X \to Y$ be a multifunction. Then Φ maps*

$H(i)$ subspaces K onto $H(i)$ subsets $\Phi(K) \subset Y$ if and only if the restriction $\Phi_K: K \to Y$ is θ-subcontinuous with respect to $\Phi(K)$ for each $H(i)$ subset $K \subset X$.

We will say that a space is *locally $H(i)$* if each point in the space has a neighborhood which is an $H(i)$ subset.

The following corollaries are analogous to Corollaries 4.6 and 4.7, respectively.

COROLLARY 5.7. *A multifunction on a locally $H(i)$ space which maps $H(i)$ subsets onto $H(i)$ subsets is θ-subcontinuous.*

COROLLARY 5.8. *A multifunction with a θ-subclosed graph on a locally $H(i)$ space which maps $H(i)$ subsets onto $H(i)$ subsets is θ-u.s.c.*

THEOREM 5.9. *Let $\Phi: X \to Y$ be a θ-u.s.c. multifunction with rigid point images and let $\alpha: X \to Y$ be a multifunction with a θ-subclosed graph. Then $E(X, Y, \Phi, \alpha)$ is θ-closed in X.*

Proof. Let $v \in \mathrm{cl}_\theta(E(X, Y, \Phi, \alpha)) - E(X, Y, \Phi, \alpha)$. There is a net $\{x_n\}$ in $E(X, Y, \Phi, \alpha) - \{v\}$ with $x_n \to_\theta v$. For each n choose $y_n \in \Phi(x_n) \cap \alpha(x_n)$. If $W \in \Sigma(\Phi(v))$ in Y there is a $V \in \Sigma(v)$ in X with $\Phi(\mathrm{cl}(V)) \subset \mathrm{cl}(W)$ because Φ is θ-u.s.c; $\{y_n\}$ is eventually in $\mathrm{cl}(W)$ since $\{x_n\}$ is eventually in $\mathrm{cl}(V)$. Since $\Phi(v)$ is rigid some subnet $\{y_{n_m}\}$ of $\{y_n\}$ θ-converges to some $y \in \Phi(v)$. Since α has a θ-subclosed graph we have $y \in \alpha(v)$. Thus, $v \in E(X, Y, \Phi, \alpha)$. This is a contradiction and the proof is complete.

Corollaries 5.10, 5.11, and 5.12 are similar to Corollaries 3.7, 3.8, and 3.9, respectively. A subset K of a space X is *θ-dense* in the space if $\mathrm{cl}_\theta(K) = X$.

COROLLARY 5.10. *Let Y be any space. If $\Phi: X \to Y$ is a θ-continuous function and $\alpha: X \to Y$ is a function with a θ-subclosed graph, then $\{x \in X: \Phi(x) = \alpha(x)\}$ is a θ-closed subset of X.*

COROLLARY 5.11. *Let $\Phi: X \to Y$ be a θ-u.s.c. multifunction with rigid point images and let $\alpha: X \to Y$ be a multifunction with a graph which is θ-closed in $X \times Y$. Then $E(X, Y, \Phi, \alpha) = X$ if $E(X, Y, \Phi, \alpha)$ is θ-dense in X.*

COROLLARY 5.12. *Let Y be any space. If $\Phi: X \to Y$ is a θ-continuous function and $\alpha: X \to Y$ is a function with a graph which is θ-closed in $X \times Y$, then $\{x \in X: \Phi(x) = \alpha(x)\} = X$ if it is θ-dense in X.*

Theorem 5.13 parallels Theorem 17 of [22]. This theorem along with Corollary 5.14 is listed without proof.

THEOREM 5.13. *Let X and Y be any spaces, $\Phi: X \to Y$ be a θ-subcontinuous multifunction with a θ-subclosed graph, and $\alpha: X \to Y$ be a multifunction with a closed graph. Then $E(X, Y, \Phi, \alpha)$ is θ-closed in X.*

COROLLARY 5.14. *If X is a space and the multifunction $\Phi: X \to X$ has a θ-subclosed graph then $\{x \in X: x \in \Phi(x)\}$ is θ-closed in X.*

We have the following parallel to Theorem 3.13. We recall that a θ-closed subset of an $H(i)$ space is an $H(i)$ subset.

THEOREM 5.15. *If X is $H(i)$ and $\Phi: X \to X$ is a multifunction which maps θ-closed subsets onto θ-closed subsets, there is a $K_0 \subset X$ with $K_0 \neq \varnothing$, K_0 θ-closed and $\Phi(K_0) = K_0$.*

Proof. The proof goes as the proof for Theorem 3.13. $Q \neq \varnothing$ since X is $H(i)$; Q is θ-closed as the intersection of θ-closed subsets.

We state Theorem 5.16 without proof.

THEOREM 5.16. *If a multifunction $\Phi: X \to Y$ has a θ-subclosed graph and $K \subset X$ in nonempty, then the restriction Φ_K of Φ to K has a θ-subclosed graph.*

It has been proved in [6] that an H-closed space is not the countable union of θ-closed nowhere dense subspaces. The author has improved upon this result in [14], showing that the closure of a nonempty open subset of an $H(i)$ space is not contained in the union of a countable family of subsets each of which has a θ-closure with empty interior. We may use this property of $H(i)$ spaces to prove a result which implies the realization of Bourbaki [3] that the set of isolated points of a countable H-closed space must be dense in the space.

THEOREM 5.17. *Let X be a countable $H(i)$ space and let $I = \{x \in X: \mathrm{cl}_\theta(\{x\})$ has nonempty interior$\}$. Then I is θ-dense in X.*

Proof. Let V be a nonempty open subset of X. Then $\mathrm{cl}(V)$ is not the union of a countable collection of subsets all of whose θ-closures in X have empty interiors in X. Thus since $\mathrm{cl}(V)$ is countable there is an $x \in \mathrm{cl}(V)$ such that $\mathrm{cl}_\theta(\{x\})$ has nonempty interior. The proof is complete.

We abstract this property of $H(i)$ spaces and say that a space X is *θ-Baire* if no nonempty open subset of X has a closure which is contained in the union of a countable family of subsets all of which have θ-closures with empty interior. We use this latter notion to give another parallel of the Uniform Boundedness Principle stated in [26], this time for multifunctions with θ-subclosed graphs. First, we state the following analogue to Theorem 3.15 above.

THEOREM 5.18. *If $\Phi: X \to Y$ is a multifunction with a θ-subclosed graph and K is an $H(i)$ subset of Y, then $\Phi^{-1}(K)$ is θ-closed in X.*

THEOREM 5.19. *Let Y be a countable union of the $H(i)$ subsets $\{Y(n)\}_{n=1}^{\infty}$ and let X be a θ-Baire space. Let F be a family of multifunctions from X to Y with θ-subclosed graphs and with the property that for each $x \in X$ there is an integer $j(x)$ satisfying $\Phi(x) \cap Y(j(x)) \neq \varnothing$ for each $\Phi \in F$. Then there is an integer m and a nonempty open $V \subset X$ such that $\Phi(x) \cap Y(m) \neq \varnothing$ for each $x \in V$ and $\Phi \in F$.*

Proof. Parallels that of Theorem 3.16 since $\Phi^{-1}(Y(n))$ is θ-closed for each Φ and n from Theorem 5.18, and X is θ-Baire.

THEOREM 5.20. *With the same hypothesis as in Theorem 5.19 along with the condition that $\Phi(\Phi^{-1}(Y(n))) \subset Y(n)$ for each n and Φ the following additional statements hold:*

(a) *Each $\Phi \in F$ is θ-u.s.c. at each point of the open set V in the conclusion of Theorem 5.19.*

(b) *There is an open set W of X such that each $\Phi \in F$ is θ-u.s.c. at each point of W and $\mathrm{cl}(W) = X$.*

Proof of (a). With the use of Theorems 5.5, 5.16, and 5.17 the proof parallels that of Theorem 3.17 (b).

Proof of (b). Let W be the union of all open subsets V of X such that each $\Phi \in F$ is θ-u.s.c. at each $x \in V$ and let A be an open and nonempty subset of X. Then X is θ-Baire and $\{\Phi_{\mathrm{cl}(A)}: \Phi \in F\}$ is a family of multifunctions from $\mathrm{cl}(A)$ to Y satisfying the same conditions relative to $\mathrm{cl}(A)$ as F satisfies relative to X. So there is a nonempty open subset B of $\mathrm{cl}(A)$ with Φ_A θ-u.s.c. at each point of $A \cap B$ for each $\Phi \in F$. Each $\Phi \in F$ is θ-u.s.c. at each $x \in A \cap B$; this gives $A \cap B \subset W$.

The proof is complete.

We will denote the set of points at which a multifunction Φ fails to be θ-u.s.c. by $\theta D(\Phi)$. The next theorem extends Theorem 5.4 in [14] to multifunctions.

THEOREM 5.21. *Let X be a space and let $\Phi: X \to Y$ be a multifunction such that for each $x \in X$, $\Phi(x)$ has a neighborhood which is an $H(i)$ subset. Then $\theta D(\Phi)$ is closed in X if Φ has a θ-subclosed graph.*

Proof. Let $x \in X$ such that Φ is θ-u.s.c. at x and let W be a neighborhood of $\Phi(x)$ with $\mathrm{cl}(W)$ an $H(i)$ subset. There is a V open about x with $\Phi(\mathrm{cl}(V)) \subset \mathrm{cl}(W)$. By arguments similar to those above Φ is θ-u.s.c. at each point of V. Thus $X - \theta D(\Phi)$ is open and the proof is complete.

The following two corollaries offer extensions of the results of Theorem 2 of [2] and Theorem 2 of [24] to multifunctions.

COROLLARY 5.22. *Let X be a θ-Baire space and let Y be the union of a countable family $\{Y(n)\}_{n=1}^{\infty}$ of $H(i)$ subsets. Then $\theta D(\theta)$ is nowhere dense in X for any multifunction $\Phi: X \to Y$ satisfying the following conditions:*

(a) *The multifunction Φ has a subclosed graph.*

(b) *For each n, $\Phi(\Phi^{-1}(Y(n))) \subset Y(n)$.*

Proof. $F = \{\Phi\}$ satisfies the conditions of Theorem 5.20. Let W be an open set satisfying the conditions of Theorem 5.20(b). Then $\theta D(\Phi) \subset X - W$ so $\theta D(\Phi)$ is nowhere dense in X since $X - W$ is nowhere dense in X. The proof is complete.

COROLLARY 5.23. *Let X, Y, and Φ satisfy the conditions of Corollary 5.22. Then $\theta D(\Phi)$ is nowhere dense and closed in X if for each $x \in X$, $\Phi(x)$ has a neighborhood which is an $H(i)$ subset.*

Proof. Theorem 5.21 and Corollary 5.22.

Our final theorem in this section is one on common fixed points of a family of multifunctions with θ-closed graphs.

THEOREM 5.24. *Let X be an $H(i)$ space and let F be a family of multifunctions from X to X with θ-subclosed graphs. If for each finite $F^* \subset F$ there is an $x \in X$ satisfying $x \in \Phi(x)$ for each $\Phi \in F^*$, then there is an $x \in X$ satisfying $x \in \Phi(x)$ for all $\Phi \in F$.*

Proof. If $\lambda: X \to X$ is the identity function then $\Omega =$

$\{E(\Phi, \lambda, X, X): \Phi \in F\}$ is a family of θ-closed subsets of X from Corollary 5.14. If $F^* \subset F$ is finite there is an $x \in X$ satisfying $x \in \Phi(x)$ for all $\Phi \in F^*$. So Ω has the finite intersection property. If Ω^* is the filterbase generated by Ω then $\mathrm{ad}_\theta \Omega^* \neq \emptyset$ since X is $H(i)$ and so $\bigcap_\gamma J \neq \emptyset$. This means that there is an $x \in X$ satisfying $x \in E(\Phi, \lambda, X, X)$ for all $\Phi \in F$. The proof is complete.

References

1. C. Berge, *Topological Spaces*, Macmillan, New York, (1963).
2. I. Baggs, *Functions with a closed graph*, Proc. Amer. Mrth. Soc., **43** (1974), 439–442.
3. N. Bourbaki, *Espaces minimaux et espaces complètement separès*, C. R. Acad. Sci. Paris, **212** (1941), 215–218.
4. G. H. Butcher and J. E. Joseph, *Two generalized compactness notions via graphs*, submitted.
5. R. F. Dickman, Jr. and J. R. Porter, *θ-perfect and θ-absolutely closed functions*, Illinois J. Math., **21** (1977), 42–60.
6. ———, *θ-closed subsets of Hausdorff spaces*, Pacific J. Math., **59** (1975), 407–414.
7. S. Fomin, *Extensions of topological spaces*, Ann. of Math., **44** (1943), 471–480.
8. L. L. Herrington, *H(i) spaces and strongly-closed graphs*, Proc. Amer. Math. Soc., **58** (1976), 277–283.
9. L. L. Herrington and P. E. Long, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc., **48** (1975), 469–475.
10. ———, *A characterization of minimal Hausdorff spaces*, Proc. Amer. Math. Soc., **57** (1976), 373–374.
11. J. E. Joseph, *On H-closed and minimal Hausdorff spaces*, Proc. Amer. Math. Soc., **60** (1976), 321–326.
12. ———, *Characterizations of minimal Hausdorff spaces*, Proc. Amer. Math. Soc., **61** (1976), 145–148.
13. ———, *Uniform boundedness generalizations* Math. Chron., to appear.
14. ———, *θ-closure and θ-subclosed graphs*, Math. Chron., to appear.
15. P. E. Long, *Functions with closed graphs*, Amer. Math. Monthly, **76** (1969), 930–932.
16. Norman Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, **68** (1961), 44–46.
17. ———, *When are compact and closed equivalent*, Amer. Math. Monthly, **72** (1965), 41–49.
18. ———, *Simple extensions of topologies*, Amer. Math. Monthly, **71** (1964), 22–25.
19. T. B. Muenzenberger and R. E. Smithson, *Continuity and closed functions*, preprint.
20. R. E. Smithson, *Multifunctions*, Nieuw Archief voor Wiskunde, (3) **20** (1972), 31–53.
21. ———, *Subcontinuity for multifunctions*, Pacific J. Math., **61** (1975), 283–288.
22. ———, *Almost and weak continuity for multifunctions*, preprint.
23. C. T. Scarborough and A. H. Stone, *Products of nearly compact spaces*, Trans. Amer. Math. Soc., **124** (1966), 131–147.
24. T. Thompson, *Characterizing certain sets with functions having closed graphs*, Boll. U.M.I. (4) **12** (1975), 327–329.
25. H.V. Veličko, *H-closed topological spaces*, Mat. Sb., **70** (112), (1966), 98–112; Amer. Math. Soc. Transl., **78** (Series 2) (1969), 103–118.
26. S. W. Willard, *General Topology*, Addison Wesley, Reading, Mass., (1970).

Received May 25, 1977.

HOWARD UNIVERSITY
WASHINGTON, D.C. 20059